Feynman–Kac Representation of Some Noncommutative Elliptic Operators

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Gaussian averages of automorphisms of a von Neumannn algebra yield Markov semigroups by the well-known procedure of subordination. We construct operator-valued martingales to realise perturbations of such semigroups through Feynman–Kac formulae. The perturbations are noncommutative vector fields, and the martingales are operator families, which are determined by an Itô equation on each vector and satisfy cocycle relations with respect to a randomised flow on the algebra. In particular this gives a probabilistic representation of some symmetric Markov semigroups considered by Davies and Lindsay.

INTRODUCTION

The Feynman–Kac formula represents a classical Schrödinger operator $H = -\frac{1}{2}\sum_i \partial_i^2 + V$ on $L^2(R^{3d})$ as a certain average over Brownian paths,

$$e^{-tH}f(x) = \int_\Omega e^{\int_0^t V(x+\omega(s))\,ds} f(x+\omega(t))\,d\mathbb{P}(\omega), \tag{0.1}$$

in which $(\Omega, \mathscr{B}, \mathbb{P})$ is the standard Wiener space. This identity holds for a wide range of potentials V (see, e.g., [ReS]). The mathematical structure of this representation is better revealed by writing

$$P_t f = \mathbb{E}[\zeta_t^V S_{B_t} f], \tag{0.2}$$

where $P_t = e^{-tH}$, $\zeta_t^V(x) = \exp \int_0^t (S_{B_s} V S_{B_s}^{-1})(x)\, ds$, $S_r f(x) = f(x+r)$ for $f \in L^2(\mathbb{R}^{3d})$, $B_s(\omega) = \omega(s)$ and $\mathbb{E}$ is the Wiener expectation. Thus the translation group on $L^2(\mathbb{R}^{3d})$ is randomised and then the shifted function is acted upon by a (randomised) multiplication operator, before being averaged to realise the semigroup generated by the Hamiltonian H. Several features of the Feynman–Kac multiplier (ζ_t^V) are worth noting. Set $j_t(\varphi) = S_{B_t} \varphi S_{-B_t}$, for $\varphi \in L^\infty(\mathbb{R}^{3d})$, and extend j_t to random functions pointwise, so that $j_t(\zeta)(\omega) = S_{\omega(t)} \zeta(\cdot, \omega)\, S_{-\omega(t)}$, then:

(i) ζ_t^V is a 1-cocycle with respect to the randomised translation group acting on the von Neumann algebra $L^\infty(\mathbb{R}^{3d})$:

$$\zeta_{s+t}^V = j_s(\zeta_t^V)\, \zeta_s^V. \tag{0.3}$$

(ii) ζ_t^V is the unique solution of the following stochastic initial value problem in $L^\infty(\mathbb{R}^{3d})$:

$$d\zeta_t^V = j_t(V)\, \zeta_t^V\, dt; \qquad \zeta_0^V = I. \tag{0.4}$$

Another form of multiplier has been introduced to study vector field perturbations of the Laplacian

$$m_{r,t}^b = \exp\left[\int_r^t j_{r,s}(b)\, dB_s - \tfrac{1}{2} \int_r^t j_{r,s}(b^2)\, ds \right], \tag{0.5}$$

where $j_{r,t}(\varphi) := S_{B_t - B_r} \varphi S_{B_r - B_t}$ and $b \in L^\infty(\mathbb{R}^{3d})$. These satisfy

$$m_{r,t}^b = m_{s,t}^{j_{r,s}(b)} m_{r,s}^b, \tag{0.6}$$

and

$$d_t m_{r,t}^b f = j_{r,t}(b)\, m_{r,t}^b f\, dB_t; \qquad m_{r,r}^b f = f, \tag{0.7}$$

for $f \in L^2(\mathbb{R}^{3d})$ (cf., [PaS]). The main difference between the two multipliers is that while ξ_t^V, as the solution of (0.4), defines an $L^\infty(\mathbb{R}^{3d})$-valued function of Brownian motion, the solution $m_{r,t}^b$ of (0.7) exists only as a strong solution in $L^2(\mathbb{R}^{3d})$.

These structures, together with various noncommutative extensions, have been studied by a number of authors (e.g., [AcF], [Arv], [HIP], [Pin]). Noncommutative extensions arise in two ways: the function space on which the semigroup acts may be replaced by an operator algebra, or the randomising may be effected by quantum stochastic processes ([AFL])—quantum Brownian motion and its associated calculus ([Mey], [Par]) being a natural tool for this. In the present note we are concerned only with the former kind. The translation group is replaced by an automorphism group of a von Neumann algebra and our main result is the construction of multipliers which yield non-commutative elliptic operators of the form $\frac{1}{2}\delta^2 + N_a\delta$, where N_a is left (or right) multiplication by the bounded algebra element a. If a is self-adjoint and central then the multiplier is given *formally* by $\exp\{\int_0^t j_s(a)\,dB_s - \frac{1}{2}\int_0^t j_s(a^2)\,ds\}$, however this is not easy to make sense of, and we have found it necessary to take a less direct route. Our construction, through Itô stochastic calculus, yields bounded operators m_t^a from $\mathfrak{h}$, the Hilbert space on which the algebra acts, to $L^2(\Omega; \mathfrak{h})$. There is no obvious sense in which one can take sections of these maps to yield operators on $\mathfrak{h}$ itself. Even in the classical case one cannot expect to obtain bounded operators on $\mathfrak{h} = L^2(\mathbb{R}^{3d})$—the paper [Pin] appears to be in error over this point.

In Section 1 we review Gaussian subordination for an automorphism group by means of Brownian motion, and describe the Itô equation for the randomised group. The multipliers are constructed in Section 2, and the basic representation is given in Section 3. In the last two sections we specialise to semi-finite algebras, and consider automorphism groups which are integrable in the sense of [DaL]. Section 4 deals with the L^∞-theory, and the final section considers Feynman–Kac representations on $L^2(\mathscr{A}, \tau)$, for a trace τ on the algebra.

