Change Point Tests Based on U-Statistics with Applications in Reliability

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Abstract: We consider the problem of testing the null hypothesis of no change against the alternative of exactly one change point. The proposed tests are based on generalized two-sample U-statistic processes. We drive the limiting null distributions of the proposed tests. Some applications in Statistical Reliability are given.

1 Introduction

Change point problems have received considerable attention in the last two decades. They are often encountered in the fields of quality control, reliability and survival analysis. A typical situation occurs in quality control when one observes the output of a production line and is interested in detecting any change in the quality of the product.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with respective distribution functions $F_1, F_2, \ldots, F_n$. We may be interested in the problem of testing the null hypothesis of no change

$$H_0\colon F_1 = F_2 = \cdots = F_n = F \ , \qquad \text{where } F \text{ is unknown}$$

against the alternative of exactly one change point

$$H_1'\colon F_1 = \cdots = F_{[n\tau]} = F \quad \text{and} \quad F_{[n\tau]+1} = \cdots = F_n = G \ ,$$

$$\text{where } F \neq G \quad \text{and} \quad \tau \in (0, 1) \quad \text{are unknown} \ .$$

where $[x]$ is the integer part of x. In H_1' we may consider various types of restrictions on the nature of the difference between F and G, parametric as well as nonparametric. In this article we will only consider the nonparametric approach. For additional results and references on change point analysis we refer to Zacks (1983), Bhattacharyya (1984), Csörgő and Horváth (1988a), Sen (1988), Lombard (1989) and Hušková and Sen (1989).

Most of the previous work in this area is focused on possible location changes, that is, when G is of the form $G(x) = F(x - \theta)$ for all x and for some unknown θ. We note here that the problem of testing H_0 against H_1' is like a two-sample problem based on two independent samples, $X_1, X_2, \ldots, X_{[n\tau]}$ and $X_{[n\tau]+1}, \ldots, X_n$, with unknown $\tau \in (0, 1)$. Motivated by this observation Sen and Srivastava (1975) proposed the test statistics $S_n^{(1)}$ and $S_n^{(2)}$ for the one-sided and the two-sided location alternatives, respectively, where

$$S_n^{(1)} = \max_{1 \le k \le n-1} \left(\frac{V_{k,n-k} - E_0(V_{k,n-k})}{\sqrt{var_0(V_{k,n-k})}} \right),$$

$$S_n^{(2)} = \max_{1 \le k \le n-1} \left| \frac{V_{k,n-k} - E_0(V_{k,n-k})}{\sqrt{var_0(V_{k,n-k})}} \right|,$$

$V_{k,n-k}$ is the usual Mann-Whitney form of the Wilcoxon statistics based on the two samples $X_1, X_2, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$ and $E_0(V_{k,n-k})$ (resp. $var_0(V_{k,n-k})$) is the expectation (resp. variance) of $V_{k,n-k}$ under H_0. We note here that $S_n^{(1)}$ and $S_n^{(2)}$ are extensions of the Wilcoxon-Mann-Whitney test statistic to the change point setup. Csörgő and Horváth (1988b) obtained the limiting distributions of statistics of the above type when the associated U-statistics are based on kernels of degree (1, 1), which include the above case. Pettitt (1979), Sen (1982), Csörgő and Horváth (1988b), and Hawkins (1989) proposed tests for H_0 against H_1' which are based on one-sample U-statistics.

It is well known that the Wilcoxon-Mann-Whitney test has good power properties for detecting location differences for moderate tailed distributions and for testing against stochastic ordering alternatives in some other cases. However, several two-sample tests based on generalized U-statistics of degrees other than (1, 1) are available in the literature which are good competitors of the Wilcoxon-Mann-Whitney test and it would be of interest to examine their extensions to the change point setup. The investigation of change point test procedures which are based on generalized U-statistics of degrees other than (1, 1) is also motivated by the following example.

