Nonnegative Idempotent Matrices and the Minus Partial Order

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ABSTRACT

We describe the structure of nonnegative matrices dominated by a nonnegative idempotent matrix under the minus order.

1. INTRODUCTION

Matrix partial orders have been an area of intense research in the past few years. The minus partial order is one of the important partial orders for matrices, and it is related to several other concepts such as rank additivity, shorted operators, and the parallel sum [4]. It is well known that idempotent matrices play an important role in the theory of generalized inverses. In particular, the structure of idempotent matrices that are entrywise nonnegative is well understood, and it is a natural problem to investigate how the minus partial order behaves with reference to nonnegative idempotent matrices. The purpose of this paper is to describe the structure of nonnegative matrices which are dominated by a given nonnegative idempotent matrix under the minus partial order. In Section 2 we introduce some definitions and prove some preliminary results. The main result is proved in Section 3.

2. DEFINITIONS AND PRELIMINARY RESULTS

We consider only real matrices. A matrix $A = [a_{ij}]$ is *nonnegative* if $a_{ij} \geqslant 0$ for all i, j, in which case we write $A \geqslant 0$. Similarly, A is *positive* if $a_{ij} > 0$ for all i, j, denoted $A > 0$. The square matrix A is *idempotent* if $A^2 = A$. The transpose of A is denoted by A^T, and the identity matrix of the appropriate order is denoted by I.

If A is an $m \times n$ matrix, then an $n \times m$ matrix G is a *generalized inverse* (or a *g-inverse*) of A if $AGA = A$. We denote an arbitrary g-inverse of A by A^-.

A matrix J is a *direct sum* of matrices $J_1, \dots, J_r$, denoted by $J = J_1 \oplus \cdots \oplus J_r$, if

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_r \end{bmatrix}.$$

If A, B are $m \times n$ matrices, then we say that A is dominated by B in the *minus order*, denoted by $A \leqslant^- B$, if rank B = rank A + rank$(B - A)$.

It is well known that the minus order is a partial order, and several characterizations of it are available in the literature (for example, see [3]). In particular, we note the following result which will be used.

THEOREM 1. *Let A, B by $m \times n$ matrices. Then the following conditions are equivalent:*

(i) $A \leqslant^- B$.

(ii) *There exists a g-inverse A^- of A such that* $(B - A)A^- = 0$ *and* $A^-(B - A) = 0$.

(iii) *Every g-inverse of B is a g-inverse of A.*

(iv) *Every g-inverse B^- of B satisfies* $AB^-(B - A) = 0$ *and* $(B - A)B^-A = 0$.

We remark that (iv) can be interpreted as saying the "parallel sum" of A and $B - A$ is zero.

We will need the following characterization of nonnegative idempotent matrices due to Flor [2] (also see Theorem 3.1, p. 65 in [1]).

THEOREM 2. *If E is a nonnegative idempotent matrix of rank r, then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where all the diagonal blocks are square; J is a direct sum of matrices $x_i y_i^T$, $x_i > 0$, $y_i > 0$, and $y_i^T x_i = 1$, $i = 1, 2, \ldots, r$; and C, D are nonnegative matrices of suitable sizes.

It follows from Theorem 2 that if E is a positive idempotent matrix, then E must be of rank one.

The main purpose of this paper is to describe nonnegative matrices dominated by a given nonnegative idempotent matrix under the minus order. We first prove some preliminary results which are needed for such a description.

LEMMA 3. *Let A, E be $n \times n$ matrices such that $E^2 = E$, and suppose that $A \leqslant^- E$. Then A is idempotent and $AE = A = EA$.*

Proof. Since $E^2 = E$, the $n \times n$ identity matrix I is a g-inverse of E. Since $A \leqslant^- E$, by Theorem 1(iii) I must be a g-inverse of A and therefore $A^2 = A$. Also, by Theorem 1(iv), $AI(E - A) = (E - A)IA = 0$, and this completes the proof. ∎

LEMMA 4. *Let X, J be $n \times n$ nonnegative matrices such that $JX = X = XJ$ and suppose $J = J_1 \oplus \cdots \oplus J_r$, where $J_i = x_i y_i^T$, $x_i > 0$, $y_i > 0$, with $y_i^T x_i = 1$, $i = 1, 2, \ldots, r$. Let $X = [X_{ij}]$ be the $r \times r$ block partitioning of X in conformity with that of J. Then for each i, j, either $X_{ij} = 0$ or X_{ij} is a positive matrix of rank 1.*

