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The Wiener–Hopf solution of a class of mixed boundary value problems arising in surface water wave phenomena

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Abstract

Two mixed boundary value problems associated with two-dimensional Laplace equation, arising in the study of scattering of surface waves in deep water (or interface waves in two superposed fluids) in the linearised set up, by discontinuities in the surface (or interface) boundary conditions, are handled for solution by the aid of the Weiner–Hopf technique applied to a slightly more general differential equation to be solved under general boundary conditions and passing on to the limit in a manner so as to finally give rise to the solutions of the original problems. The first problem involves one discontinuity while the second problem involves two discontinuities. The reflection coefficient is obtained in closed form for the first problem and approximately for the second. The behaviour of the reflection coefficient for both the problems involving deep water against the incident wave number is depicted in a number of figures. It is observed that while the reflection coefficient for the first problem steadily increases with the wave number, that for the second problem exhibits oscillatory behaviour and vanishes at some discrete values of the wave number. Thus, there exist incident wave numbers for which total transmission takes place for the second problem.

1. Introduction

If a part of the surface of deep water is covered by an *inertial surface* composed of a thin uniform distribution of non-interacting particles (e.g. broken ice, unstretched mat) and the remaining part is *free*, then the surface boundary condition becomes discontinuous in the sense that there are one condition on the free surface and another condition on the inertial surface. The line separating the free surface and the inertial surface becomes a line of discontinuity. Peters[1], Weitz and Keller [2] first developed this model to study wave–ice interaction. When half the surface of water is covered by an inertial surface and the other half of the surface is free, Peters [1] investigated the case when waves from the free surface region are normally incident on the straight line separating the free surface and the inertial surface. Weitz and Keller [2] treated the same problem for water of arbitrary finite depth and oblique incidence of waves.

Gabov et al. [3] considered two infinitely extended, immiscible superposed fluids for which half of the interface is covered by an inertial surface and the other half of the interface is a free separating boundary of the two fluids,

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and investigated the scattering of the interface waves travelling from the free interface region and normally incident on the line separating the free interface and the inertial interface.

The above two physical problems are mathematically equivalent. This can be shown by considering the case of deep water and the case of two superposed fluids separately.

Case (a). Deep water. We first consider a deep water and choose the y-axis vertically downwards into the water so that its surface at the rest position coincides with the plane $y = 0$. Let the semi-infinte plane represented by $y = 0, x < 0$ be the free surface and the semi-infinte plane $y = 0, x > 0$ be covered by an inertial surface of area density σ, say. Let $\operatorname{Re}\psi(x, y)e^{-iwt}$ represent the velocity potential describing the irrotational motion in the fluid where t denotes the time. Let the factor e^{-iwt} be suppressed always henceforth. Then the complex-valued potential function $\psi(x, y)$ is harmonic in the fluid region. The linearised free surface condition is

$$\psi_y + a\psi = 0 \quad \text{on } y = 0, \quad x < 0, \tag{1.1}$$

where

$$a = w^2/g, \tag{1.2}$$

g being the acceleration due to gravity. The condition of no motion at infinite depth gives

$$\psi, \nabla\psi \to 0 \quad \text{as } y \to \infty. \tag{1.3}$$

Thus an incident progressive surface wave field is represented by the potential function $e^{-ay+iax}$. Again, the linearised inertial surface condition is

$$\psi_y + b\psi = 0 \quad \text{on } y = 0, \quad x > 0, \tag{1.4}$$

where

$$b = \frac{\rho w^2}{\rho g - w^2\sigma}, \tag{1.5}$$

ρ being the density of water. It may be noted that for $\sigma < \rho g/w^2$ (i.e. $b > 0$), the form of (1.4) is merely a modification of the usual free surface condition (1.1) corresponding to $\sigma = 0$, and it allows progressive waves at the inertial surface. However, for $\sigma \geq \rho g/w^2$ (i.e. $b < 0$ or $|b| = \infty$), the form of (1.4) is different and does not allow progressive waves at the inertial surface (cf. [4]).

Let $\phi(x, y)$ denote the scattered potential function due to the incident wave field $e^{-ay+iax}$ propagating from infinity along the free surface and normally incident on the line separating the free surface and the inertial surface. Then $\phi(x, y)$ is harmonic in the fluid region and satisfies the boundary conditions

$$\phi_y + a\phi = 0 \quad \text{on } y = 0, \quad x < 0, \tag{1.6a}$$

$$\phi_y + b\phi = -(b - a)e^{iax} \quad \text{on } y = 0, \quad x > 0, \tag{1.6b}$$

so that $x = 0$ is a point of discontinuity in the boundary condition on the boundary $y = 0$. ϕ also satisfies the same conditions as ψ satisfies as $y \to \infty$. Moreover, it also satisfies some edge conditions as $(x^2 + y^2)^{1/2} \to 0$ and infinity requirements involving the unknown complex-valued reflection and transmission coefficients as $|x| \to \infty$. These will be stated later. These unknown coefficients form a part of the problem.

Case (b). Two superposed fluids. Again, we consider two superposed fluids of densities ρ_1, ρ_2 where ρ_1 is the density of the lower fluid and ρ_2 $(< \rho_1)$ is the density of the upper fluid. Let the y-axis be chosen vertically downwards into the lower fluid so that the rest position of the common interface is $y = 0$. Let the half plane represented by $y = 0, x < 0$ be the free separating boundary of the two fluids and the half plane $y = 0, x > 0$ be covered by an inertial surface of the area density σ. Let $\operatorname{Re}\psi_1(x, y)e^{-iwt}$, $\operatorname{Re}\psi_2(x, y)e^{-iwt}$, respectively, denote the velocity potentials in the lower and upper fluids describing irrotational motion. Then ψ_1 is harmonic in $y > 0$ and ψ_2 is harmonic in $y < 0$. The linearised conditions at the free separating boundary are

$$\rho_1(g\psi_{1y} + w^2\psi_1) = \rho_2(g\psi_{2y} + w^2\psi_2) \quad \text{on } y = 0, \ x < 0, \tag{1.7}$$

$$\psi_{1y} = \psi_{2y} \quad \text{on } y = 0, \ x < 0, \tag{1.8}$$

and the conditions at the bottom and top are

$$\psi_1, \nabla\psi_1 \to 0 \quad \text{as } y \to \infty, \qquad \psi_2, \nabla\psi_2 \to 0 \quad \text{as} y \to -\infty. \tag{1.9}$$