Example 1: Let F be a life distribution function, $\overline{F} = 1 - F$ and f be the corresponding survival and density functions, respectively. The hazard rate function of F is $r_F(\cdot) = f(\cdot)/\overline{F}(\cdot)$. Suppose that the quality of items produced in an assembly line is measured by their hazard rates. An item with a smaller hazard rate would *tend to survive longer. Due to wear and tear of the machinery it might be of interest to test H_0 against H_1' where the difference between F and G is expressed as $F \underset{hr}{\preceq} G$, i.e., F is less than G in the hazard rate ordering which holds if and only if, $r_F(\cdot) \geq r_G(\cdot)$. In the corresponding two-sample problem, Kochar (1979) proposed a testing procedure based on a generalized U-statistic of degree (2, 2). A change point extension of Kochar (1979)'s test will be discussed in the sequel.*

In this article we consider test procedures based on generalized U-statistics of degrees other than (1, 1). These procedures can be used to test H_0 against

$$H_1 \colon H_1' \text{ holds with } F \underset{PO}{\preceq} G \text{ and } F \neq G \ ,$$

where $\underset{PO}{\preceq}$ is a specified partial ordering of the considered family of distribution functions like, for example, the stochastic and hazard rate orderings.

Next we give some basic defintions of generalized two-sample U-statistics of degree (m, m). Let $Y_1, Y_2, \ldots, Y_{n_1}$ and $Z_1, Z_2, \ldots, Z_{n_2}$ be two independent samples of sizes n_1 and n_2 drawn from the Y and the Z populations, respectively. Let $\phi(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m)$ be a kernel of degree (m, m), $1 \leq m \leq \min(n_1, n_2)$. Without any loss of generality we may assume that ϕ is symmetric in its first (resp. last) m arguments. The roles of these two sets of arguments need not be (and frequently are not) symmetric. A kernel ϕ of degree (m, m) is said to be skew-symmetric if

$$\phi(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m) = -\phi(z_1, z_2, \ldots, z_m; y_1, y_2, \ldots, y_m) \ .$$

In most of the applications we employ skew-symmetric kernels. For this reason we will confine our attention to this type of kernels.

The generalized two-sample U-statistic based on the Y and Z samples and the kernel ϕ of degree (m, m) is defined as

$$U_{n_1, n_2} = \left\{ \binom{n_1}{m} \binom{n_2}{m} \right\}^{-1} \sum \sum \phi(Y_{i_1}, \ldots, Y_{i_m}; Z_{j_1}, \ldots, Z_{j_m}) \ , \tag{1}$$

where the above summations extend over $1 \leq i_1 < i_2 < \cdots < i_m \leq n_1$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n_2$. Since the kernels are assumed to be skew-symmetric, it follows that under H_0, $E(U_{n_1, n_2}) = 0$. For the kernel ϕ of (1) we define

$$\phi_1(y) = E\phi(y, Y_2, \ldots, Y_m; Z_1, Z_2, \ldots, Z_m) \ . \tag{2}$$

We will need the following condition

$$E|\phi(Y_1, Y_2, \ldots, Y_m; Z_1, Z_2, \ldots, Z_m)|^v < \infty \ , \qquad \text{for some } v \geq 2 \ . \tag{3}$$

Let

$$\sigma_1^2 = E\phi_1^2(Y_1) \qquad \text{and} \qquad \sigma_2^2 = E\phi_1^2(Z_1) .$$

Condition (3) insures that σ_1, σ_2 and $E|\phi_1(Y_1)\cdot\phi_1(Z_1)|$ are finite.

2 Tests Against the One-Change Point Alternative

Assume first that the change point τ is known and $[n\tau] = k, m \leq k \leq n - m$. Let $U_{k,n-k}$ be a generalized two-sample U-statistic based on a skew-symmetric kernel ϕ of degree (m, m) and the two samples $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$. The kernel ϕ is selected in such a way that $U_{k,n-k}$ is an appropriate test statistic for testing H_0 against H_1' when τ is known and $[n\tau] = k$ and large values of $U_{k,n-k}$ are significant. The two-sample U-statistic tests proposed here are based on the U-statistic process $\{U_n(t), 0 \leq t \leq 1\}$, where

$$U_n\left(\frac{k}{n}\right) = m^{-1}n^{-2}k(n-k)\cdot U_{k,n-k} , \qquad m \leq k \leq n-m , \tag{4}$$

$U_n(0) = U_n(1) = 0$ and $U_n(\cdot)$ is defined elsewhere by interpolation. One of the following test staistics may be used for testing H_0 against H_1.

$$T_{1n} = \sum_{k=m}^{n-m} U_n^2\left(\frac{k}{n}\right) ,$$

$$T_{2n} = \sqrt{n} \max_{m \leq k \leq n-m} U_n\left(\frac{k}{n}\right) ,$$

$$T_{3n} = \sqrt{n} \max_{m \leq k \leq n-m} \left|U_n\left(\frac{k}{n}\right)\right| ,$$

$$T_{4n} = n^2 \sum_{k=m}^{n-m} \frac{U_n^2\left(\frac{k}{n}\right)}{k(n-k)} ,$$

$$T_{5n} = n^{3/2} \max_{m \leq k \leq n-m} \frac{U_n\left(\frac{k}{n}\right)}{\sqrt{k(n-k)}} ,$$

$$T_{6n} = n^{3/2} \max_{m \le k \le n-m} \frac{\left| U_n\left(\frac{k}{n}\right) \right|}{\sqrt{k(n-k)}} .$$

The limiting null distributions of T_{in}, $i = 1, 2, \ldots, 6$ are given in the following Theorem which is proved in Section 4.

