Chernoff-type inequality and variance bounds

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Abstract

After a brief review of the work on Chernoff-type inequalities, bounds for the variance of functions $g(X, Y)$ of a bivariate random vector (X, Y) are derived when the marginal distribution of X is normal, gamma, binomial, negative binomial or Poisson assuming that the variance of $g(X, Y)$ is finite. These results follow as a consequence of Chernoff inequality, Stein-identity for the normal distribution and their analogues for other distributions as obtained by Cacoullos, Papathanasiou, Prakasa Rao, Sreehari among others. Some interesting inequalities in real analysis are derived as special cases.

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1. Introduction

Chernoff (1981) derived an inequality giving an upper bound for the variance of a function of a standard normal random variable. Chen (1982) extended the result for multivariate normal distribution. Cacoullos (1982) and Klassen (1985) obtained a generalization of the inequality to other distributions and derived upper and lower bounds. Similar results were obtained by Cacoullos and Papathanasiou (1985, 1989), Prakasa Rao (1990) and Srivastava and Sreehari (1987, 1990). Bounds for the variance of a function g of an infinitely divisible random variable (r.v.) X are given in Vitale (1989). Prakasa Rao (1992) obtained extensions of Chernoff-type inequalities and used them to derive inequalities for nonlinear functions of stochastic integrals.

Borovkov and Utev (1983) characterized the normal distribution via Chernoff-inequality. Prakasa Rao and Sreehari (1986) extended the result to characterize the multivariate normal distribution. Prakasa Rao (1993) derived an integro-differential inequality

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for random vectors following Utev (1989) and obtained a characterization of the multivariate normal distribution. For more references and an extensive survey, see Utev (1992). Prakasa Rao and Sreehari (1987) obtained a characterization of the Poisson distribution and Chen and Lou (1987) studied characterizations of probability distributions by Poincare-type inequalities. Hu (1986) and Purkayastha and Bhandari (1990) characterized the uniform distribution by Chernoff-type inequalities. Freimer and Mudholkar (1992) obtained an analogue of Chernoff-type inequality to characterize the double exponential distribution. Prakasa Rao (1990) studied the case of elliptical distributions. Korwar (1989) and Johnson (1993) considered similar problems for Pearsonian family.

Our aim in this paper is to obtain lower and upper bounds on the variance of a function $g(X,Y)$ of a bivariate random vector (X,Y) and to derive results for functions of the type $h(X,Y) = XY$ where X is independent of Y. Variance bounds for such functions are of interest. For instance, it is known that r.vs. like $W_1|X_\alpha|^p$ and $W_2|t|^p$ are infinitely divisible (i.d.) when W_1 and W_2 are r.vs independent of X_α, the symmetric stable r.v. with characteristic exponent α and t, the Student's t r.v. respectively, and $p \geqslant 2$ (see Shanbhag and Sreehari, 1977).

2. Stein–Chernoff bounds and an extension

Suppose X is standard normal. Let $g(\cdot)$ be an absolutely continuous real valued function such that

$$E[g(X)]^2 < \infty \quad \text{and} \quad E[g'(X)]^2 < \infty. \tag{2.1}$$

Stein (1973) proved that

$$\mathrm{Cov}(X, g(X)) = E[Xg(X)] = E[g'(X)]. \tag{2.2}$$

Here g' denotes the derivative of g. Hudson (1978) and more recently Chou (1988) extended the result to exponential families. Prakasa Rao (1979) derived some characterization results for discrete and continuous exponential families via these identities. Cacoullos and Papathanasiou (1989) obtained a generalisation of the covariance identity (2.2). Chernoff (1981) proved that

$$\mathrm{Var}[g(X)] \leqslant E[g'(X)]^2. \tag{2.3}$$

In fact, it can be shown that (See Chen, 1982)