Thus an incident progressive interface wave field propagating on the free separating boundary is represented by

$$\psi_1^{\text{inc}}(x, y) = \mathrm{e}^{-ay+\mathrm{i}ax}, \quad y > 0, \qquad \psi_2^{\text{inc}}(x, y) = -\mathrm{e}^{-ay+\mathrm{i}ax}, \quad y < 0, \tag{1.10}$$

where now

$$a = \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}\frac{w^2}{g}. \tag{1.11}$$

Again, the linearised conditions at the inertial interface are

$$\rho_1(g\psi_{1y} + w^2\psi_1) - \rho_2(g\psi_{2y} + w^2\psi_2) = \sigma w^2\psi_{1y} = \sigma w^2\psi_{2y} \quad \text{on } y = 0, \ x > 0. \tag{1.12}$$

Let $\chi_j(x, y)(j = 1, 2)$ denote the scattered potentials due to the incident wave field represented by $\psi_j^{\text{inc}}(x, y)(j = 1, 2)$ propagating from infinity along the free separating boundary and normally incident on the line separating the free interface and inertial interface. Then $\chi_j(x, y)(j = 1, 2)$ satisfy

$$\nabla^2\chi_1 = 0 \quad \text{in } y > 0, \qquad \nabla^2\chi_2 = 0 \quad \text{in } y < 0, \tag{1.13}$$

the interface conditions

$$\rho_1(g\chi_{1y} + w^2\chi_1) - \rho_2(g\chi_{2y} + w^2\chi_2) = \begin{cases} 0 & \text{on } y = 0, \ x < 0, \quad (1.14) \\ \sigma w^2\chi_{1y} - a\mathrm{e}^{\mathrm{i}ax} & \text{on } y = 0, \ x > 0, \quad (1.15) \end{cases}$$

$$\chi_{1y} = \chi_{2y} \text{ on } y = 0, \tag{1.16}$$

and the bottom and top conditions

$$\chi_1, \nabla\chi_1 \to 0 \quad \text{as } y \to \infty, \qquad \chi_2, \nabla\chi_2 \to 0 \quad \text{as } y \to -\infty. \tag{1.17}$$

We now show that

$$\chi_2(x, y) = -\chi_1(x, -y), \quad y < 0. \tag{1.18}$$

To show this, let

$$\chi(x, y) = \chi_1(x, y) + \chi_2(x, -y), \quad y > 0, \tag{1.19}$$

then $\chi(x, y)$ satisfies the boundary value problem described by

$$\nabla^2\chi = 0 \quad \text{in } y > 0, \qquad \chi_y = 0 \quad \text{on } y = 0, \qquad \chi, \nabla\chi \to 0 \quad \text{as } y \to \infty \tag{1.20}$$

where Eqs. (1.13) and conditions (1.16) and (1.17) have been utilized. Now, by uniqueness theorem of harmonic functions, we find that

$$\chi(x, y) \equiv 0. \tag{1.21}$$

Thus Eq. (1.18) is proved. Hence it is sufficient to solve for the function $\chi_1(x, y)$ which is harmonic in the region $y > 0$. The boundary conditions satisfied by $\chi_1(x, y)$ on $y = 0$ are obtained from (1.14) and (1.15) as

$$\chi_{1y} + a\chi_1 = 0 \quad \text{on } y = 0,\ x < 0, \tag{1.22}$$

$$\chi_{1y} + b\chi_1 = -(b - a)\mathrm{e}^{\mathrm{i}ax} \quad \text{on } y = 0,\ x > 0, \tag{1.23}$$

where a is given by (1.11) and

$$b = \frac{(\rho_1 + \rho_2)w^2}{(\rho_1 - \rho_2)g - aw^2}. \tag{1.24}$$

We note that, in the absence of the upper fluid, a, b assume the values given by (1.2) and (1.5), respectively.

Thus the problem for deep water and the problem for two superposed fluids are mathematically similar, the only difference being that the constants a, b for deep water and for two superposed fluids are to be two different sets of constants. The mathematical formulation of the corresponding physical problem is described as BVP I in Section 2.

Instead of the inertial surface occupying half the surface of deep water (or half the interface of two superposed fluids) if it occupies an infinite strip of finite width l, say, on the surface (interface), so that it is sandwitched between two free surfaces (interfaces), one on the left of $x = 0$ and the other on the right side of $x = l$, then we have two points of discontinuity in the surface condition. The mathematical formulation of the corresponding physical problem is described as BVP II in the following section.

2. Boundary value problems

The mathematical formulations of the two wave scattering problems involving discontinuities in the boundary conditions are now stated in the form of two boundary value problems (BVPs) described below.

BVP I. To solve the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \quad y > 0, \quad -\infty < x < \infty, \tag{2.1}$$

along with the surface boundary conditions, as given by

$$\phi_y + a\phi = 0 \quad \text{on } y = 0 \text{ for } x < 0, \text{ with } a > 0, \tag{2.2}$$

$$\phi_y + b\phi = -(b - a)\mathrm{e}^{\mathrm{i}ax} \quad \text{on } y = 0 \text{ for } x > 0, \text{ with } -\infty < b < \infty, \tag{2.3}$$

producing a discontinuity at the origin in the surface boundary condition, the edge conditions

$$\phi = \mathrm{O}(1) \text{ and } \nabla\phi = \mathrm{O}(1) \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0 \text{ for finite } |b|,$$

or

$$\phi = \mathrm{O}(1) \text{ and } r^{1/2}\nabla\phi = \mathrm{O}(1) \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0 \text{ for } |b| = \infty, \tag{2.4}$$

which are given by the physics of the problem, the conditions

$$\phi,\ \nabla\phi \to 0 \quad \text{as } y \to \infty, \tag{2.5}$$

and the conditions as $|x| \to \infty$, as given by

$$\begin{aligned} \phi(x, y) + \mathrm{e}^{-ay+\mathrm{i}ax} &\sim A_1\mathrm{e}^{-ay-\mathrm{i}ax} + \mathrm{e}^{-ay+\mathrm{i}ax} \quad \text{as } x \to -\infty, \\ &\sim \begin{cases} B_1\mathrm{e}^{-by+\mathrm{i}bx} & \text{as } x \to \infty \text{ for } b > 0, \\ 0 & \text{as } x \to \infty \text{ for } b < 0 \text{ or } |b| = \infty. \end{cases} \end{aligned} \tag{2.6}$$