Theorem 1: Assume that ϕ is skew-symmetric of degree (m, m), $B(\cdot)$ is a Brownian bridge, H_0 and condition (3) *hold and let $\sigma^2 = E\phi_1^2(X_1)$.*

1. *If $v = 2$ in condition* (3), *then as $n \to \infty$,*

$$T_{1n}/\sigma^2 \xrightarrow{D} \int_0^1 B^2(s)ds := t_1 ,$$

$$T_{2n}/\sigma \xrightarrow{D} \sup_{0 < s < 1} B(s) := t_2 ,$$

and

$$T_{3n}/\sigma \xrightarrow{D} \sup_{0 < s < 1} |B(s)| := t_3 .$$

2. *If $v > 2$ in condition* (3), *then as $n \to \infty$,*

$$T_{4n}/\sigma^2 \xrightarrow{D} \int_0^1 \frac{B^2(s)}{s(1-s)} ds := t_4 ,$$

$$P\{T_{5n}/\sigma \le a(y, \log n)\} \to \exp\{-\exp - y\} , \qquad -\infty < y < \infty$$

and

$$P\{T_{6n}/\sigma \le a(y, \log n)\} \to \exp\{-2 \exp - y\} , \qquad 0 < y < \infty ,$$

where $a(y, s) = (2 \log s)^{-1/2}\{y + 2 \log s + \frac{1}{2} \log \log s - \frac{1}{2} \log \pi\}$.

The critical values of t_1, t_3 and t_4 are respectively given in Tables 4, 1 and 5 of Section 8 of Chapter 3 of Shorack and Wellner (1986). The critical values of t_2 are obtained from the well known result $\Pr\{t_2 \ge x\} = \exp\{-2x^2\}$, $x > 0$.

3 Applications

Example 2: This is a continuation of Example 1 in which we wish to test H_0 *against*

$$H_{1,hr}: H_1' \text{ holds with } F \underset{hr}{\preceq} G \text{ and } F \neq G \ .$$

In the corresponding two-sample problem, Kochar (1979) proposed the kernel

$$\phi(x_1, x_2; y_1, y_2) = \begin{cases} -1 & \text{if } xxyy \text{ or } yxxy \\ 0 & \text{if } xyxy \text{ or } yxyx \\ 1 & \text{if } yyxx \text{ or } xyyx \ , \end{cases} \tag{5}$$

where, for example, yyxx represents $y_1 \leq y_2 \leq x_1 \leq x_2$, $y_2 \leq y_1 \leq x_2 \leq x_1$, $y_2 \leq y_1 \leq x_1 \leq x_2$ *or* $y_1 \leq y_2 \leq x_2 \leq x_1$. *It is shown in Kochar (1979) that the two-sample test based on the kernel* ϕ *of (5) performs better than the Wilcoxon test for detecting hazard rate ordering.*

Tests for H_0 *against* $H_{1,hr}$ *can be based on the process* $U_n(\cdot)$ *of (4) with the kernel* ϕ *of (5) which is skew symmetric of degree (2, 2). Under* H_0 *and condition (3) we obtain the results of Theorem 1 with* $\sigma^2 = \frac{8}{105}$ *(see Theorem 2.1 of Kochar (1979) for details).*

Example 3: Testing Against a Change in the Location Parameter

We are interested in testing H_0 *against* H_1' *with* $G(x) = F(x + \theta)$, $\theta \geq 0$ *and* F *is unknown. In the corresponding two-sample problem Deshpandé and Kochar (1982) proposed the kernel*

$$\phi(x_1, \ldots, x_m; y_1, \ldots, y_m) = \begin{cases} 1 & \text{if } \min(y_1, \ldots, y_m) \leq \min(x_1, \ldots, x_m) \\ & \text{and } \max(y_1, \ldots, y_m) \leq \max(x_1, \ldots, x_m) \\ -1 & \text{if } \min(x_1, \ldots, x_m) \leq \min(y_1, \ldots, y_m) \\ & \text{and } \max(x_1, \ldots, x_m) \leq \max(y_1, \ldots, y_m) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and showed that large values of m *are more suitable for thin tailed distributions. Note that when* $m = 1$, *the kernel* ϕ *of (6) reduces to the Wilcoxon-Mann-Whitney kernel.*