$$E[\{g(X) - g(0)\}^2] \leqslant E[g'(X)]^2. \tag{2.4}$$

Relation (2.2) implies that

$$E[X\{g(X) - Eg(X)\}] = E[g'(X)] \tag{2.5}$$

because $EX = 0$ and hence, by the Cauchy–Schwartz inequality,

$$\{E[g'(X)]\}^2 \leqslant E(X^2)E[g(X) - Eg(X)]^2 = \mathrm{Var}[g(X)] \tag{2.6}$$

because $EX^2 = 1$. Combining (2.4) and (2.6) we have the inequality

$$[Eg'(X)]^2 \leqslant V[g(X)] \leqslant E[g(X) - g(0)]^2 \leqslant E[g'(X)]^2. \tag{2.7}$$

Here $V(Z)$ stands for Var(Z). It is easy to extend this result for r.vs. which are $N(\mu, \sigma^2)$. Let $X = \sigma Z + \mu$ where Z is $N(0, 1)$ and define $g(z) = h(x)$. Note that $g'(z) = \sigma h'(x)$ and applying (2.7), we obtain that

$$V(X)[Eh'(X)]^2 \leqslant V[h(X)] \leqslant E[h(X) - h(\mu)]^2 \leqslant V(X)E[h'(X)]^2. \tag{2.8}$$

Cacoullos (1982) obtained the lower bound in (2.7) in Proposition 3.2 of his paper. Hereafter we call the inequality (2.8) as *Chernoff-Cacoullos inequality*.

Consider a random vector (X, Y) such that the conditional distribution of X given $Y = y$ is $N(\mu_y, \sigma_y^2)$. Suppose that $g(x, y)$ is a function differentiable with respect to x. Let g_x denote the partial derivative of g with respect to x. Castillo and Galambos (1989) considered some bivariate distributions where conditional distributions are normal and proved some characterization results. Suppose that

$$E[g^2|Y = y] < \infty \quad \text{and} \quad E[g_X^2|Y = y] < \infty \quad \text{a.s.} \tag{2.9}$$

It is clear from (2.8) that

$$\begin{aligned} \sigma_Y^2[E_X\{g_X|Y\}]^2 &\leqslant V_X[g(X, Y)|Y] \leqslant E_X[\{g(X, Y) - g(\mu_Y, Y)\}^2|Y] \\ &\leqslant \sigma_Y^2 E_X[g_X^2|Y] \quad \text{a.s.} \end{aligned} \tag{2.10}$$

Taking expectations with respect to Y, we have

$$\begin{aligned} E_Y[\sigma_Y^2\{E_X(g_X|Y)\}^2] &\leqslant E_Y[V_X\{g(X, Y)|Y\}] \\ &\leqslant E_{X,Y}[g(X, Y) - g(\mu_Y, Y)]^2 \leqslant E_{X,Y}[\sigma_Y^2 g_X^2]. \end{aligned} \tag{2.11}$$

2.1. Special cases

Suppose $g(X, Y) = h(XY)$ and X is independent of Y. Clearly μ_Y and σ_Y^2 are independent of Y and hence can be denoted as μ and σ^2, respectively. Further $g_x = yh'(x, y)$. From (2.11) we then have,

$$\begin{aligned} E_Y[\sigma^2\{E_X(Yh'(XY)|Y)\}^2] &\leqslant E_Y[V_X(h(XY)|Y)] \\ &\leqslant E_{X,Y}[h(XY) - h(\mu Y)]^2 \\ &\leqslant E_{X,Y}[\sigma^2 Y^2\{h'(XY)\}^2]. \end{aligned} \tag{2.12}$$

Since X is independent of Y, the inequality (2.12) can be rewritten in the form

$$\begin{aligned} V(X)E_Y[Y^2\{E_X(h'(XY))\}^2] &\leqslant E_Y[V_X(h(XY))] \\ &\leqslant E_{X,Y}[h(XY)-h(\mu Y)]^2 \\ &\leqslant V(X)E_{X,Y}[Y^2\{h'(XY)\}^2]. \end{aligned} \tag{2.13}$$