Notation. Throughout the paper $(\Omega, \mathscr{B}, \mathbb{P})$ will denote the Wiener probability space: thus Ω is the complete metric space of continuous real-valued functions on the half-line $[0, \infty)$ which vanish at 0, with metric defined through the seminorms $p_n(\omega) = \sup\{|\omega(t)|: t \in [0, n]\}$, $n \in \mathbb{N}$, $\mathscr{B}$ is the Borel σ-algebra and $\mathbb{P}$ is Wiener measure. The coordinate process $\omega \in \Omega \mapsto \omega(t)$ is a standard Brownian motion which will be denoted B_t—it generates the Wiener filtration $(\mathscr{B}_{r,t})_{t \geqslant r \geqslant 0}$: $\mathscr{B}_{r,t} = \sigma\{B_s - B_r: r \leqslant s \leqslant t\}$. We shall be considering a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space $\mathfrak{h}$, and will denote the von Neumann algebra (respectively Hilbert space) tensor products $\mathscr{A} \otimes L^\infty(\Omega)$ and $\mathfrak{h} \otimes L^2(\Omega)$ by $\mathscr{M}$ and $\mathscr{H}$ respectively. Elements of $\mathscr{M}$, $\mathscr{H}$ and $\mathscr{M}_*$ may be considered as (equivalence classes of) functions on Ω: $\mathscr{M} = L^\infty(\Omega; \mathscr{A})$, $\mathscr{H} = L^2(\Omega; \mathfrak{h})$; and $\mathscr{M}_* = L^1(\Omega; \mathscr{A}_*)$. Here $\mathscr{A}_*$ and $\mathscr{M}_*$ denote the preduals of the von Neumann algebras $\mathscr{A}$ and $\mathscr{M}$ respectively (see [Sak], not [Tak]!).

1. GAUSSIAN SUBORDINATION

Let $(\alpha_t: t \in \mathbb{R})$ be a weak*-continuous group of *-automorphisms of the von Neumann algebra $\mathscr{A}$ acting on the Hilbert space $\mathfrak{h}$. We first randomise this group. Thus for each $t \geqslant 0$ and $a \in \mathscr{A}$ define $j_t(a): \Omega \to \mathscr{A}$ by

$$(j_t a)(\omega) = \alpha_{\omega(t)} a.$$

$j_t(a)$ then defines an element of $\mathcal{M} = L^\infty(\Omega; \mathcal{A})$ and each j_t defines a weak*-continuous injective *-homomorphism of $\mathcal{A}$ into $\mathcal{M}$. Moreover the family $(j_t : t \geqslant 0)$ is (pointwise) weak*-continuous in t and so defines a W^*-stochastic process in the sense of [AFL].

We extend the maps (j_t), as well as the shifts and conditional expectation maps on Wiener space, to $\mathcal{M}$ as the unique weak*-continuous linear extensions of the respective prescriptions

$$J_t : a \otimes \varphi \mapsto j_t(a)\, \varphi$$

$$\Theta_t : a \otimes \varphi \mapsto a \otimes \varphi \circ \gamma_t$$

$$\mathbb{E}_t : a \otimes \varphi \mapsto a \otimes \mathbb{E}[\varphi \mid \mathcal{B}_t] \qquad a \in \mathcal{A}, \qquad \varphi \in L^\infty(\Omega),$$

where γ_t is the shift on paths defined by $(\gamma_t \omega)(s) = \omega(t+s) - \omega(t)$, and $\mathcal{B}_t = \mathcal{B}_{0,t}$. $(J_t : t \geqslant 0)$ is then a (pointwise) weak*-continuous family of *-automorphisms of $\mathcal{M}$, $(\Theta_t : t \geqslant 0)$ a weak*-continuous semigroup of injective *-homomorphisms of $\mathcal{M}$ and $(\mathbb{E}_t : t \geqslant 0)$ is a weak*-continuous family of projections on $\mathcal{M}$, related to the shifts through

$$\mathbb{E}_t \Theta_t = \mathbb{E}_0 \tag{1.1}$$

and satisfying the tower relation $\mathbb{E}_s \mathbb{E}_t = \mathbb{E}_{\min\{s,t\}}$ and $\mathbb{E}_t(fgh) = f(\mathbb{E}_t\, g)\, h$ when $f, h \in \mathcal{M}_t := \mathbb{E}_t(\mathcal{M})$, $g \in \mathcal{M}$—in particular $\mathbb{E}_0$ maps $\mathcal{M}$ to $\mathcal{A} \otimes \mathbb{C}$ identified with the algebra $\mathcal{A}$ itself. Viewing $\mathcal{M}$ as consisting of $\mathcal{A}$-valued functions,

$$(J_t f)(\omega) = \alpha_{\omega(t)}[f(\omega)], \qquad \Theta_t f(\omega) = f(\gamma_t \omega), \qquad v(\mathbb{E}_t f) = \mathbb{E}[v \circ f \mid \mathcal{B}_t],$$

for $f \in \mathcal{M}$, $v \in \mathcal{A}_*$. It is convenient to work with two parameters, so let $J_{s,t} = J_s^{-1} J_t$, and $j_{s,t} = J_{s,t} \mid \mathcal{A}$.

We summarise how the randomised automorphisms combine with the shift and conditional expectation maps.

Proposition 1.1. *For $t \geqslant 0$ let $\Gamma_t = J_t \Theta_t : \mathcal{M} \to \mathcal{M}$. Then $(\Gamma_t : t \geqslant 0)$ is a weak*-continuous semigroup of injective *-homomorphisms on $\mathcal{M}$ which extend (j_t), in the sense that $\Gamma_t\, j_s(a) = j_{s+t}(a)$, $a \in \mathcal{A}$, and satisfy*

$$\mathbb{E}_s \Gamma_s = j_s \mathbb{E}_0. \tag{1.2}$$

Proof. The semigroup property follows from the relation

$$\Theta_s J_t = J_{s,s+t} \Theta_s, \tag{1.3}$$

which may be verified directly on $\mathscr{M}$. (1.2) follows by weak*-continuous linear extension of the relation

$$\mathbb{E}_s J_s(a \otimes \varphi) = \mathbb{E}_s[\, j_s(a)\, \varphi\,] = j_s(a)\, \mathbb{E}[\, \varphi \mid \mathscr{F}_s\,] = J_s \mathbb{E}_s(a \otimes \varphi),$$

for $a \in \mathscr{A}$, $\varphi \in L^\infty(\Omega)$, together with (1.1). The remaining properties are inherited from (J_t) and (Θ_t). ∎

Remark. The relation (1.3) is a (commutative!) extension of the following cocycle relation considered by W. S. Bradshaw ([Bra]):

$$J_s \Theta_s j_t = j_{s+t}.$$

PROPOSITION 1.2. *For* $t \geqslant 0$ *let* $P_t^0 = \mathbb{E}_0 j_t \colon \mathscr{A} \to \mathscr{A}$. $(P_t^0 \colon t \geqslant 0)$ *is a weak*-continuous semigroup of completely positive, identity preserving normal contractions on* $\mathscr{A}$.

Proof. By the tower property of conditional expectations, (1.1) and (1.2),

$$\begin{aligned} P_s^0 P_t^0 &= \mathbb{E}_0 j_s \mathbb{E}_0 j_t \\ &= \mathbb{E}_0 \mathbb{E}_s \Gamma_s j_t \\ &= \mathbb{E}_0 j_{s+t} = P_{s+t}^0. \end{aligned}$$

The remaining properties follow from the corresponding properties of $\mathbb{E}_0$ and (j_t). ∎

We shall refer to (P_t^0) as the *Gaussian semigroup* corresponding to the automorphism group (α_t).