In (2.6), A_1 and B_1 are two unknown complex constants to be determined, and $|A_1|$, $|B_1|$ represent the reflection and transmission coefficients, respectively, corresponding to the incident surface (or interface) wave field represented by $\mathrm{e}^{-ay+iax}$ propagating in the region $x < 0$. Here $\phi(x, y)$ denotes the scattered field so that the total field is

$$\phi^t(x, y) = \phi(x, y) + \mathrm{e}^{-ay+iax}. \tag{2.7}$$

BVP II. This is the same as BVP I except that the surface boundary condition (2.3) is replaced by two conditions, as given by

$$\phi_y + b\phi = -(b-a)\mathrm{e}^{iax} \quad \text{on } y = 0 \ \text{ for } 0 < x < l, \tag{2.3a}$$

$$\phi_y + a\phi = 0 \quad \text{on } y = 0 \ \text{ for } x > l \tag{2.3b}$$

producing two discontinuities in the surface boundary condition at the points (0, 0) and $(l, 0)$, $l(> 0)$ being finite. For the edge condition (2.4), r now represents the distance from the two discontinuities at (0,0) and $(0, l)$, and instead of (2.6) ϕ now satisfies

$$\begin{aligned}\phi^t(x, y) &\sim A_2\mathrm{e}^{-ay-iax} + \mathrm{e}^{-ay+iax} \quad \text{as } x \to -\infty \\ &= \begin{cases} B_2\mathrm{e}^{-by+ibx} + B_3\mathrm{e}^{-by-ib(x-l)} + \mu_1(x, y), & 0 < x < l \ \text{ for } b > 0, \\ \mu_2(x, y), \quad 0 < x < l \ \text{ for } b < 0 \ \text{ or } \ |b| = \infty, \end{cases} \\ &\sim B_4\mathrm{e}^{-ay+ia(x-l)} \quad \text{as } x \to \infty.\end{aligned} \tag{2.6a}$$

In (2.6a), A_2, B_2, B_3, B_4 are four unknown complex constants to be determined, and the functions $\mu_1(x, y)$, $\mu_2(x, y)$ represent two unknown, ordinary, non-wavy solutions of the Laplace equation. Physically, A_2 represents the reflection coefficient (complex) in the region $x < 0$, B_2 and B_3, respectively, represent the transmission and reflection coefficients (complex) in the region $0 < x < l$ and B_4 represents the transmission coefficients (complex) in the region $x > l$, corresponding to the incident wave field $\mathrm{e}^{-ay+iax}$, propagating in the region $x < 0$ as in BVP I.

The BVP I can be viewed as a special case of a more general problem of wave scattering by a surface discontinuity tackled for its solution by Gabov et al. [3] and the BVP II is a further generalisation of BVP I by introducing a second discontinuity on the surface at a distance l away from the first.

We observe that both the BVPs can be handled for their solution by the aid of the Weiner–Hopf technique after generalising the Laplace equation (2.1) to the Helmholtz equation

$$\phi_{xx} + \phi_{yy} + \epsilon^2\phi = 0, \quad y > 0, \quad -\infty < x < \infty, \tag{2.8}$$

where ϵ is a complex number with small positive real and imaginary parts as well as by generalising some conditions (as described shortly) and ultimately passing on to the limit as $\epsilon \to 0$ in a manner similar to Gabov et al. [3]. Thus the generalised form of BVP I is to solve the Helmholtz equation (2.8) along with the boundary conditions

$$\phi_y + a\phi = 0 \quad \text{on } y = 0 \ \text{ for } x < 0, \tag{2.9}$$

$$\phi_y + b\phi = -(b-a)\mathrm{e}^{ikx} \quad \text{on } y = 0 \ \text{ for } x > 0 \tag{2.10}$$

where

$$k = (a^2 + \epsilon^2)^{1/2} \quad \text{with } k = a \ \text{ for } \epsilon = 0, \tag{2.11}$$

the edge conditions

$$\phi = \mathrm{O}(1) \ \text{ and } \ \nabla\phi = \mathrm{O}(1) \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0 \ \text{ for finite } |b|, \tag{2.12a}$$

or

$$\phi = \mathrm{O}(1) \ \text{ and } \ r^{1/2}\nabla\phi = \mathrm{O}(1) \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0 \ \text{ for } |b| = \infty, \tag{2.12b}$$

and the conditions at infinity, as given by

$$|\phi| + |\nabla\phi| \leq \text{const. } e^{-\delta(\epsilon)r} \quad \text{as } r = (x^2 + y^2)^{1/2} \to \infty, \tag{2.13}$$

where $\delta(\epsilon)$ is such that

$$\delta(\epsilon) \to 0+ \quad \text{as } \epsilon \to 0. \tag{2.14}$$

The generalisation of BVP II is similar to BVP I with the exception that conditions (2.9) and (2.10) are replaced, respectively, by

$$\phi_y + a\phi = 0 \quad \text{on } y = 0 \text{ for } x < 0 \text{ and } x > l, \tag{2.15}$$

$$\phi_y + b\phi = -(b - a)e^{ikx} \quad \text{on } y = 0 \quad \text{for } 0 < x < l \tag{2.16}$$

and in the edge conditions (2.12), r denotes the distance from the two discontinuities at $(0, 0)$ and $(l, 0)$ so that $r = (x^2 + y^2)^{1/2}$ or $\{(x - l)^2 + y^2\}^{1/2}$. It may be noted that condition (2.13) is stronger than condition (2.5) as $y \to \infty$, and is consistent with the behaviour of ϕ as $|x| \to \infty$.