In particular, if $\mu = 0$ and X is independent of Y, then

$$\begin{aligned} V(X)E_Y[Y^2\{E_X(h'(XY))\}^2] &\leqslant E_Y[V_X(h(XY))] \\ &\leqslant E_{X,Y}[h(XY)-h(0)]^2 \\ &\leqslant V(X)E_{X,Y}[Y^2\{h'(XY)\}^2]. \end{aligned} \tag{2.14}$$

This result was derived in Thoerem 2.1 of Prakasa Rao (1992) when X is $N(0,1)$. We shall now discuss two examples. In both the examples, the random variable X is assumed to follow $N(0,\sigma^2)$ and Y is a r.v. independent of X.

(i) Suppose $h(x) = \sin x$. Then, by the inequality (2.14), it follows that

$$\begin{aligned} \sigma^2 E_Y[Y^2\{E_X \cos(XY)\}^2] &\leqslant E_Y[V_X(\sin(XY))] \\ &\leqslant E_{X,Y}[\sin(XY)]^2 \\ &\leqslant \sigma^2 E_{X,Y}[Y^2\{\cos(XY)\}^2]. \end{aligned}$$

(ii) Suppose $h(x) = \exp(x)$. Then

$$\begin{aligned} \sigma^2 E_Y[Y^2\{E_X(e^{XY})\}^2] &\leqslant E_Y[V_X(e^{XY})] \\ &\leqslant E_{X,Y}[e^{XY}-1]^2 \\ &\leqslant \sigma^2 E_{X,Y}[Y^2 e^{2XY}] \end{aligned}$$

which reduces to

$$\begin{aligned} \sigma^2 E[Y^2(e^{\sigma^2 Y^2})] &\leqslant E[e^{2\sigma^2 Y^2} - e^{\sigma^2 Y^2}] \\ &\leqslant E[e^{2\sigma^2 Y^2} - 2e^{\sigma^2 Y^2/2} + 1] \\ &\leqslant \sigma^2 E[Y^2 e^{2\sigma^2 Y^2}]. \end{aligned}$$

By taking $\sigma = 1$ in the above inequality, we have the following result for any r.v. Y:

$$\begin{aligned} E[Y^2 e^{Y^2}] \leqslant E[e^{2Y^2} - e^{Y^2}] &\leqslant E[e^{2Y^2} - 2e^{Y^2/2} + 1] \\ &\leqslant E[Y^2 e^{2Y^2}] \end{aligned}$$

or equivalently, for any non-negative r.v. Z,

$$E[Ze^Z] \leqslant E[e^{2Z} - e^Z] \leqslant E[e^{2Z} - 2e^{Z/2} + 1] \leqslant E[Ze^{2Z}]. \tag{2.15}$$

In particular, we have

$$E[(1-Z)e^{2Z}] \leqslant 2E[e^{Z/2}] - 1 \tag{2.16}$$

for any non-negative random variable Z. As a further special case, let $P(Z = z) = 1$ where $z \geqslant 0$. Then, we have the inequality

$$(1 - z)e^{2z} \leqslant 2e^{z/2} - 1$$

for all $z \geqslant 0$.

Other inequalities dealing with exponential function can be derived from (2.15). Obviously, what is of interest in (2.15) and other inequalities are the intermediate bounds.