Tests for H_0 against a change in the location parameter can be based on the process $U_n(\cdot)$ of (4) *with the kernel ϕ of* (6) *which is skew-symmetric of degree* (m, m). *Under H_0 and condition* (3) *we obtain the results of Theorem* 1 *with*

$$\sigma^2 = 2m^2(2m-1)^{-2}(4m-1)^{-1}\left\{1 - \frac{1}{\binom{4m-1}{2m-1}}\right\},$$

(*see Theorem* 2.1 *of Deshpandé and Kochar* (1982) *for details*).

Example 4: Testing Against a Change in the Scale Parameter

We consider here the problem of testing H_0 against H_1' with $G(x) = F(\theta x)$, $\theta \geq 1$ and F is symmetric around zero and unknown. In the corresponding two-sample problem Kochar and Gupta (1986) *proposed tests based on the kernel*

$$\phi(x_1, \ldots, x_m; y_1, \ldots, y_m) = \begin{cases} 1 & \text{if } \min \text{ as well as } \max \text{ of } (x_1, \ldots, x_m, y_1, \\ & \ldots, y_m) \text{ are some } x\text{'s} \\ -1 & \text{if } \min \text{ as well as } \max \text{ of } (x_1, \ldots, x_m, y_1, \\ & \ldots, y_m) \text{ are some } y\text{'s} \\ 0 & \text{otherwise} . \end{cases} \tag{7}$$

Kochar and Gupta (1986) *showed that the corresponding U-statistic when $m = 2$ is equivalent to the two-sample Mood's statistic for the scale problem. For other choices of m these tests perform better for thin tailed distributions. For additional properties of these tests we refer to Kochar and Woodworth* (1992).

Tests for H_0 against a change in the scale parameter can be based on the process $U_n(\cdot)$ of (4) *with the kernel ϕ of* (7) *which is skew-symmetric of degree* (m, m). *Under H_0 and condition* (3) *we obtain the results of Theorem* 1 *with*

$$\sigma^2 = \frac{2m^2}{(2m-1)^2}\left\{\frac{1}{4m-1} - \frac{1}{2m^2} + \frac{\{(2m-1)!\}^2}{(4m-1)!}\right\},$$

(*see Corollary* 2.1 *of Kochar and Gupta* (1986) *for details*).

4 Proofs

Theorem 1 follows from the results of Section 4 of Csörgö and Horváth (1988b) and the following result.

Lemma 1: Assume that ϕ is skew symmetric of degree (m, m) and H_0 and condition (3) hold. Let $\sigma^2 = E\phi_1^2(X_1)$ and $\xi_i = \sigma^{-1}\phi_1(X_i)$, $i = 1, 2, \ldots, n$, where $\phi_1(\cdot)$ is as in (2). Then,

$$\max_{m<k\leq n-m}\left|\sqrt{n}\sigma^{-1}U_n\left(\frac{k}{n}\right) - n^{-1/2}\left\{\sum_{i=1}^{k}\xi_i - \frac{k}{n}\sum_{i=1}^{n}\xi_i\right\}\right| \stackrel{P}{=} O(n^{-1/2}) \ . \tag{8}$$

Proof: The proof for $m > 2$ is a routine extension of that of $m = 2$ which is given below.

Recall that

$$U_n\left(\frac{k}{n}\right) = m^{-1}n^{-2}k(n-k)U^*_{k,n-k}\left\{\binom{k}{2}\binom{n-k}{2}\right\}^{-1} , \tag{9}$$

where

$$U^*_{k,n-k} = \sum_{1\leq i<j\leq k}\ \sum_{k+1\leq r<s\leq n}\phi(X_i, X_j; X_r, X_s) \ . \tag{10}$$

Let $\Phi(\cdot;\cdot)$: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be such that $\Phi(\mathbf{x};\mathbf{y}) = \phi(x_1, x_2; y_1, y_2)$, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Define $\mathbf{Z} = (X_1, X_2)$, $K = \binom{k}{2}$, $N - K = \binom{n-k}{2}$ and

$$\Phi_1(\mathbf{x}) = E\Phi(\mathbf{x};\mathbf{Z}) = -E\Phi(\mathbf{Z};\mathbf{x}) \ .$$

Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_K$ be the pairs $\{(X_i, X_j), 1 \leq i < j \leq k\}$ in lexicographic (dictionary) ordering and $\mathbf{Z}_{K+1}, \ldots, \mathbf{Z}_N$ be the pairs $\{(X_r, X_s), k+1 \leq r < s \leq n\}$ in lexicographic ordering. Note that

$$U^*_{k,n-k} = \sum_{1\leq i\leq K}\ \sum_{K+1\leq j\leq N}\Phi(\mathbf{Z}_i;\mathbf{Z}_j) \ .$$

Following Csörgő and Horváth (1988b), we write

$$U^*_{k,n-k} = U_N^{(1)} - U_K^{(2)} - U_K^{(3)} \; , \tag{11}$$

where

$$U_N^{(1)} = \sum_{1 \le i < j \le N} \Phi(\mathbf{Z}_i; \mathbf{Z}_j) \; ,$$

$$U_K^{(2)} = \sum_{1 \le i < j \le K} \Phi(\mathbf{Z}_i; \mathbf{Z}_j) \; ,$$

and

$$U_K^{(3)} = \sum_{K+1 \le i < j \le N} \Phi(\mathbf{Z}_i; \mathbf{Z}_j) \; .$$

By (2.9) of Janson and Wichura (1983) and the equation immediately preceeding it (see also (4.4)–(4.7) of Csörgő and Horváth (1988b)), we obtain

$$\max_{2 \le k \le n-2} \left| U_N^{(1)} - \sum_{i=1}^{N} (N - 2i + 1)\Phi_1(\mathbf{Z}_i) \right| \overset{P}{=} O(n^2) \tag{12}$$

$$\max_{2 \le k \le n-2} \left| U_K^{(2)} - \sum_{i=1}^{K} (K - 2i + 1)\Phi_1(\mathbf{Z}_i) \right| \overset{P}{=} O(n^2) \tag{13}$$

and

$$\max_{2 \le k \le n-2} \left| U_K^{(3)} - \sum_{i=K+1}^{N} (N + K - 2i + 1)\Phi_1(\mathbf{Z}_i) \right| \overset{P}{=} O(n^2) \; . \tag{14}$$

By (11)–(14) we obtain

$$\max_{2 \le k \le n-2} \left| U^*_{k,n-k} - \left\{ \binom{n-k}{2} \sum_{i=1}^{K} \Phi_1(\mathbf{Z}_i) - \binom{k}{2} \sum_{i=K+1}^{N} \Phi_1(\mathbf{Z}_i) \right\} \right|$$
$$\overset{P}{=} O(n^2) \; . \tag{15}$$

Next, we note that

$$\sum_{i=1}^{K} \Phi_1(\mathbf{Z}_i) = \sum_{1 \le i < j \le k} \Phi_1((X_i, X_j)) \;, \tag{16}$$

$$\sum_{i=K+1}^{N} \Phi_1(\mathbf{Z}_i) = \sum_{k+1 \le i < j \le n} \Phi_1((X_i, X_j)) \;, \tag{17}$$

and

$$\phi_1(x) = E\Phi_1((x, X_1)) = E\Phi_1((X_1, x)) \;. \tag{18}$$

By (16)–(18) and Theorem 1 of Hall (1979) we obtain

$$\max_{2 \le k \le n-2} \left| \sum_{i=1}^{K} \Phi_1(\mathbf{Z}_i) - \binom{k-1}{1} \sum_{i=1}^{k} \phi_1(X_i) \right| \overset{P}{=} O(n) \;, \tag{19}$$

and

$$\max_{2 \le k \le n-2} \left| \sum_{i=K+1}^{N} \Phi_1(\mathbf{Z}_i) - \binom{n-k-1}{1} \sum_{i=k+1}^{n} \phi_1(X_i) \right| \overset{P}{=} O(n) \;. \tag{20}$$

By (15), (19) and (20) we obtain

$$\max_{2 \le k \le n-2} \left| U^*_{k,n-k} - \binom{n-k}{2}\binom{k-1}{1} \sum_{i=1}^{k} \phi_1(X_i) + \binom{k}{2}\binom{n-k-1}{1} \sum_{i=k+1}^{n} \phi_1(X_i) \right|$$
$$\overset{P}{=} O(n^2) \;. \tag{21}$$

By (9) and (21) we obtain (8).

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