With this as background, we proceed to present in Sections 3 and 4, the Wiener–Hopf technique applied to the two generalised BVPs satisfying the Helmholtz equation (1.8) involving the complex parameter ϵ, the surface conditions (2.9) and (2.10) (for BVP I) or (2.15) and (2.16) (for BVP II), the edge conditions (2.12) and the infinity requirement (2.13). For the BVP I, the reflection and transmission coefficients have been obtained explicitly. For the BVP II, which is a generalisation of BVP I to the case when the inertial surface is in the form of a horizontal strip of breadth l, the reflection and transmission coefficients have been obtained asymptotically for large l. The reflection coefficient for both the problems in the region $x < 0$ is depicted graphically against the wave number aL for $b > 0$ to visualize the effect of the inertial surface on the incident wave train progressing along the free surface, where L is a characteristic length used to non-dimensionalise a, $b(> 0)$ and σ/ρ for deep water.

3. The Wiener–Hopf technique involving BVP I

Let $\phi_1(x, y) \equiv \phi(x, y; \epsilon)$ denote the function satisfying the generalised BVP I described by (2.8)–(2.10), (2.12) and (2.13). We introduce the Fourier transform $\Phi_1(\alpha, y)$ as defined by

$$\Phi_1(\alpha, y) = \int_{-\infty}^{\infty} \phi_1(x, y)e^{i\alpha x}\, dx,$$

where $\alpha = \sigma + i\tau$, σ and τ being real. Then

$$\Phi_1(\alpha, y) = \Phi_{1-}(\alpha, y) + \Phi_{1+}(\alpha, y),$$

where

$$\Phi_{1-}(\alpha, y) = \int_{-\infty}^{0} \phi_1(x, y)e^{i\alpha x}\, dx, \quad \Phi_{1+}(\alpha, y) = \int_{0}^{\infty} \phi_1(x, y)e^{i\alpha x}\, dx. \tag{3.1}$$

By using condition (2.13) it is observed that the functions $\Phi_{1+}(\alpha, y)$ and $\Phi_{1-}(\alpha, y)$ are analytic functions of α in the overlapping half-planes $\tau > -\delta(\epsilon)$ and $\tau < \delta(\epsilon)$, respectively, and by using the edge conditions (2.12) along with the Abelian theorem (see Noble [5]), it can be shown that

$$|\Phi_{1+}(\alpha, y)| = \mathrm{O}(|\alpha|^{-1}) \quad \text{as } |\alpha| \to \infty \ \text{ in } \tau > -\delta(\epsilon),$$
$$|\Phi_{1-}(\alpha, y)| = \mathrm{O}(|\alpha|^{-1}) \quad \text{as } |\alpha| \to -\infty \ \text{ in } \tau < \delta(\epsilon). \tag{3.2}$$

To use the Wiener–Hopf procedure, conditions (2.9) and (2.10) are rewritten as

$$\phi_{1y} + a\phi_1 = \begin{cases} 0 & \text{on } y = 0 \text{ for } x < 0, \\ f(x) & \text{on } y = 0 \text{ for } x > 0 \end{cases} \tag{3.3}$$

and

$$\phi_{1y} + b\phi_1 = \begin{cases} g(x) & \text{on } y = 0 \text{ for } x < 0, \\ -(b-a)\mathrm{e}^{\mathrm{i}kx} & \text{on } y = 0 \text{ for } x > 0 \end{cases} \tag{3.4}$$

where $f(x)$ (for $x > 0$) and $g(x)$ (for $x < 0$) are unknown functions having the behaviours

$$f(x) = \mathrm{O}(1) \quad \text{as } x \to +0,$$

and

$$g(x) = \mathrm{O}(1) \quad \text{as } x \to -0 \ \text{ for finite } |b|$$

while

$$g(x) = \mathrm{O}(|x|^{-1/2}) \quad \text{as } x \to -0 \ \text{ for } |b| = \infty \tag{3.5}$$

obtained from the edge conditions (2.12) for BVP I.

Now applying the Fourier transform to the pde (2.8), we obtain that

$$\frac{\mathrm{d}^2\Phi_1(\alpha, y)}{\mathrm{d}y^2} - \gamma^2\Phi_1(\alpha, y) = 0, \quad y \geq 0,$$

with $\gamma^2(\alpha) = \alpha^2 - \epsilon^2$, whose appropriate solution is given by

$$\Phi_1(\alpha, y) = D_1(\alpha)\mathrm{e}^{-\gamma y}, \ y \geq 0, \tag{3.6}$$

where $D_1(\alpha)$ is an arbitrary function of the transform parameter α, and we denote by $\gamma(\alpha)$ that branch of the function $(\alpha^2 - \epsilon^2)^{1/2}$ that takes the value $-\mathrm{i}\epsilon$ for $\alpha = 0$. Applying the Fourier transform to conditions (3.3) and (3.4) we obtain that

$$\Phi_1'(\alpha, 0) + a\Phi_1(\alpha, 0) = F_+(\alpha), \tag{3.7}$$

$$\Phi_1'(\alpha, 0) + b\Phi_1(\alpha, 0) = G_-(\alpha) + \frac{b-a}{\mathrm{i}(\alpha + k)} \tag{3.8}$$

in which the two unknown functions $F_+(\alpha) \equiv \int_0^\infty f(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x$ and $G_-(\alpha) \equiv \int_{-\infty}^0 g(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x$ can be shown to be analytic in the two overlapping half plane $\tau > -\delta(\epsilon)$ and $\tau < \delta(\epsilon)$, respectively, with $|F_+(\alpha)| = \mathrm{O}(|\alpha|^{-1})$ as $|\alpha| \to \infty$ in $\tau > -\delta(\epsilon)$ and

$$|G_-(\alpha)| = \begin{cases} \mathrm{O}(|\alpha|^{-1}) & \text{as } |\alpha| \to \infty \ \text{ for finite } |b|, \\ \mathrm{O}(|\alpha|^{-1/2}) & \text{as } |\alpha| \to \infty \ \text{ for } |b| = \infty, \end{cases}$$