3. Bounds for functions of gamma distributed random variables

If X follows the gamma distribution with parameters (α, λ) and $g(\cdot)$ is a real valued function satisfying

$$E[g^2(X)] < \infty \quad \text{and} \quad E[X\{g'(X)\}^2] < \infty \tag{3.1}$$

where g' is the derivative of g, then Cacoullos (1982), Cacoullos and Papathanasiou (1985, 1989) and Srivastava and Sreehari (1990) proved that

$$\frac{1}{\alpha}[E\{Xg'(X)\}]^2 \leqslant V[g(X)] \leqslant \frac{1}{\lambda}E[X\{g'(X)\}^2]. \tag{3.2}$$

In particular, if X is exponential with scale parameter $\lambda = 1$, then

$$[E\{Xg'(X)\}]^2 \leqslant V[g(X)] \leqslant E[X\{g'(X)\}^2]. \tag{3.3}$$

Consider a random vector (X, Y) and let $g(x, y)$ be a real valued function. Suppose $g_x(x, y)$ denotes the partial derivative of g with respect to x. Assume that the conditional distribution of X given $Y = y$ is gamma (α_y, λ_y). If

$$E_{X,Y}[g^2(X, Y)] < \infty \quad \text{and} \quad E_Y\left[\frac{1}{\lambda_Y}E_X\{g_X^2(X, Y)|Y\}\right] < \infty \tag{3.4}$$

then, as in the normal case, we get (using (3.3))

$$\begin{aligned} E_Y\left[\frac{1}{\alpha_Y}(E_X\{Xg_X(X, Y)|Y\})^2\right] &\leqslant E_Y[V_X(g(X, Y)|Y)] \\ &\leqslant E_Y\left[\frac{1}{\lambda_Y}E_X\{X(g_X^2(X, Y))|Y\}\right]. \end{aligned} \tag{3.5}$$

In particular, if X and Y are independent and $g(x, y) = h(xy)$, where h is a differentiable function, then (3.5) reduces to

$$\begin{aligned} \frac{1}{\alpha}E_Y[Y^2\{E_X(Xh'(XY))\}^2] &\leqslant E_Y[V_X(h(XY))] \\ &\leqslant \frac{1}{\lambda}E_{X,Y}[XY^2\{h'(XY)\}^2]. \end{aligned} \tag{3.6}$$

Shanbhag and Sreehari (1979) proved that if Z is gamma with parameters $(\alpha, 1)$ and W is a r.v. independent of Z, then, for $\beta \geqslant \max(1, \alpha)$, the r.v. WZ^{β} is i.d. Further if $\alpha = 1$ and W is non-negative, then $\exp(WZ)$ is i.d. It is easily seen that the inequality (3.6) is applicable to the above r.vs under certain conditions.

Suppose $P[0 < W < \frac{1}{2}] = 1$ and Z has the p.d.f. $f(z) = \exp(-z)$ for $z > 0$. If W and Z are independent, then, by taking $h(u) = \exp\{u\}$, we get from (3.6) that

$$E_W[W^2\{E_Z(Ze^{WZ})\}^2] \leqslant E_W[E_Z(e^{2ZW}) - \{E_Z(e^{ZW})\}^2] \leqslant E_W[W^2 E_Z(Ze^{2ZW})]. \tag{3.7}$$

It is easy to check that, for any $0 < w < \frac{1}{2}$,

$$E[e^{wZ}] = \frac{1}{1-w},$$

and

$$E[Ze^{wZ}] = \frac{1}{(1-w)^2}.$$

Hence, inequality (3.7) implies that

$$E_W\left[W^2\left(\frac{1}{(1-W)^2}\right)^2\right] \leqslant E_W\left[\frac{1}{1-2W} - \left(\frac{1}{1-W}\right)^2\right] \leqslant E_W\left[W^2\frac{1}{(1-2W)^2}\right],$$

i.e.,

$$E_W\left[\frac{W^2}{(1-W)^4}\right] \leqslant E_W\left[\frac{W^2}{(1-2W)(1-W)^2}\right] \leqslant E_W\left[\frac{W^2}{(1-2W)^2}\right] \tag{3.8}$$

for any random variable W with $P[0 < W < \frac{1}{2}] = 1$.