Proof. Let i, j be fixed, $1 \leqslant i, j \leqslant r$. Since $JX = X = XJ$, we have

$$J_i X_{ij} = X_{ij} = X_{ij} J_j. \tag{1}$$

Therefore rank $X_{ij} \leqslant$ rank $J_j = 1$. If rank $X_{ij} = 0$ then $X_{ij} = 0$. Suppose rank $X_{ij} = 1$, and let $X_{ij} = uv^T$ be a nonnegative rank factorization. Then from (1)

$$J_i uv^T = uv^T = uv^T J_j. \tag{2}$$

Since u, v must be nonzero vectors, it follows from (2) that $J_i u = u$ and $v^T = v^T J_j$. Since J_i, J_j are positive matrices, u, v must be positive vectors. Thus $X_{ij} = uv^T$ is positive and the proof is complete. ∎

3. THE MAIN RESULT

THEOREM 5. *Let A, E be $n \times n$ nonnegative matrices such that $E^2 = E$, and suppose $A \leqslant^- E$. Then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad PAP^T = \begin{bmatrix} U & UD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CU & CUD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$U = \begin{bmatrix} \hat{J} & \hat{J}\hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{C}\hat{J} & \hat{C}\hat{J}\hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

J and $\hat{J}$ are direct sums of positive idempotent matrices of rank 1, and $C, D, \hat{C}, \hat{D}$ are nonnegative matrices of the appropriate sizes.

Proof. Since $E \geqslant 0$ and $E^2 = E$, by Theorem 2 there exists a permutation matrix R such that

$$RER^T = \begin{bmatrix} \bar{J} & \bar{J}\bar{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{C}\bar{J} & \bar{C}\bar{J}\bar{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $\tilde{J} = J_1 \oplus \cdots \oplus J_r$, $J_i = x_i y_i^T$, x_i, y_i are positive vectors, and $y_i^T x_i = 1$, $i = 1, 2, \ldots, r$. For convenience we assume that E itself is of the form given in (3), as this will not affect the conclusion of the theorem. Let

$$A = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}$$

be the corresponding partitioning of A. Since $A \leqslant E$, by Lemma 3 we have $AE = A = EA$, and from these equations it follows that

$$A = \begin{bmatrix} Y_{11} & Y_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y_{31} & Y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equation $AE = A$ now gives

$$\begin{bmatrix} Y_{11}\tilde{J} & Y_{11}\tilde{J}\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y_{31}\tilde{J} & Y_{31}\tilde{J}\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y_{31} & Y_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $Y_{11} = Y_{11}\tilde{J}$, $Y_{12} = Y_{11}\tilde{J}\tilde{D}$, and hence $Y_{12} = Y_{11}\tilde{D}$. Similarly, from $A = EA$ we conclude that $Y_{11} = \tilde{J}Y_{11}$, $Y_{31} = \tilde{C}\tilde{J}Y_{11}$, and hence $Y_{31} = \tilde{C}Y_{11}$. It follows that $Y_{32} = Y_{31}\tilde{J}\tilde{D} = \tilde{C}Y_{11}\tilde{J}\tilde{D} = \tilde{C}Y_{11}\tilde{D}$. Therefore, setting $X = Y_{11}$, we have

$$A = \begin{bmatrix} X & X\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{C}X & \tilde{C}X\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now take a closer look at the equation $X\tilde{J} = X = \tilde{J}X$. Let

$$\tilde{J} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_r \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1r} \\ X_{21} & X_{22} & \cdots & X_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{bmatrix}$$

be conformal partitions. By Lemma 4, each X_{ij} is either zero or a positive, rank one matrix. We also note from the equation $A^2 = A$ that $X^2 = X$. Construct the $r \times r$ matrix $Z = [z_{ij}]$ by setting $z_{ij} = y_i^T X_{ij} x_j$, $1 \leqslant i, j \leqslant r$. We claim that Z is idempotent. This is seen as follows. For any $i, k \in \{1, 2, \dots, r\}$, we have

$$\begin{aligned}
\sum_{j=1}^{r} z_{ij} z_{jk} &= \sum_{j=1}^{r} y_i^T X_{ij} x_j y_j^T X_{jk} x_k \\
&= y_i^T \left\{ \sum_{j=1}^{r} X_{ij} J_j X_{jk} \right\} x_k \qquad \left(\text{since } x_j y_j^T = J_j\right) \\
&= y_i^T \left\{ \sum_{j=1}^{r} X_{ij} X_{jk} \right\} x_k \qquad \left(\text{since } X_{ij} J_j = X_{ij}\right) \\
&= y_i^T X_{ik} x_k \qquad \left(\text{since } X^2 = X\right) \\
&= z_{ik},
\end{aligned}$$

and therefore the claim is proved.