in $\tau < \delta(\epsilon)$. Using (3.6) in (3.7) and (3.8) and eliminating $D_1(\alpha)$ we obtain the following two-part Wiener–Hopf functional relation, for the determination of the two functions $F_+(\alpha)$ and $G_-(\alpha)$, as given by

$$\frac{\gamma(\alpha)-b}{\gamma(\alpha)-a}F_+(\alpha)-G_-(\alpha)=\frac{b-a}{\mathrm{i}(\alpha+k)} \tag{3.9}$$

valid in the strip $c < \tau < d$ where c and d are chosen such that

$$-\delta(\epsilon) < -\min(\mathrm{Im}\,k, \mathrm{Im}\,\alpha_0) < c < 0 < d < \min(\mathrm{Im}\,k, \mathrm{Im}\,\alpha_0) < \delta(\epsilon) \tag{3.10}$$

with $\alpha_0 = (b^2+\epsilon^2)^{1/2}$ that takes the value b for $\epsilon = 0$.

Now three cases arise according as $b > 0$, $b < 0$ and $|b| = \infty$, and we treat the Wiener–Hopf relation (3.9), in these three cases, in different manners as described below.

Case 1: $b > 0$. We note that the coefficient of $F_+(\alpha)$ in (3.9) is $[(\alpha^2-\alpha_0^2)/(\alpha^2-k^2)]M(\alpha)$ where $M(\alpha) = (\gamma(\alpha)+a)/(\gamma(\alpha)+b)$. The function $M(\alpha)$ is analytic in the strip $-\delta(\epsilon) < \tau < \delta(\epsilon)$ and hence, in the strip $c < \tau < d$, which can be factorised as

$$M(\alpha) = M_+(\alpha)M_-(\alpha), \tag{3.11}$$

where $M_+(\alpha) = M_-(-\alpha)$, $|M_+(\alpha)| = \mathrm{O}(1)$ as $|\alpha| \to \infty$ in $\tau > c$, $M_+(\alpha)$ being analytic in the upper half plane $\tau > c$ and $M_-(\alpha)$ is analytic in the lower half plane $\tau < d$.

Following Noble [5], $M_+(\alpha)$ is obtained as

$$M_+(\alpha)=\frac{(a-\mathrm{i}\epsilon)^{1/2}}{(b-\mathrm{i}\epsilon)^{1/2}}\frac{\exp\left[\int_0^\alpha\{(\xi+k)/2+(\xi/a)\wedge_+(\xi)+(k/a)\wedge_-(k)\}\,\mathrm{d}\xi/(\xi^2-k^2)\right]}{\exp\left[\int_0^\alpha\{(\xi+\alpha_0)/2+(\xi/b)\wedge_+(\xi)+(\alpha_0/b)\wedge_-(\alpha_0)\}\,\mathrm{d}\xi/(\xi^2-\alpha_0^2)\right]},$$

where

$$\wedge_+(\xi)=\frac{\mathrm{i}}{\pi\gamma(\xi)}\ln\frac{\gamma(\xi)-\xi+\epsilon}{\gamma(\xi)+\xi-\epsilon},\qquad \wedge_-(\xi)=\wedge_+(-\xi). \tag{3.12}$$

As is customary in Wiener–Hopf analysis, Eq. (3.9) is rewritten in the form

$$\begin{aligned}&\frac{\alpha+\alpha_0}{\alpha+k}M_+(\alpha)F_+(\alpha)-\frac{2k(b-a)}{\mathrm{i}(\alpha_0+k)}\frac{1}{M_-(-k)}\frac{1}{\alpha+k}\\&=\frac{\alpha-k}{\alpha-\alpha_0}\frac{G_-(\alpha)}{M_-(\alpha)}\\&+\frac{b-a}{\mathrm{i}}\left[\frac{\alpha-k}{(\alpha-\alpha_0)(\alpha+k)}\left\{\frac{1}{M_-(\alpha)}-\frac{1}{M_-(-k)}\right\}+\frac{\alpha_0-k}{\alpha_0+k}\frac{1}{M_-(-k)}\frac{1}{\alpha-\alpha_0}\right].\end{aligned} \tag{3.13}$$

The left-hand side of (3.13) is analytic in $\tau > c$ and the right-hand side is analytic in $\tau < d$, and as $|\alpha| \to \infty$ in the respective half planes, each side tends to zero. Applying the principle of analytic continuation and Liouville's theorem, we find that each side of (3.13) vanishes identically. Thus we find the unknown function $F_+(\alpha)$, as given by

$$F_+(\alpha)=\frac{2k(b-a)}{\mathrm{i}(\alpha_0+k)M_+(k)}\frac{1}{(\alpha+\alpha_0)M_+(\alpha)}.$$

Now, the use of (3.6) in (3.7) produces $D_1(\alpha)$ as

$$D_1(\alpha)=-\frac{F_+(\alpha)}{\gamma(\alpha)-a}$$

so that $\Phi_1(\alpha, y)$ is obtained explicitly. Thus, by Fourier inversion $\Phi_1(x, y)(y > 0)$ is obtained as

$$\phi_1(x, y) = -\frac{k(b-a)}{\mathrm{i}\pi(\alpha_0 + k)M_+(k)} \int\limits_C \frac{\mathrm{e}^{-\mathrm{i}\alpha x - \gamma y}}{(\gamma - a)(\alpha + \alpha_0)M_+(\alpha)}\,\mathrm{d}\alpha, \tag{3.14}$$

where C is a line parallel to the real axis in the complex α-plane and lies in the strip $c < \tau < d$.