4. Bounds for functions of a random variable with Pareto distribution

Suppose a r.v. X follows the Pareto distribution of Type I with density function

$$\begin{aligned} f(x) &= ax_0^a x^{-(a+1)}, \quad x \geqslant x_0 > 0, \\ &= 0 \qquad\qquad\quad \text{otherwise}, \end{aligned} \tag{4.1}$$

where $a > 0$. Then the variance of X exists only if $a > 2$ in which case

$$\mu = EX = ax_0/(a-1) \quad \text{and} \quad V(X) = \frac{ax_0^2}{(a-2)(a-1)^2}.$$

Further, for a differentiable real valued function $g(\cdot)$, we have, by results of Srivastava and Sreehari (1990) and Cacoullos and Papathanasiou (1985, 1989), that

$$\frac{a-2}{ax_0^2}E^2[X(X-x_0)g'(X)] \leqslant V[g(X)] \leqslant \frac{1}{a-1}E[X(X-x_0)\{g'(X)\}^2] \tag{4.2}$$

provided

$$E[g^2(X)] < \infty \text{ and } E[\{Xg'(X)\}^2] < \infty. \tag{4.3}$$

As in the preceeding sections, we can derive bounds corresponding to bivariate random vectors. We consider a particular case which has some applications.

Suppose X represents the declared income of a person. Then the real income Z of the person is given by $Z = XY$ where $Y \geqslant 1$ a.s. It is assumed that Y is independent of X and a proper model for the distribution of X is Pareto type I. Suppose $g(x, y) = h(xy)$ and h is differentiable. Then $g_x(x, y) = yh'(xy)$ and under suitable assumptions, we get from (4.2) that

$$\begin{aligned} \frac{a-2}{ax_0^2} E_Y[Y^2\{E_X(X(X-x_0)h'(XY))\}^2] &\leqslant E_Y[V_X(h(XY))] \\ &\leqslant \frac{1}{a-1} E_Y[Y^2 E_X\{X(X-x_0)(h'(XY))^2\}]. \end{aligned} \tag{4.4}$$

5. Bounds for functions of negative binomial random variables

Suppose X is a negative binomial r.v. with parameters (r, p). Then we have the inequality

$$\frac{q}{r} E^2[(X+r)\Delta h(X)] \leqslant V[h(X)] \leqslant \frac{q}{p} E[(X+r)\Delta^2 h(X)] \tag{5.1}$$

for all functions h for which $E[h^2(X)] < \infty$ and $E[(X+r)\Delta^2 h(X)] < \infty$ where $q = 1 - p$. Inequality (5.1) follows from the results of Cacoullos and Papathanasiou (1985) and Srivastava and Sreehari (1987, 1990). Here $\Delta h(X) = h(X+1) - h(X)$.

Suppose the conditional distribution of X given Y is negative binomial (r_Y, p_Y) and $g(X, Y)$ is a measurable function of X and Y. Let $\Delta_1 g(X, Y)$ denote $g(X+1, Y) - g(X, Y)$. Then, by the inequality (5.1), we have

$$\begin{aligned} \frac{q_Y}{r_Y} E_X^2[(X+r_Y)\Delta_1 g(X,Y)|Y] &\leqslant V(g(X,Y)|Y) \\ &\leqslant \frac{q_Y}{p_Y} E[(X+r_Y)\Delta_1^2 g(X,Y)|Y] \quad \text{a.s.} \end{aligned}$$

and hence

$$\begin{aligned} E_Y\left[\frac{q_Y}{r_Y} E_X^2\{(X+r_Y)\Delta_1 g(X,Y)|Y\}\right] &\leqslant E_Y[V(g(X,Y)|Y] \\ &\leqslant E_Y\left[\frac{q_Y}{p_Y} E\{(X+r_Y)\Delta_1^2 g(X,Y)|Y\}\right]. \end{aligned} \tag{5.2}$$