We next establish a stochastic integral representation of the flow (j_t) by appealing to the Itô Lemma. For each $a \in \mathscr{A}$, $\{ j_{s,t}(a) \colon t \geqslant s\}$ may be viewed as a strongly measurable, $\mathscr{A}$-valued stochastic process adapted to the Brownian filtration. Let δ denote the weak*-generator of the automorphism group (α_t).

PROPOSITION 1.3. *Let* $x \in \mathrm{Dom}(\delta^2)$ *and* $v \in \mathfrak{h}$. *Then*

$$j_{s,t}(x)\, v = xv + \int_s^t j_{s,\tau}(\delta x)\, v\, dB_\tau + \tfrac{1}{2} \int_s^t j_{s,\tau}(\delta^2 x)\, v\, d\tau, \tag{1.4}$$

where the first integral is an Itô integral for a function taking values in the Hilbert space $\mathfrak{h}$, *and the second is a Bochner integral defined pointwise for each path.*

Proof. First suppose that $s=0$, let $u \in \mathfrak{h}$ and define a function $f: \mathbb{R} \to \mathbb{C}$ by $f(r) = \langle u, \alpha_r(x)\, v \rangle$. By our assumption on x, $f \in C^2(\mathbb{R})$ and $f^{(n)}(r) = \langle u, \alpha_r(\delta^n x)\, v \rangle$ for $n = 1, 2$. Hence by the Itô Lemma

$$\langle u, j_t(x)\, v \rangle = \langle u, xv \rangle + \int_0^t \langle u, j_\tau(\delta x)\, v \rangle\, dB_\tau + \tfrac{1}{2} \int_0^t \langle u, j_\tau(\delta^2 x)\, v \rangle\, d\tau$$

$$= \left\langle u, xv + \int_0^t j_\tau(\delta x)\, v\, dB_\tau + \tfrac{1}{2} \int_0^t j_\tau(\delta^2 x)\, v\, d\tau \right\rangle.$$

Letting u run through a countable dense subset of $\mathfrak{h}$ we see that (1.4) holds almost everywhere or as an identity in $\mathscr{H} = L^2(\Omega; \mathfrak{h})$, when $s = 0$. The full relation now follows from identity (1.3) restricted to $\mathscr{A}$: $j_{s,t} = \Theta_s j_{t-s}$, and the corresponding property of Brownian increments. ∎

Notice that (1.4) provides the pathwise continuous version of the Itô integral $\int_s^t \alpha_{\omega(\tau)-\omega(s)}(\delta x)\, v\, dB_\tau(\omega)$, namely

$$\alpha_{\omega(t)-\omega(s)}(x)\, v - xv - \tfrac{1}{2} \int_s^t \alpha_{\omega(\tau)-\omega(s)}(\delta^2 x)\, v\, d\tau.$$

In order to use the above representation to identify the weak*-generator of the Gaussian semigroup (P_t^0) precisely, we need the following invariance result.

LEMMA 1.4. *Let $x \in \operatorname{Dom} \delta$. Then, for each $s \geqslant 0$,*

$$P_s^0 x \in \operatorname{Dom} \delta \qquad \text{and} \qquad \delta(P_s^0 x) = P_s^0(\delta x).$$

Proof. Let $v \in \mathscr{A}_*$ and let (t_n) be a real sequence converging to zero. Then

$$\langle v, t_n^{-1}(\alpha_{t_n} - \mathrm{id.})\, P_s^0 x \rangle = \int \langle t_n^{-1}[(\alpha_{t_n})_* - \mathrm{id.}]\, v, \alpha_{\omega(s)} x \rangle\, d\mathbb{P}(\omega)$$

$$= \int \langle v, \alpha_{\omega(s)}\{t_n^{-1}(\alpha_{t_n} - \mathrm{id.})\}\, x \rangle\, d\mathbb{P}(\omega).$$

Since the integrand is bounded by $\|v\|_{\mathscr{A}_*} \|\delta x\|_{\mathscr{A}}$, weak*-$\lim_{n\to\infty} t_n^{-1}(\alpha_{t_n} - \mathrm{id.})\, P_s^0 x$ exists and equals $P_s^0(\delta x)$ by Lebesgue's Dominated Convergence Theorem. The result follows. ∎

THEOREM 1.5. *Let (α_t) be a weak*-continuous automorphism group on a von Neumann algebra $\mathscr{A}$. If δ is the generator of (α_t) then $\frac{1}{2}\delta^2$ is a weak*-pre-generator of the corresponding Gaussian semigroup.*

Proof. By Proposition 1.3, if $x \in \mathrm{Dom}(\delta^2)$, then for $u, v \in \mathfrak{h}$,

$$\begin{aligned}\langle u, (P_t^0 x)\, v\rangle &= \mathbb{E}\langle u, j_t(x)\, v\rangle \\ &= \langle u, xv\rangle + \tfrac{1}{2}\mathbb{E}\int_0^t \langle u, j_\tau(\delta^2 x)\, v\rangle\, d\tau \\ &= \left\langle u, xv + \int_0^t P_\tau^0(\tfrac{1}{2}\delta^2 x)\, v\, d\tau \right\rangle.\end{aligned}$$

Since (P_t^0) is a semigroup, this implies that its generator is an extension of $\frac{1}{2}\delta^2$. Now let $\mathscr{A}_\infty(\delta)$ denote the collection of analytic vectors for the generator δ of the group (α_t). $\mathscr{A}_\infty(\delta)$ is weak*-dense in $\mathscr{A}$ ([BrR], p. 178) and, by Lemma 1.4, it is invariant under (P_t^0). Therefore, by [BrR] Corollary 3.1.20, $\mathscr{A}_\infty(\delta)$ is a core for the generator of (P_t^0). In particular $\frac{1}{2}\delta^2$ is a pre-generator for (P_t^0). ∎

2. EXPONENTIAL MARTINGALES

In this section we define an analogue of exponential martingales. Technical problems associated with infinite dimensionality (of the algebra and Hilbert space) force us to approach these somewhat indirectly. We obtain bounded operators from $\mathfrak{h}$ to $\mathscr{H} = L^2(\Omega; \mathfrak{h})$, as strong (operator) sense solutions of stochastic differential equations.