Case 2: $b < 0$. In this case the coefficient of $F_+(\alpha)$ in (3.9) is written as $L(\alpha)/(\alpha^2 - k^2)$ where $L(\alpha) = (\gamma + a)(\gamma + |b|)$ which is analytic in the strip $c < \tau < d$. $L(\alpha)$ can be factorised as

$$L(\alpha) = L_+(\alpha)L_-(\alpha), \tag{3.15}$$

where $L_+(\alpha) = L_-(-\alpha)$, $L_+(\alpha)$ is analytic in $\tau > c$, $L_-(\alpha)$ is analytic in $\tau < d$ and that $L_+(\alpha) = \mathrm{O}(|\alpha|)$ as $|\alpha| \to \infty$ in $\tau > c$. Following the same procedure as used to obtain $M_+(\alpha)$ above, $L_+(\alpha)$ is found to be

$$L_+(\alpha) = \{(a - \mathrm{i}\epsilon)(-b - \mathrm{i}\epsilon)\}^{1/2} \exp\left[\int\limits_0^\alpha \left\{\frac{\xi + k}{2} + \frac{\xi}{a}\wedge_+(\xi) + \frac{k}{a}\wedge_-(k)\right\}\frac{\mathrm{d}\xi}{\xi^2 - k^2} + \int\limits_0^\alpha \left\{\frac{\xi + \alpha_0}{2} - \frac{\xi}{b}\wedge_+(\xi) - \frac{\alpha_0}{b}\wedge_-(\alpha_0)\right\}\frac{\mathrm{d}\xi}{\xi^2 - \alpha_0^2}\right], \tag{3.16}$$

where $\wedge_+(\xi)$ is the same expression as given in relation (3.12).

Using a similar procedure as in Case 1, the function $\phi_1(x, y)$ $(y > 0)$ in this case is obtained as

$$\phi_1(x, y) = -\frac{k(a-b)}{\mathrm{i}\pi L_+(k)} \int\limits_C \frac{\mathrm{e}^{-\mathrm{i}\alpha x - \gamma y}}{(\gamma - a)L_+(\alpha)}\,\mathrm{d}\alpha \tag{3.17}$$

where C is the same contour as in (3.14).

Case 3: $|b| = \infty$. In this case condition (2.10) assumes the form

$$\phi = -\mathrm{e}^{\mathrm{i}kx} \quad \text{on } y = 0 \ \text{ for } x > 0$$

so that the functional relation (3.9) is modified as

$$\frac{\gamma(\alpha) + a}{\alpha^2 - k^2}F_+(\alpha) + G_-(\alpha) = -\frac{1}{\mathrm{i}(\alpha + k)} \quad \text{for } c < \tau < d. \tag{3.18}$$

Here we factorise $P(\alpha) = \gamma(\alpha) + a$ $(a > 0)$ in the form

$$P(\alpha) = P_+(\alpha)P_-(\alpha), \tag{3.19}$$

where $P_+(\alpha) = P_-(-\alpha)$, $P_+(\alpha)$ is analytic in $\tau > c$, $P_-(\alpha)$ is analytic in $\tau < d$, $P_+(\alpha) = \mathrm{O}(|\alpha|^{1/2})$ as $|\alpha| \to \infty$ in $\tau > c$. $P_+(\alpha)$ is obtained as

$$P_+(\alpha) = (a - \mathrm{i}\epsilon)^{1/2} \exp\left[\int\limits_0^\alpha \left\{\frac{\xi + k}{2} + \frac{\xi}{a}\wedge_+(\xi) + \frac{k}{a}\wedge_-(k)\right\}\frac{\mathrm{d}\xi}{\xi^2 - k^2}\right]. \tag{3.20}$$

Following a similar procedure as in case 1, the function $\phi_1(x, y)$ $(y > 0)$ in this case is obtained as

$$\phi_1(x, y) = -\frac{k}{\mathrm{i}\pi P_+(k)} \int\limits_C \frac{\mathrm{e}^{-\mathrm{i}\alpha x - ry}}{(\gamma - a)P_+(\alpha)}\,\mathrm{d}\alpha. \tag{3.21}$$

The representations (3.14), (3.17) and (3.21) for $\phi_1(x, y)$ are now analysed after passing on to the limit $\epsilon \to 0$ so as to obtain the solution of BVP I for $b > 0$, $b < 0$ and $|b| = \infty$, respectively.

As $\epsilon \to 0$, the functions $M_\pm(\alpha)$ in (3.11), $L_\pm(\alpha)$ in (3.15) and $P_\pm(\alpha)$ in (3.19) reduce, respectively, to

$$M_+^0(\alpha) = M_-^0(-\alpha) = \frac{(\alpha + a)^{1/2} \exp\left[\frac{1}{i\pi}\int_0^{\alpha/a} \frac{\ln\xi}{\xi^2 - 1}\,d\xi\right]}{(\alpha + b)^{1/2} \exp\left[\frac{1}{i\pi}\int_0^{\alpha/b} \frac{\ln\xi}{\xi^2 - 1}\,d\xi\right]},$$

$$L_+^0(\alpha) = L_-^0(-\alpha) = (\alpha + a)^{1/2}(\alpha - b)^{1/2} \exp\left[\frac{1}{i\pi}\left\{\int_0^{\alpha/a} + \int_0^{\alpha/|b|}\right\}\frac{\ln\xi}{\xi^2 - 1}\,d\xi\right],$$

$$P_+^0(\alpha) = P_-^0(-\alpha) = (\alpha + a)^{1/2} \exp\left[\frac{1}{i\pi}\int_0^{\alpha/a} \frac{\ln\xi}{\xi^2 - 1}\,d\xi\right]. \tag{3.22}$$

Again, as $\epsilon \to 0$, we note that

$$k \to a,\ \alpha_0 \to b \ \text{ and } r \to \alpha \operatorname{sgn} \operatorname{Re}\alpha. \tag{3.23}$$

Using (3.23) and (3.22) in (3.14), (3.17) and (3.20), we obtain the solution of the BVP I as

$$\phi_1(x, y) = \begin{cases} -\dfrac{a(b-a)}{i\pi(a+b)M_+^0(a)} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-\alpha(\operatorname{sgn}\operatorname{Re}\alpha)y - i\alpha x}}{(\alpha \operatorname{sgn}\operatorname{Re}\alpha - a)(\alpha + b)M_+^0(\alpha)}\,d\alpha & \text{for } b > 0, \\ \dfrac{a(a-b)}{i\pi L_+^0(a)} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-\alpha(\operatorname{sgn}\operatorname{Re}\alpha)y - i\alpha x}}{(\alpha \operatorname{sgn}\operatorname{Re}\alpha - a)L_+^0(\alpha)}\,d\alpha & \text{for } b < 0, \\ -\dfrac{a}{i\pi P_+^0(a)} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-\alpha(\operatorname{sgn}\operatorname{Re}\alpha)y - i\alpha x}}{(\alpha \operatorname{sgn}\operatorname{Re}\alpha - a)P_+^0(\alpha)}\,d\alpha & \text{for } |b| = \infty \end{cases} \tag{3.24}$$

where the path of the integral is indented above (below), the poles on the negative (positive) real axis.