Since Z is a nonnegative idempotent matrix, by Theorem 2 there exists a permutation matrix Q_1 such that

$$Q_1 Z Q_1^T = \begin{bmatrix} J_1 & J_1 D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1 J_1 & C_1 J_1 D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J_1 is a direct sum of positive idempotent matrices, each of rank one. Let Q be the permutation matrix obtained from Q_1 by replacing the 1 in the ith column by the identity matrix of the same order as X_{ii}, $i = 1, 2, \dots, r$, and replacing each zero by a zero matrix of appropriate size. Then by block matrix multiplication it can be seen that QXQ^T permutes the blocks X_{ij} of X in the same way as $Q_1 Z Q_1^T$ permutes the entries z_{ij} of Z. Therefore, keeping in mind that each X_{ij} is either zero or positive, we have

$$QXQ^T = \begin{bmatrix} \hat{J} & \hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{C} & \hat{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\hat{J}$ is a direct sum of positive matrices and $\hat{D}$, $\hat{C}$, $\hat{M}$ are nonnegative matrices.

The equation $(QXQ^T)^2 = QXQ^T$ now leads to $\hat{J}^2 = \hat{J}$, $\hat{D} = \hat{J}\hat{D}$, $\hat{C} = \hat{C}\hat{J}$, and $\hat{M} = \hat{C}\hat{J}\hat{D}$. It follows that $\hat{J}$ must be a direct sum of positive idempotent matrices of rank 1 and we have

$$QXQ^T = \begin{bmatrix} \hat{J} & \hat{J}\hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{C}\hat{J} & \hat{C}\hat{J}\hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set

$$P = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where the partitioning is conformal with that of A and E. Then

$$PEP^T = \begin{bmatrix} Q\tilde{J}Q^T & Q\tilde{J}\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{C}\tilde{J}Q^T & \tilde{C}\tilde{J}\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad PAP^T = \begin{bmatrix} QXQ^T & QX\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{C}XQ^T & \tilde{C}X\tilde{D} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The construction of Q and the structure of $\tilde{J}$ (as a block sum) show that $Q\tilde{J}Q^T$ is also a direct sum of positive, rank one idempotent matrices. In fact if Q_1 corresponds to the permutation σ, then

$$Q\tilde{J}Q^T = \begin{bmatrix} J_{\sigma(1)} & 0 & \cdots & 0 \\ 0 & J_{\sigma(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\sigma(r)} \end{bmatrix}.$$

Finally, set $U = QXQ^T$, $C = \tilde{C}Q^T$, $D = Q\tilde{D}$, and $J = Q\tilde{J}Q^T$ to get forms of PEP^T and PAP^T as asserted. That completes the proof. ■

COROLLARY 6. *Let A, E be $n \times n$ nonnegative matrices with no zero row or column. Suppose $E^2 = E$ and $A \leqslant^- E$. Then the following assertions hold.*

(i) *There exists a permutation matrix P such that PEP^T and PAP^T are both direct sums of positive idempotent matrices of rank 1.*

(ii) *For any i, j if $e_{ij} > 0$ then $a_{ij} > 0$, where $A = [a_{ij}]$ and $E = [e_{ij}]$.*

(iii) *Let $\{1, 2, \dots, n\} = S_1 \cup \cdots \cup S_r$ be the partition of $\{1, 2, \dots, n\}$ induced by the direct sum representation of PEP^T, and let $\{1, 2, \dots, n\} = T_1 \cup \cdots \cup T_k$ be the partition induced by the direct sum representation of PAP^T. Then the former partition is a refinement of the latter, and if $S_i = T_j$ for some i, j then the corresponding blocks in the direct sum representations are identical.*

The proof of Corollary 6 is essentially contained in the proof of Theorem 5. Statement (ii) of the corollary follows from $AE = A = EA$ together with the fact that the diagonal of A is positive, and statement (iii) follows from (ii).

The next example shows that in the case of Theorem 5, the natural converse implication is not true.

EXAMPLE 7. Let

$$A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then A, E are both nonnegative idempotent matrices. Also the condition in Theorem 5 is trivially satisfied. However, A is not dominated by E in the minus order, since $\operatorname{rank} E = 2$, which does not equal $\operatorname{rank} A + \operatorname{rank}(E - A)$, which is 3.

The following example illustrates the structure described in Corollary 6.

EXAMPLE 8. Let

$$E = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then it can be verified that the hypotheses in Corollary 6 are satisfied. Also, if P is the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$PBP^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$PAP^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which are in the form asserted in Corollary 6.

In the remainder of the paper we show that the set of nonnegative matrices dominated in the minus order by a given nonnegative idempotent matrix of rank r can be "identified" with the set of $r \times r$ nonnegative idempotent matrices. This jibes well with the simple observation that the set of nonnegative matrices dominated by the $r \times r$ identity matrix is precisely the set of all $r \times r$ nonnegative idempotent matrices.