In the proof of the first result, we need to make the identification $L^2(\Omega_{[s}; \mathscr{H}_s) = L^2(\Omega; \mathfrak{h})$ in which $\Omega_{[s} = \{\omega: [s, \infty) \to \mathbb{R} \mid \omega \text{ is continuous}\}$ and $\mathscr{H}_s = L^2(\Omega_{s]}; \mathfrak{h})$. This is given by $f(\omega_1)(\omega_2) = f(\omega_1 \circ \omega_2)$, where

$$(\omega_1 \circ \omega_2)(t) = \begin{cases} \omega_2(t) & t \leqslant s \\ \omega_2(s) + \omega_1(t) - \omega_1(s) & t \geqslant s. \end{cases}$$

Proposition 2.1. *For $s \geqslant 0$, $a \in \mathscr{M}_s$, $v \in \mathscr{H}_s$ the stochastic integral equation*

$$f(a, v; s, t) = v + \int_s^t J_{s,\tau}(a)\, f(a, v; s, \tau)\, dB_\tau \tag{2.1}$$

has a unique solution.

Proof. Assume, as induction hypothesis, that the recursive procedure

$$f_{s,t}^{(0)} = v; \qquad f_{s,t}^{(n)} = \int_s^t J_{s,\tau}(a)\, f_{s,\tau}^{(n-1)}\, dB_\tau \qquad (n \geqslant 1)$$

defines adapted, measurable $\mathfrak{h}$-valued processes $(f_{s,t}^{(n)})_{t \geqslant s}$ satisfying

$$\|f_{s,t}^{(n)}\|_{\mathscr{H}}^2 = \mathbb{E}\,\|f_{s,t}^{(n)}\|_{\mathfrak{h}}^2 \leqslant [\,\|a\|_{\mathscr{M}}^2\,(t-s)]^n\,\|v\|_{\mathscr{H}}^2/n!, \tag{2.2}$$

where $\|a\|_{\mathscr{M}}$ and $\|v\|_{\mathscr{H}}$ are the norms

$$\|a\|_{\mathscr{M}} := \operatorname*{ess\,sup}_{\omega \in \Omega} \|a(\omega)\|_{\mathscr{A}}$$

and

$$\|v\|_{\mathscr{H}} := (\mathbb{E}\,\|v\|_{\mathfrak{h}}^2)^{1/2} = \left\{\int \|v(\omega)\|_{\mathfrak{h}}^2\,d\mathbb{P}(\omega)\right\}^{1/2}.$$

The inductive hypothesis is clearly satisfied when $n=0$, so assume it is satisfied when $n=k$. Then $(J_{s,\tau}(a)\,f_{s,\tau}^{(k)} : \tau \geqslant s)$ is adapted, measurable and satisfies

$$\begin{aligned}
\int_s^t \mathbb{E}\,\|J_{s,\tau}(a)\,f_{s,\tau}^{(k)}\|^2\,d\tau \leqslant \|a\|_{\mathscr{M}}^2 \int_s^t \mathbb{E}\,\|f_{s,\tau}^{(k)}\|^2\,d\tau \\
\leqslant \|a\|_{\mathscr{M}}^2\,\mathbb{E}\,\|v\|_{\mathscr{H}}^2 \int_s^t [\,\|a\|_{\mathscr{M}}^2\,(\tau-s)]^k/k!\,d\tau \\
= [\,\|a\|_{\mathscr{M}}^2\,(t-s)]^{k+1}\,\|v\|_{\mathscr{H}}^2/(k+1)!
\end{aligned}$$

so that $(f_{s,t}^{(k+1)} : t \geqslant s)$ is well-defined and satisfies (2.2). The recursive procedure is therefore justified, moreover

$$\sum_{n \geqslant 0} \|f_{s,t}^{(n)}\|_{\mathscr{H}} \leqslant \|v\|_{\mathscr{H}} \sum_{n \geqslant 0} (\|a\|_{\mathscr{M}} \sqrt{t-s})^n/\sqrt{n!} < \infty.$$

Hence $\sum_{n \geqslant 0} f_{s,t}^{(n)}$ defines an $\mathfrak{h}$-valued L^2-process $(f_{s,t} : t \geqslant s)$. Since, for each $N \geqslant 1$,

$$\sum_{n=0}^{N} f_{s,t}^{(n)} = v + \int_s^t J_{s,\tau}(a) \sum_{n=0}^{N-1} f_{s,\tau}^{(n)}\,dB_\tau,$$

$(f_{s,t})$ satisfies the equation (2.1). Now let (g_t) be any solution of (2.1). Then

$$g_t - v = \int_s^t J_{s,\tau}(a)\,g_\tau\,dB_\tau = \int_s^t J_{s,\tau}(a)(g_\tau - v)\,dB_\tau + \int_s^t J_{s,\tau}(a)\,v\,dB_\tau$$

so that, by the triangle inequality and Itô isometry,

$$\mathbb{E}\,\|g_t - v\|_{\mathfrak{h}}^2 \leqslant 2(t-s)\,\|a\|_{\mathscr{M}}^2\,\mathbb{E}\,\|v\|_{\mathfrak{h}}^2 + 2\,\|a\|_{\mathscr{M}}^2 \int_s^t \mathbb{E}\,\|g_\tau - v\|_{\mathfrak{h}}^2\,d\tau.$$

Applying the Gronwall Lemma therefore leads to the bound

$$\mathbb{E}\,\|g_t - v\|_{\mathfrak{h}}^2 \leqslant (t-s)\,\|a\|_{\mathcal{M}}^2\,(2\mathbb{E}\,\|v\|_{\mathfrak{h}}^2)\exp\{2\,\|a\|_{\mathcal{M}}^2\,(t-s)\}. \tag{2.3}$$

Now let h_t be the difference $f_{s,t} - g_t$. By Itô isometry

$$\mathbb{E}\,\|h_t\|^2 = \int_s^t \mathbb{E}\,\|J_{s,\tau}(a)\,h_\tau\|^2\,d\tau$$

$$\leqslant \|a\|_{\mathcal{M}}^2 \int_s^t \mathbb{E}\,\|h_\tau\|^2\,d\tau.$$

Iterating this relation and applying the bound (2.3) to the estimates $\|h_{t_n}\|^2 \leqslant 2\{\|f_{s,t_n} - v\|^2 + \|g_{s,t_n} - v\|^2\}$ gives

$$\mathbb{E}\,\|h_t\|^2 \leqslant \|a\|_{\mathcal{M}}^{2n} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} 8\mathbb{E}\,\|v\|_{\mathfrak{h}}^2\,(t_n - s)\,e^{2\,\|a\|_{\mathcal{M}}^2\,(t_n - s)}\,dt_1 \cdots dt_n$$

$$\leqslant 8\mathbb{E}\,\|v\|_{\mathfrak{h}}^2 \exp\{2\,\|a\|_{\mathcal{M}}^2\,(t-s)\}\,[\,\|a\|_{\mathcal{M}}^2\,(t-s)]^n/n!$$

for each n. Hence $g_t = f_{s,t}$ almost everywhere, in other words the solution is unique. ∎

Corollary 2.2. *Let $a \in \mathcal{A}$ and $u \in \mathfrak{h}$, and keep the above notation. Then, for $t \geqslant s$,*

$$f(a, u; s, t) = \Theta_s^{(2)} f(a, u; 0, t-s), \tag{2.4}$$

where $\Theta_s^{(2)}$ is the shift on $\mathscr{H} = L^2(\Omega; \mathfrak{h})$: $\Theta_s^{(2)} f(\omega) = f(\gamma_s \omega)$.