To evaluate the integrals in (3.24), we introduce the polar coordinates (r, β) where $x = r\cos\beta$, $y = r\sin\beta$ $(0 \le \beta \le \pi)$. The poles of the integrands are on the real axis. For $x < 0$ (> 0) we deform the contour over the bisectors of the first and second (third and fourth) quadrants of the complex α-plane. The integrands decrease exponentially on the bisectors and we retain up to the order of r^{-1} for the integrals on the bisectors. Thus we finally obtain the following asymptotic results, valid for large r.

For $x < 0$,

$$\phi_1(x, y) \approx \begin{cases} -\dfrac{2(b-a)}{i\pi b(a+b)M_+^0(a)M_+^0(0)}\dfrac{\sin\beta}{r} - \dfrac{2a(b-a)}{\{(a+b)M_+^0(a)\}^2}e^{-ay - iax} & \text{for } b > 0, \\ \dfrac{2(a-b)}{i\pi L_+^0(a)L_+^0(0)}\dfrac{\sin\beta}{r} + \dfrac{2a(b-a)}{\{L_+^0(a)\}^2}e^{-ay - iax} & \text{for } b < 0, \\ \dfrac{2}{i\pi P_+^0(a)P_+^0(0)}\dfrac{\sin\beta}{r} - \dfrac{2a}{\{P_+^0(a)\}^2}e^{-ay - iax} & \text{for } |b| = \infty, \end{cases} \tag{3.25}$$

while for $x > 0$

$$\phi_1(x,y) \approx \begin{cases} -\mathrm{e}^{-ay+\mathrm{i}ax} + \dfrac{2(b-a)}{\mathrm{i}\pi b(a+b)M_+^0(a)M_+^0(0)}\dfrac{\sin\beta}{r} \\ \quad + \dfrac{4abM_+^0(b)}{(a+b)^2M_+^0(a)}\mathrm{e}^{-by+\mathrm{i}bx} & \text{for } b > 0, \\ -\mathrm{e}^{-ay+\mathrm{i}ax} - \dfrac{2(b-a)}{\mathrm{i}\pi L_+^0(a)L_+^0(0)}\dfrac{\sin\beta}{r} & \text{for } b < 0, \\ -\mathrm{e}^{-ay+\mathrm{i}ax} + \dfrac{2}{\mathrm{i}\pi P_+^0(a)P_+^0(0)}\dfrac{\sin\beta}{r} & \text{for } |b| = \infty. \end{cases} \tag{3.26}$$

Since the total field is given by (cf. Eq. (2.7))

$$\phi_1^t(x,y) = \mathrm{e}^{-ay+\mathrm{i}ax} + \phi_1(x,y),$$

we observe that the behaviour of the total field as $|x| \to \infty$, given by (2.6), is satisfied by (3.25) and (3.26). The complex constants A_1 and B_1 which are the reflection and transmission coefficients (complex), respectively, are now determined explicitly. We note that for $b > 0$, there occurs reflection and transmission of the incoming wave train by the discontinuity at (0, 0) into the regions $x < 0$ and $x > 0$, respectively, while for $b < 0$ and $|b| = \infty$, there is no transmitted wave in the region $x > 0$. This is expected, since in the latter cases the inertial surface is too heavy to allow for the propagation of the incoming wave train after it encounters the discontinuity at the origin. We note that the first terms in the right-hand side of (3.25) and the second terms in the right-hand side of (3.26) arise due to interaction of the incident wave train with the discontinuity at the origin and they die out at large distance from the origin. These do not represent any wave.

Now comparing (2.6) with (3.25) and (3.26) we find that the complex reflection and transmission coefficients are given by

$$A_1, B_1 = \begin{cases} -\dfrac{2a(b-a)}{\{(a+b)M_+^0(a)\}^2}, \dfrac{4abM_+^0(b)}{(a+b)^2M_+^0(a)} & \text{for } b > 0, \\ \dfrac{2a(b-a)}{\{L_+^0(a)\}^2}, 0 & \text{for } b < 0, \\ -\dfrac{2a}{\{P_+^0(a)\}^2}, 0 & \text{for } |b| = \infty. \end{cases} \tag{3.27}$$

Hence the reflection and transmission coefficients (real) are obtained as

$$|A_1|, |B_1| = \begin{cases} \left|\dfrac{b-a}{a+b}\right|, \dfrac{2|ab|^{1/2}}{|a+b|} & \text{for } b > 0, \\ 1, \quad 0 & \text{for } b < 0, \\ 1, \quad 0 & \text{for } |b| = \infty. \end{cases} \tag{3.28}$$

In deriving the results in (3.28), we have used from (3.22)

$$|M_+^0(a)| = \left|\frac{2a}{a+b}\right|^{1/2}, \qquad |M_+^0(b)| = \left|\frac{a+b}{2b}\right|^{1/2},$$
$$|L_+^0(a)| = |2a(a-b)|^{1/2}, \qquad |P_+^0(a)| = (2a)^{1/2}. \tag{3.29}$$

The results in (3.28) for $b > 0$ have been recently obtained by Chakrabarti [6] by a different technique. It is also verified from (3.28) that the principle of conservation of energy, viz.

$$|A_1|^2 + |B_1|^2 = 1$$

holds good.