Suppose X and Y are independent and $g(x, y) = y^x$. Further suppose that $0 < Y < 1$ a.s. Then we have, from (5.2), that

$$\frac{q}{r}E_Y[E_X^2\{(X+r)(Y-1)Y^X\}]\leqslant E_Y[E_X(Y^{2X})-E_X^2(Y^X)]$$
$$\leqslant\frac{q}{p}E_Y[E_X\{(X+r)(Y-1)^2Y^{2X}\}],$$

i.e.,

$$\frac{q}{r}E_Y[(Y-1)^2E_X^2\{(X+r)Y^X\}]\leqslant E_Y[H(Y^2)-H^2(Y)]$$
$$\leqslant\frac{q}{p}E_Y[(Y-1)^2E_X\{(X+r)Y^{2X}\}],$$

where $H(s)$ is the probability generating function of X. On further simplification, we get

$$\frac{q}{r}E_Y[(Y-1)^2\{YH'(Y)+rH(Y)\}^2]\leqslant E_Y[H(Y^2)-H^2(Y)]$$
$$\leqslant\frac{q}{p}E_Y[(Y-1)^2\{Y^2H'(Y^2)+rH(Y^2)\}].$$

Recalling that $H(s)=p^r(1-qs)^{-r}$ we get that

$$\frac{q}{r}E_Y\left[(Y-1)^2\left\{\frac{Yrqp^r}{(1-qY)^{r-1}}+\frac{rp^r}{(1-qY)^r}\right\}^2\right]\leqslant E_Y\left[\frac{pr}{(1-qY^2)^r}-\frac{p^{2r}}{(1-qY)^{2r}}\right]$$
$$\leqslant\frac{q}{p}E_Y\left[(Y-1)^2\left\{\frac{Y^2rqp^r}{(1-qY^2)^{r-1}}+\frac{rp^r}{(1-qY^2)^r}\right\}\right],$$

i.e.,

$$rqp^{2r}E_Y\left[\frac{(Y-1)^2}{(1-qY)^{2(r+1)}}\right]\leqslant E_Y\left[\frac{p^r}{(1-qY^2)^r}-\frac{p^{2r}}{(1-qY)^{2r}}\right]$$
$$\leqslant rqp^{r-1}E_Y\left[\frac{(Y-1)^2}{(1-qY^2)^{r-1}}\right]. \tag{5.3}$$

It may be noted that the above inequality holds for all r.vs Y such that $0<Y<1$ a.s. and all $p, 0<p<1, q=1-p$ and $r\geqslant 1$. In particular, for $r=1$ (i.e. geometric assumption)

$$qp^2E_Y\left[\frac{(Y-1)^2}{(1-qY)^4}\right]\leqslant E_Y\left[\frac{p}{(1-qY^2)}-\frac{p^2}{(1-qY)^2}\right]\leqslant qE_Y\left[\frac{(Y-1)^2}{(1-qY^2)^2}\right]. \tag{5.4}$$

For any real number $a, 0<a<1$, we then have

$$\frac{qp^2(a-1)^2}{(1-qa)^4}\leqslant\frac{p}{(1-qa^2)}-\frac{p^2}{(1-qa)^2}\leqslant q\,\frac{(a-1)^2}{(1-qa^2)^2}. \tag{5.5}$$

If we take $g(x,y)=e^{xy}$ instead of $g(x,y)=y^x$, we then get, for any r.v. Y such that $0<Y<-\frac{1}{2}\log q$ a.s.,

$$rqp^{2r}E[(e^Y-1)^2/(1-qe^Y)^{2(r+1)}]\leqslant E\left[\frac{pr}{(1-qe^{2Y})^r}-\frac{p^2r}{(1-qe^Y)^{2r}}\right]$$

$$\leqslant rqp^{r-1}E[(e^Y-1)^2/(1-qe^{2Y})^{r+1}] \tag{5.6}$$

provided all the above expectations exist.