Proof. The effect of the shift on an Itô integral is given by

$$\Theta_s^{(2)} \int_0^r h_\sigma\,dB_\sigma = \int_s^{s+r} (\Theta_s^{(2)} h_{\tau-s})\,dB_\tau.$$

Therefore, putting $g_{s,t} = \Theta_s^{(2)} f(a, u; 0, t-s)$, we have

$$g_{s,t} = \Theta_s^{(2)} \left\{ u + \int_0^{t-s} j_\sigma(a)\,f(a, u; 0, \sigma)\,dB_\sigma \right\}$$

$$= u + \int_s^t [\Theta_s\,j_{\tau-s}(a)]\,\Theta_s^{(2)} f(a, u; 0, \tau - s)\,dB_\tau$$

$$= u + \int_s^t j_{s,\tau}(a)\,g_{s,\tau}\,dB_\tau.$$

The result therefore follows by uniqueness for this stochastic integral equation. ∎

Another immediate consequence of uniqueness is the following result.

COROLLARY 2.3. *Let $a \in \mathcal{M}_s$, $v \in \mathcal{H}_s$. For any c in $\mathcal{B}(\mathcal{H}_s)$ which commutes with the family $\{J_{s,r}(a): r \geqslant s\}$, we have*

$$f(a, cv; s, t) = cf(a, v; s, t). \tag{2.5}$$

We now cast this in terms of operators.

PROPOSITION 2.4. *For each $a \in \mathcal{M}_s$ there is a unique family of bounded operators $m^a_{s,t}: \mathcal{H}_s \to \mathcal{H}_t \subset \mathcal{H}$, $t \geqslant s \geqslant 0$, satisfying the (strong operator sense) stochastic differential equation*

$$m^a_{s,s} = I_{\mathcal{H}_s}; \qquad d_t m^a_{s,t} = J_{s,t}(a)\, m^a_{s,t}\, dB_t.$$

These operators satisfy

$$m^a_{r,t} = m^{J_{r,s}(a)}_{s,t}\, m^a_{r,s}, \tag{2.6}$$

$$\|m^a_{r,t} - m^a_{r,s}\| \leqslant \sqrt{2(t-s)}\, \|a\| \exp\{\|a\|^2_{\mathcal{M}}\,(t-r)\}, \tag{2.7}$$

$$m^a_{s,t} a' = a' m^a_{s,t} \tag{2.8}$$

for $a' \in \mathcal{A}'$, the commutant of $\mathcal{A}$, and $r \leqslant s \leqslant t$.

Proof. By Proposition 2.1, $m^a_{s,t} v = f(a, v; s, t)$ defines maps $\mathcal{H}_s \to \mathcal{H}_t \subset \mathcal{H}$. Linearity of these maps follows from the uniqueness part of Proposition 2.1, and boundedness follows from (2.3). To prove (2.6) fix $v \in \mathcal{H}$, and define $\{g_{r,t}: t \geqslant r\}$ by

$$g_{r,t} = \begin{cases} f(a, v; r, t) & \text{for} \quad r \leqslant t \leqslant s \\ m^{J_{r,s}(a)}_{s,t} f(a, v; r, s) & \text{for} \quad t \geqslant s. \end{cases}$$

Then $(g_{r,t})$ satisfies the same stochastic integral equation as $(m^a_{r,t} v)$ and so (2.6) follows by uniqueness.

In particular,

$$\|m^a_{r,t} - m^a_{r,s}\| \leqslant \|m^{J_{r,s}(a)}_{s,t} - I_{\mathcal{H}_s}\|\, \|m^a_{r,s}\|,$$

and (2.7) follows from (2.3) and a further application of Gronwall's Lemma. (2.8) also follows by a uniqueness argument. ∎

Our next goal is to display the sense in which operators $(m^a_{r,s})$ form a cocycle with respect to the free flow (J_t). For this we make the following assumption: the automorphism group (α_t) is unitarity implemented, $\alpha_t(a) = U_t a U_t^*$ where (U_t) is a strongly continuous one parameter group of unitary operators on $\mathfrak{h}$. Randomise and extend to $\mathscr{H}$ by continuous linear extension of the prescription

$$u_t\colon v \otimes \varphi \mapsto \varphi U_{B_t} v \qquad (t \geqslant 0).$$

This defines a family of unitaries on $\mathscr{H}$ satisfying the (a.e.) identity

$$(u_t f)(\omega) = U_{\omega(t)}[f(\omega)]. \tag{2.9}$$

Note that u_t leaves $\mathscr{H}_t$ invariant, for each t.

THEOREM 2.5. *Let (α_t) be a weak*-continuous automorphism group on $\mathscr{A}$. If (α_t) is unitarily implemented then, for each $a \in \mathscr{A}$, the exponential martingale constructed in Proposition 2.4 satisfies the cocycle identity*

$$u_s m^a_{s,t} u_s^* m^a_s = m^a_t \qquad (s \leqslant t),$$

where we abbreviate $m^a_{0,t}$ to m^a_t.

Proof. Let $v \in \mathfrak{h}$ and define $(g_t\colon t \geqslant 0)$ by

$$g_t = \begin{cases} f(a, v; 0, t) & 0 \leqslant t \leqslant s \\ u_s m^a_{s,t} u_s^* f(a, v; 0, s) & t \geqslant s. \end{cases}$$