We now introduce some notation. Let E be an $n \times n$ nonnegative idempotent matrix of rank r. Let $\mathscr{M}_E = \{A : A \geqslant 0, \; A \leqslant^- E\}$, and let $\mathscr{I}_r$ be the set of $r \times r$ nonnegative idempotent matrices. With this notation we have the following result.

THEOREM 9. *Let E be a nonnegative idempotent matrix of rank r. Then there exists a map $\phi : \mathscr{M}_E \to \mathscr{I}_r$ satisfying the following properties:*

(i) *ϕ is one-to-one.*

(ii) *ϕ is onto.*

(iii) *If $A, B \in \mathscr{M}_E$ and $AB = BA = 0$, then $A + B \in \mathscr{M}_E$, $\phi(A + B) = \phi(A) + \phi(B)$, and $\phi(A)\phi(B) = 0$.*

Proof. We first describe the construction of the map ϕ. Let $A \in \mathscr{M}_E$. By the proof of Theorem 5 we may write, without loss of generality, that

$$E = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} X & XD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CX & CXD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

where $X \geqslant 0$, where $JX = X = XJ$, and where $J = J_1 \oplus \cdots \oplus J_r$, $J_i = x_i y_i^T$, $x_i > 0$, $y_i > 0$ with $y_i^T x_i = 1$, $i = 1, 2, \dots, r$. Let $X = [X_{ij}]$ be the corresponding block partitioning of X. By Lemma 4, each X_{ij} is either zero or a positive matrix of rank 1. Since $JX = X = XJ$, we have

$$J_i X_{ij} = X_{ij} = X_{ij} J_j. \tag{5}$$

We define $\phi(A) = Z$, where $Z = [z_{ij}] = [y_i^T X_{ij} x_j]$ is the matrix constructed in the proof of Theorem 5. As observed in that proof, Z is idempotent and thus ϕ is clearly a map from $\mathscr{M}_E$ into $\mathscr{I}_r$.

(i): Suppose $A, B \in \mathscr{M}_E$ and $\phi(A) = \phi(B)$. As in the proof of Theorem 5, we conclude that

$$A = \begin{bmatrix} X & XD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CX & CXD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} Y & YD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CY & CYD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

Let $X = [X_{ij}]$, $Y = [Y_{ij}]$ be block partitionings of X, Y in conformity with that of J. Since $\phi(A) = \phi(B)$, we have, in view of the definition of ϕ, that

$$y_i^T X_{ij} x_j = y_i^T Y_{ij} x_j, \qquad 1 \leqslant i, j \leqslant r.$$

Thus,

$$x_i y_i^T X_{ij} x_j y_j^T = x_i y_i^T Y_{ij} x_j y_j^T, \qquad 1 \leqslant i, j \leqslant r,$$

and, hence $J_i X_{ij} J_j = J_i Y_{ij} J_j$, $1 \leqslant i, j \leqslant r$. It follows from (5) and a similar equation for Y_{ij} that $X_{ij} = Y_{ij}$, $1 \leqslant i, j \leqslant r$. Thus $X = Y$ and hence $A = B$. Therefore ϕ is one-to-one.

(ii): To show that ϕ is onto, let $L = [l_{ij}]$ be a nonnegative idempotent $r \times r$ matrix. Let $X_{ij} = l_{ij} x_i y_j^T$, $i, j = 1, 2, \dots, r$. Set $X = [X_{ij}]$, a block partitioned matrix, and let

$$A = \begin{bmatrix} X & XD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CX & CXD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then it can be verified that $X^2 = X$ and $JX = X = XJ$. Thus $A \in \mathscr{M}_E$. Also, $\phi(A) = L$, and therefore ϕ is onto.

(iii): Suppose $A, B \in \mathscr{M}_E$ and $AB = BA = 0$. Then clearly $A + B$ is idempotent and $[E - (A + B)](A + B) = (A + B)[E - (A + B)] = 0$, thus $A + B \in \mathscr{M}_E$. We assume, without loss of generality, that E, A, B have the form given in (4) and (6). Then $AB = BA = 0$ leads to $XY = YX = 0$. Furthermore, $X + Y$ is idempotent. Let $X = [X_{ij}]$, $Y = [Y_{ij}]$ be the block partitioning of X, Y, compatible with that of J, and let

$$z_{ij} = y_i^T X_{ij} x_j, \qquad w_{ij} = y_i^T Y_{ij} x_j$$

for all i, j. Then

$$\phi(A + B) = [z_{ij} + w_{ij}] = [z_{ij}] + [w_{ij}] = \phi(A) + \phi(B).$$

Since $XY = 0$, it is easy to verify that $\phi(A)\phi(B) = 0$, completing the proof. ■

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