6. Bounds for functions of binomial and Poisson random variables

Inequalities similar to those derived above can be obtained in the case of Poisson and binomial r.vs. (see, for example, Srivastava and Sreehari, 1990). Let $B(n,p)$ denote the binomial distribution with parameters n and p. If the conditional distribution of X given Y is Poisson (λ_Y), then we have

$$E_Y[\lambda_Y\{E_X(\Delta_1 g(X,Y)|Y)\}^2] \leqslant E_Y[V_X(g(X,Y)|Y)]$$
$$\leqslant E_Y[\lambda_Y E_X\{(\Delta_1 g(X,Y))^2|Y\}]. \tag{6.1}$$

If the conditional distribution of X given Y is $B(n_Y, p_Y)$, then

$$E_Y\left[\frac{p_Y}{1-p_Y}\frac{1}{n_Y}(E_X\{(n_Y-X)\Delta_1 g(X,Y)|Y\})^2\right] \leqslant E_Y[V_X(g(X,Y)|Y)]$$
$$\leqslant E_Y[p_Y E_X\{(n_Y-X)(\Delta_1 g(X,Y))^2|Y\}]. \tag{6.2}$$

In particular, if $X|Y \simeq B(Y,p)$, then

$$\frac{p}{1-p}E_Y\left[\frac{1}{Y}(E_X\{(Y-X)\Delta_1 g(X,Y)|Y\})^2\right] \leqslant E_Y[V_X(g(X,Y)|Y)]$$
$$\leqslant p\,E_Y[E_X\{(Y-X)(\Delta_1 g(X,Y))^2|Y\}]. \tag{6.3}$$

If, further, X and Y are independent and $g(X,Y)=Y^X$, the inequality (6.3), after some simplifications, (here $X \simeq B(n,p)$) reduces to

$$npqE[(Y-1)^2(q+pY)^{2(n-1)}] \leqslant E[(q+pY^2)^n-(q+pY)^{2n}]$$
$$\leqslant npq\,E[(Y-1)^2(q+pY^2)^{n-1}] \tag{6.4}$$

for any r.v. Y such that $0<Y<1$ a.s. As a special case, it follows that

$$npq(y-1)^2(q+py)^{2(n-1)} \leqslant (q+py^2)^n-(q+py)^{2n}$$
$$\leqslant npq(y-1)^2(q+py^2)^{n-1} \tag{6.5}$$

whenever $0<y<1$, $0<p<1$, $q=1-p$ and $n \geqslant 1$. It is easy to see that these inequalities are sharp. For instance, if $n=1$, then both the upper and lower bounds lead to the same expression and equality occurs throughout. Let

$$\alpha = q+py^2 \quad \text{and} \quad \beta=(q+py)^2. \tag{6.6}$$

Note that $\beta \leqslant \alpha \leqslant \sqrt{\beta}$ and $(1+\alpha)/2 \geqslant \sqrt{\beta}$. Then the above set of inequalities can be written in the form

$$n(\alpha-\beta)\beta^{n-1} \leqslant \alpha^n-\beta^n \leqslant n(\alpha-\beta)\alpha^{n-1} \tag{6.7}$$

which is well known whenever $0 \leqslant \alpha$, $\beta \leqslant 1$, $\beta \leqslant \alpha \leqslant \sqrt{\beta}$ and $(1+\alpha)/2 \geqslant \sqrt{\beta}$. In fact, given any α and β in $(0,1)$ satisfying the above inequalities, one can find p and y in $(0,1)$ satisfying the above relations (6.6). This gives an alternate method of deriving the inequality (6.7).

Remark. The above inequalities viz. (2.13), (3.6), (4.4) can be used to characterize the corresponding distributions. For example, if X is a random variable such that for some random variable Y independent of X and all differentiable functions $h(\cdot)$ the inequality (2.13) holds, then the moments μ_n' of the symmetrized r.v. $\tilde{X}$ satisfy the recurrence relation

$$\mu_{n+1}' = nV(X)\mu_{n-1}',$$

where $\mu_n' = E|X - X'|^n = E|\tilde{X}|^n$ as in the case of Borovkov and Utev (1983). Hence X is normal.

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