Then, for $t \geqslant s$

$$\begin{aligned} g_t &= u_s \left\{ u_s^* f(a, v; 0, s) + \int_s^t j_{s,\tau}(a)\, m^a_{s,\tau} u_s^* f(a, v; 0, s)\, dB_\tau \right\} \\ &= f(a, v; 0, s) + \int_s^t j_\tau(a)\, u_s m^a_{s,\tau} u_s^* f(a, v; 0, s)\, dB_\tau \\ &= v + \int_0^s j_\tau(a)\, f(a, v; 0, \tau)\, dB_\tau + \int_s^t j_\tau(a)\, g_\tau\, dB_\tau \\ &= v + \int_0^t j_\tau(a)\, g_\tau\, dB_\tau, \end{aligned}$$

and similarly, for $t \leqslant s$. Hence, by uniqueness for this stochastic integral equation (Proposition 2.1), $g_t = f(a, v; 0, t) = m_t^a v$, or

$$u_s m_{s,t}^a u_s^* m_s^a v = m_t^a v.$$

Since this holds for each $v \in \mathfrak{h}$, the result follows. ∎

Remark. If $\alpha_t(a) = T_t a T_{-t}$ so that $j_{s,t}(a) = T_{B(t)-B(s)} a T_{B(s)-B(t)}$, then the result of Theorem 2.5 remains true with obvious modifications. This is because the c_0-property of T_t implies that $\|T_t\| \leqslant e^{\beta |t|/4}$ for some $\beta > 0$ and $t \in \mathbb{R}$ and in such a case, the proof of Proposition 2.1 goes through essentially unchanged. The only change one has to make is to replace (2.2) by

$$\|f_{s,t}^{(n)}\|_{\mathscr{H}}^2 \leqslant \frac{[\|a\|_{\mathscr{A}}^2 \psi(t-s)]^n}{n!} \|v\|_{\mathscr{H}}^2,$$

where

$$\begin{aligned}\psi(t-s) &= \int_s^t \mathbb{E}[\exp(\beta |B_\tau - B_s|)]\, d\tau \\ &= \int_s^t d\tau\, 2e^{\beta^2(\tau-s)/2} \left[\int_0^\infty e^{-1/2(y-\beta\sqrt{\tau-s})^2} \frac{dy}{\sqrt{2\pi}}\right] \\ &= \int_0^{t-s} d\tau\, e^{\beta^2\tau/2} \left[1 + 2\int_0^{\beta\sqrt{\tau}} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}\right].\end{aligned}$$

3. FEYNMAN–KAC SEMIGROUPS ON $\mathscr{A}$

In this section we show how the exponential martingales, each determined by an element a of the algebra, may be employed as multipliers to yield Feynman–Kac type perturbations of Gaussian semigroups. In the notation of the previous sections define, for $x \in \mathscr{A}$, maps $P_t^a x\colon \mathfrak{h} \to \mathfrak{h}$ by

$$P_t^a x\colon v \mapsto \mathbb{E}_0^{(2)}[j_t(x)\, m_t^a v],$$

where, for $s \geqslant 0$, $\mathbb{E}_s^{(2)}$ is the expectation map (orthogonal projection) from $\mathscr{H}$ to $\mathscr{H}_s$. Clearly $P_t^a x$ are linear maps and, since they satisfy

$$\|(P_t^a x)\, v\| \leqslant \|x\|\, \|m_t^a\|\, \|v\|, \tag{3.1}$$

each $P_t^a x$ is bounded. Moreover if y' belongs to the commutant of $\mathscr{A}$ then, by (2.8),

$$\begin{aligned} y'(P_t^a x)\, v &= \mathbb{E}_0^{(2)}[\, j_t(x)\, y' m_t^a v] \\ &= \mathbb{E}_0^{(2)}[\, j_t(x)\, m_t^a y' v] = (P_t^a x)\, y' v, \qquad v \in \mathfrak{h}. \end{aligned}$$

Therefore $P_t^a x \in \mathscr{A}$, and so each P_t^a maps $\mathscr{A}$ to $\mathscr{A}$. Clearly each P_t^a is linear and by (3.1), also bounded. We extend each $P_t^a x$ to $\mathscr{H}$ by ampliation

$$((P_t^a x)\, f)(\omega) = P_t^a x[\, f(\omega)].$$

LEMMA 3.1. *For* $w \in \mathscr{H}_s$, $s, t \geqslant 0$, $x \in \mathscr{A}$,

$$\mathbb{E}_s^{(2)}[\, j_{s,\, s+t}(x)\, m_{s,\, s+t}^a w] = (P_t^a x)\, w.$$

Proof. We exploit the Hilbert space counterpart to (1.1): $\mathbb{E}_t^{(2)} \Theta_t^{(2)} = \mathbb{E}_0^{(2)}$. If $w = u \otimes \varphi$, where $\varphi \in L^\infty(\mathscr{B}_s)$ and $u \in \mathfrak{h}$, Corollaries 2.2 and 2.3 imply that

$$\begin{aligned} \mathbb{E}_s^{(2)}[\, j_{s,\, s+t}(x)\, m_{s,\, s+t}^a w] &= \mathbb{E}_s^{(2)}[\, \Theta_s^{(2)}\, j_t(x)\, m_t^a u]\, \varphi \\ &= \mathbb{E}_0^{(2)}[\, j_t(x)\, m_t^a u]\, \varphi \\ &= (P_t^a x)\, u\varphi = (P_t^a x)\, w, \end{aligned}$$

so the identity holds for such w. But these are total in $\mathscr{H}_s$, so the result follows by linearity and continuity. ∎

THEOREM 3.2. *If the weak*-continuous automorphism group* (α_t) *is unitarily implemented, then* (P_t^a) *defined by*

$$(P_t^a x)\, v = \mathbb{E}_0^{(2)}[\, j_t(x)\, m_t^a v], \qquad v \in \mathfrak{h},$$

is a weak-continuous semigroup of identity preserving operators on* $\mathscr{A}$, *whose weak*-generator is an extension of* $\frac{1}{2}\delta^2 + R_a \delta$, *where* R_a *is the right multiplication operator:* $R_a x = xa$.

Proof. Let $w = u_s^* m_s^a v$, where u_s is given (in terms of the implementing unitary group) by (2.9), and apply the tower property of conditional expectations, Theorem 2.5, and Lemma 3.1:

$$\begin{aligned} \mathbb{E}_0^{(2)}[\, j_{s+t}(x)\, m_{s+t}^a v] &= \mathbb{E}_0^{(2)} \mathbb{E}_s^{(2)}[\, u_s\, j_{s,\, s+t}(x)\, u_s^* m_{s+t}^a v] \\ &= \mathbb{E}_0^{(2)} u_s \mathbb{E}_s^{(2)}[\, j_{s,\, s+t}(x)\, m_{s,\, s+t}^a u_s^* m_s^a v] \\ &= \mathbb{E}_0^{(2)}[\, u_s (P_t^a x)\, w] \\ &= \mathbb{E}_0^{(2)}[\, j_s(P_t^a x)\, m_s^a v] = (P_s^a P_t^a x)\, v. \end{aligned}$$

A weak*-continuous semigroup whose generator extends $\frac{1}{2}\delta^2 + L_a\delta$ is

$$(m_t^{a^*})^* j_t = \overline{P_t^{(a^*)}},$$

where $\overline{P} = CPC$, C being the isometric operator $x \mapsto x^*$ on $\mathscr{A}$. We conjecture that both these semigroups are positivity preserving, and therefore *Markov* (in the sense of [DaL]), when a is self-adjoint and central.

4. THE SEMIFINITE CASE: L^∞-THEORY

In the next two sections the algebra $\mathscr{A}$ will be semifinite with faithful normal trace τ. We shall consider it acting in standard fashion on the Segal space of τ-measurable square traceable operators $L^2(\mathscr{A}, \tau)$, by strong sense left multiplication ([Nel], [DaL]). The previous sections will be applied with $\mathfrak{h} = L^2(\mathscr{A}, \tau)$. In this context we can establish the reality of Feynman–Kac semigroups (P_t^a) when a is self-adjoint and central. To do this we exploit the action of right multiplication of $\mathscr{A}$ on $L^2(\mathscr{A}, \tau)$:

LEMMA 4.1. *Let $a, b \in \mathscr{M}_s$, $v \in \mathscr{H}_s$. Then, in the notation of Section 2,*

(i) $f(a, v; s, t)\, b = f(a, vb; s, t)$,

and, if $\{J_{s,r}(a): r \geqslant s\}$ commutes with $\mathscr{M}$, then

(ii) $(f(a^*, v^*; s, t))^* = f(a, v; s, t)$.

Proof. Straightforward application of the uniqueness part of Proposition 2.1. ∎

COROLLARY 4.2. *Let $a \in \mathscr{A}$ be self-adjoint and central then, for each weak*-continuous automorphism group $(\alpha_t)_{t \in \mathbb{R}}$ of $\mathscr{A}$, the associated Feynman–Kac semigroup (P_t^a) is real: $P_t^a = \overline{P_t^a}$, or $P_t^a x^* = (P_t^a x)^*$.*

Proof. Combining Lemma 4.1 with Corollary 2.3 we have the following relations for $u, v \in L^2 \cap L^\infty$,

$$f(a, u; 0, t)\, v = uf(a, v; 0, t); \qquad [f(a^*, u^*; 0, t)]^* = f(a, u; 0, t),$$

when a is central. Therefore,

$$\begin{aligned}\langle v, (P_t^a x^*)^* v\rangle &= \mathbb{E}\langle j_t(x^*)\, m_t^a v, v\rangle = \mathbb{E}\tau([m_t^a v]^*\, j_t(x)\, v)\\ &= \mathbb{E}\tau(j_t(x)\, v[m_t^{a^*} v^*]) = \mathbb{E}\tau(j_t(x)[m_t^{a^*} v]\, v^*)\\ &= \langle v, (P_t^{a^*} x)\, v\rangle \qquad v \in L^2 \cap L^\infty.\end{aligned}$$

The result follows. ∎

We next suppose that the automorphism group (α_t) is *integrable* with respect to τ ([DaL]). Thus (α_t) satisfies the invariance condition

$$\tau(a^*a) < \infty \Rightarrow \tau(\alpha_t(a^*a)) < \infty,$$

and the continuity condition

$$\tau(a^*a) < \infty \Rightarrow \tau([\alpha_t(a) - a]^* [\alpha_t(a) - a]) \to 0 \qquad \text{as} \quad t \to 0.$$

Then the group (α_t) extends to a strongly continuous group $(T_t^{(p)})$ on each $L^p(\mathscr{A}, \tau)$ for $(1 \leqslant p < \infty)$, moreover under the duality of Segal spaces

$$T_t^{(p)*} = A_t T_{-t}^{(p')} \qquad \text{for} \quad 1 \leqslant p < \infty,$$

where each A_t is the multiplication operator by a bounded, self-adjoint element of the centre of $\mathscr{A}$ ([DaL], Theorem 4.5). In particular, for $a \in \mathscr{A}$,

$$\alpha_t(a) = T_t^{(2)} a T_{-t}^{(2)}.$$

Let δ_p denote the generator of $(T_t^{(p)})$. Under the following *smoothness assumption* on the L^1-generator: $u \in \operatorname{Dom} \delta_1 \mapsto \tau(\delta_1 u)$ is L^1-bounded, we may say more.

THEOREM 4.3. *Let (α_t) be a weak*-continuous automorphism group on a semi-finite algebra $\mathscr{A}$, which is integrable with respect to the faithful, normal semifinite trace τ on $\mathscr{A}$. If $1 \in \operatorname{Dom} \delta_1^*$ then $\alpha_t = \operatorname{ad} U_t$ where (U_t) is the strongly continuous unitary group with skew-adjoint generator $(\delta_2 - M_{b/2})$ in which $b = \delta_1^* 1$.*

Proof. Under the above assumptions it follows from [DaL], Proposition 6.6, that $\delta_2^* = -\delta_2 + M_b$, so $\delta_2 - M_{b/2}$ is skew-adjoint and (U_t) well-defined. Let $Z_t = e^{-(t/2)b}$ then, since b is central,

$$T_s^{(2)} Z_s a Z_{-s} T_{-s}^{(2)} = T_s^{(2)} a T_{-s}^{(2)} = \alpha_s(a) \qquad \forall a \in \mathscr{A}, \qquad s \in \mathbb{R}.$$

Therefore, for each n,

$$\begin{aligned} (T_{t/n}^{(2)} Z_{t/n})^n x (T_{-t/n}^{(2)} Z_{-t/n})^n &= (T_{t/n}^{(2)} Z_{t/n})^{n-1} \alpha_{t/n}(x) (T_{-t/n}^{(2)} Z_{-t/n})^{n-1} \\ &= \alpha_{t/n}(\cdots (\alpha_{t/n}(x)) \cdots) \\ &= \alpha_t(x). \end{aligned}$$

Hence, by the Trotter product formula ([Kat]),

$$U_t x U_t^* = e^{t(\delta_2 - M_{b/2})} x e^{-t(\delta_2 - M_{b/2})}$$
$$= \operatorname{weak}\lim_{n\to\infty} (e^{(t/n)\delta_2} e^{-(t/n)M_{b/2}})^n x (e^{(t/n)M_{b/2}} e^{-(t/n)\delta_2})^n$$
$$= \alpha_t(x),$$

and the result follows. ▮

Note that, for $x \in \operatorname{Dom}\delta$, $\delta x \supset [iA, x]$, where $iA = \delta_2 - M_{b/2}$, and $u_t(=U_{B_t})$ satisfies the (strong operator sense) Itô equation $du_t = u_t iA\,dB_t + u_t(-\frac{1}{2}A^2)\,dt$. Moreover, if $1 \in \operatorname{Dom}\delta\delta_1^*$, then for $x \in \operatorname{Dom}\delta^2$, $\frac{1}{2}\delta^2 x \supset AxA - \frac{1}{2}(A^2 x + xA^2)$, where the right hand side has dense domain $\operatorname{Dom}\delta_2^2 = \operatorname{Dom}\delta_2^*\delta_2$. In the next section we take a different point of view, namely we consider semigroups on $L^2(\mathscr{A}, \tau)$ generated by bounded vector field perturbations of the symmetric Markov generator $-\frac{1}{2}\delta_2^*\delta_2$ ([DaL]).

5. FEYNMAN–KAC SEMIGROUPS ON $L^2(\mathscr{A}, \tau)$

Let (α_t) be a τ-integrable automorphism group of the semifinite algebra $\mathscr{A}$, with corresponding groups $(T_t^{(p)})$ on L^p. Define $j_t^{(2)} = T_{B_t}^{(2)} : \mathfrak{h} \to \mathscr{H} = L^2(\Omega; \mathfrak{h})$ and let δ_2 be the generator of $(T_t^{(2)})$.

Proposition 5.1. *For each $x \in \operatorname{Dom}(\delta_2^2)$ the Hilbert space valued process $(j_t^{(2)}x)$ satisfies the Itô equation*

$$j_t^{(2)}x = x + \int_0^t j_s^{(2)}(\delta_2 x)\,dB_s + \tfrac{1}{2}\int_0^t j_s^{(2)}(\delta_2^2 x)\,ds.$$

Proof. Straightforward and similar to the proof of Proposition 1.3. ▮

For each $t \geqslant 0$ let Q_t^a be the unique bounded operator on $\mathfrak{h}$ determined by

$$\langle u, Q_t^a v\rangle = \langle j_t^{(2)} v^*, m_t^a u^*\rangle = \mathbb{E}\tau[m_t^a u^* j_t^{(2)} v].$$

Proposition 5.2. *The family (Q_t^a) forms a c_0-semigroup on $\mathfrak{h}$ whose generator extends $\frac{1}{2}\delta_2^2 + R_a\delta_2$. Moreover the semigroup is compatible with the corresponding semigroup on L^∞:*

$$Q_t^a x = P_t^a x \qquad x \in L^2 \cap L^\infty.$$

Proof. Let $x, y, z \in L^2 \cap L^\infty$. Then, by Lemma 4.1,

$$\begin{aligned}\langle x, Q_t^a(z)\, y\rangle = \langle xy^*, Q_t^a z\rangle &= \mathbb{E}\tau[\, j_t^{(2)}(z)\, m_t^a(\, yx^*)\,] \\ &= \mathbb{E}\tau[\, j_t(z)\, m_t^a\, y . x^*\,] \\ &= \langle x, j_t(z)\, m_t^a\, y\rangle = \langle x, P_t^a z . y\rangle.\end{aligned}$$

Hence by density, (Q_t^a) and (P_t^a) are compatible—in particular (Q_t^a) is a one-parameter semigroup. Writing out the Itô product

$$\begin{aligned}\langle u, Q_t^a v\rangle &= \langle u, v\rangle + \mathbb{E}\int_0^t \{\langle j_s(\tfrac{1}{2}\delta_2^2 v^*), m_s^a u^*\rangle \\ &\quad + \langle j_s(\delta_2 v^*), j_s(a)\, m_s^a u^*\rangle\}\; ds \\ &= \langle u, v\rangle + \int_0^t \mathbb{E}\langle j_s(a^*\delta_2 v^* + \tfrac{1}{2}\delta_2^2 v^*), m_s^a u^*\rangle\; ds \\ &= \left\langle u, v + \int_0^t Q_s^a(\tfrac{1}{2}\delta_2^2 v + (\delta_2 v)\, a)\; ds\right\rangle.\end{aligned}$$

In particular, the semigroup is weakly, and hence also strongly continuous and the generator extends $\frac{1}{2}\delta_2^2 + R_a\delta_2$. ∎

If the automorphism group (α_t) satisfies the *smoothness condition*

$$1 \in \operatorname{Dom} \delta_1^*, \tag{5.1}$$

then $\delta_2^* = -\delta_2 + M_b$ where $b = \delta_1^* 1$, and we may sharpen the previous result.

THEOREM 5.3. *Let (α_t) be an integrable automorphism group on $(\mathscr{A}, \tau)$ whose generator δ satisfies the smoothness condition* (5.1). *Then, for each $a \in \mathscr{A}$, the elliptic operators $\frac{1}{2}\delta_2^2 + R_a\delta$ and $\frac{1}{2}\delta_2^2 + L_a\delta$ generate (holomorphic) c_0-semigroups, (Q_t^a) and $(\overline{Q_t^a})$ respectively, on $L^2(\mathscr{A}, \tau)$, and both are expressible as Feynman–Kac type averages with respect to a standard Brownian motion.*

Proof. The smoothness assumption implies that $\delta_2^2 = -\delta_2^*\delta_2 + M_b\delta_2$, so each of the operators is of the form $-\frac{1}{2}\delta^*\delta + N\delta$, where N is a bounded multiplication operator and $\delta = \delta_2$. Let $V|\delta|$ be the polar decomposition of δ into partial isometry V and non-negative operator $|\delta|$. Then, for $x \in \operatorname{Dom} \delta^*\delta$ and $\lambda > 0$,

$$\begin{aligned}\|N\,\delta x\| &= \|NV\,|\delta|\,(\delta^*\delta + \lambda)^{-1}\,(\delta^*\delta + \lambda)\,x\| \\ &\leqslant \|N\|\;\|\,|\delta|\,(|\delta|^2 + \lambda)^{-1}\|\;\|(\delta^*\delta + \lambda)\,x\| \\ &\leqslant (2/\lambda)^{1/2}\,\|N\|\;\{\|\tfrac{1}{2}\delta^*\delta x\| + \lambda\,\|x\|/2\}.\end{aligned}$$

Letting $\lambda \to \infty$ we see that $N\delta$ is relatively bounded with respect to $-\frac{1}{2}\delta^*\delta$, with relative bound equal to zero. But $-\frac{1}{2}\delta^*\delta$ generates a holomorphic contraction semigroup and hence (by [Kat], Ch. 9, Corollary 2.5) $-\frac{1}{2}\delta^*\delta + N\delta$ generates a holomorphic c_0-semigroup and the proof is complete. ∎

Specialising to symmetric Markov semigroups ([DaL]) we have the following:

COROLLARY 5.4. *Let δ be a smooth generator of a τ-integrable automorphism group on a semifinite von Neumann algebra $\mathscr{A}$. Then*

$$(P_t x)\, v = \int e^{\omega(t)\delta}(x)\, m^{-\delta_1^*(1)/2}(\omega)\, v\, d\mathbb{P}(\omega)$$

$x \in L^2 \cap L^\infty$, $v \in L^2$, represents the symmetric Markov semigroup with L^2 generator $-\frac{1}{2}\delta_2^*\delta_2$.

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