4. The three-part Wiener–Hopf technique involving BVP II

Let $\phi_2(x, y) \equiv \phi_2(x, y; \epsilon)$ denote the function satisfying the generalised BVP II. The Fourier transform of $\phi_2(x, y)$ is written in the form

$$\Phi_2(\alpha, y) = \int_{-\infty}^{\infty} \phi_2(x, y)e^{i\alpha x}\,dx$$

$$= \Phi_{2-}(\alpha, y) + e^{i\alpha l}\Phi_{2+}(\alpha, y) + \int_0^l \phi_2(x, y)e^{i\alpha x}\,dx$$

where

$$\Phi_{2-}(\alpha, y) = \int_{-\infty}^{0} \phi_2(x, y)e^{i\alpha x}\,dx, \qquad \Phi_{2+}(\alpha, y) = \int_l^{\infty} \phi_2(x, y)e^{i\alpha(x-l)}\,dx. \tag{4.1}$$

By using condition (2.13) it is observed that $\Phi_{2+}(\alpha, y)$ and $\Phi_{2-}(\alpha, y)$ are analytic functions of α in the half planes $\tau > -\delta(\epsilon)$ and $\tau < \delta(\epsilon)$, respectively. Again, by using the edge conditions (2.12) (for BVP II) along with the Abelian theorem, it can be shown that

$$\begin{aligned} |\Phi_{2+}(\alpha, y)| &= O(|\alpha|^{-1}) \quad \text{as } |\alpha| \to \infty \ \text{ in } \tau > -\delta(\epsilon), \\ |\Phi_{2-}(\alpha, y)| &= O(|\alpha|^{-1}) \quad \text{as } |\alpha| \to \infty \ \text{ in } \tau < \delta(\epsilon). \end{aligned} \tag{4.2}$$

To use the Wiener–Hopf procedure, as in BVP I, conditions (2.15) and (2.16) are written in the form

$$\phi_{2y} + a\phi_2 = \begin{cases} 0 & \text{on } y = 0 \ \text{ for } x < 0 \text{ and } x > l, \\ f_1(x) & \text{on } y = 0 \ \text{ for } 0 < x < l \end{cases} \tag{4.3}$$

and

$$\phi_{2y} + b\phi_2 = \begin{cases} u(x) & \text{on } y = 0 \ \text{ for } x < 0, \\ -(b - a)e^{ikx} & \text{on } y = 0 \ \text{ for } 0 < x < l, \\ v(x) & \text{on } y = 0 \ \text{ for } x > l \end{cases} \tag{4.4}$$

where $f_1(x)$ (for $0 < x < l$), $u(x)$ (for $x < 0$) and $v(x)$ (for $x > l$) are unknown functions having the behaviour at the points $x = 0$ and $x = l$, which are similar to the ones given by conditions (3.5). Specifically,

$$f_1(x) = O(1) \quad \text{as } x \to +0, \ \text{ and } x \to l - 0, \tag{4.5}$$

$$u(x) = \begin{cases} O(1) & \text{as } x \to -0 \ \text{ for finite } |b|, \\ O(x^{-1/2}) & \text{as } x \to -0 \ \text{ for } |b| = \infty, \end{cases} \tag{4.6}$$

$$v(x) = \begin{cases} O(1) & \text{as } x \to l + 0 \ \text{ for finite } |b|, \\ O(|x - l|^{-1/2}) & \text{as } x \to l + 0 \ \text{ for } |b| = \infty, \end{cases} \tag{4.7}$$

obtained from the edge conditions (2.12) for BVP II.

Now an appropriate solution for $\Phi_2(\alpha, y)$ is taken to be

$$\Phi_2(\alpha, y) = D_2(\alpha)\mathrm{e}^{-\gamma y}, \quad y \geq 0, \tag{4.8}$$

where $D_2(\alpha)$ is an arbitrary function of α, and is determined from the relations obtained by Fourier-transforming conditions (4.3) and (4.4), as given by

$$\begin{aligned} &\Phi_2'(\alpha, 0) + a\Phi_2(\alpha, 0) = F_1(\alpha), \\ &\Phi_2'(\alpha, 0) + b\Phi_2(\alpha, 0) = U_-(\alpha) + \mathrm{e}^{\mathrm{i}\alpha l}V_+(\alpha) - \frac{b-a}{\mathrm{i}(\alpha+k)}\{\mathrm{e}^{\mathrm{i}(\alpha+k)l} - 1\}. \end{aligned} \tag{4.9}$$

In (4.9) the three unknown functions $U_-(\alpha)$, $V_+(\alpha)$ and $F_1(\alpha)$ are defined by

$$\begin{aligned} &U_-(\alpha) = \int_{-\infty}^{0} u(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x, \qquad V_+(\alpha) = \int_{l}^{\infty} v(x)\mathrm{e}^{\mathrm{i}\alpha(x-l)}\,\mathrm{d}x, \\ &F_1(\alpha) = \int_{0}^{l} f_1(x)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x. \end{aligned} \tag{4.10}$$

It can be shown that $U_-(\alpha)$ is analytic in the half-plane $\tau < \delta(\epsilon)$, $V_+(\alpha)$ is analytic in the half-plane $\tau > -\delta(\epsilon)$ and $F_1(\alpha)$ is an integral function of α. Use of the edge conditions (4.5)–(4.7) ensures that

$$\begin{aligned} &U_-(\alpha) = \begin{cases} \mathrm{O}(|\alpha|^{-1}) & \text{as } |\alpha| \to \infty \text{ in } \tau < \delta(\epsilon) \quad \text{for finite } |b|, \\ \mathrm{O}(|\alpha|^{-1/2}) & \text{as } |\alpha| \to \infty \text{ in } \tau < \delta(\epsilon) \quad \text{for } |b| = \infty, \end{cases} \\ &V_+(\alpha) = \begin{cases} \mathrm{O}(|\alpha|^{-1}) & \text{as } |\alpha| \to \infty \text{ in } \tau > -\delta(\epsilon) \quad \text{for finite } |b|, \\ \mathrm{O}(|\alpha|^{-1/2}) & \text{as } |\alpha| \to \infty \text{ in } \tau > -\delta(\epsilon) \quad \text{for } |b| = \infty, \end{cases} \\ &|\mathrm{e}^{-\mathrm{i}\alpha l}F_1(\alpha)| = \mathrm{O}(|\alpha|^{-1}) \quad \text{as } |\alpha| \to \infty \text{ in } \tau < \delta(\epsilon), \\ &|F_1(\alpha)| = \mathrm{O}(|\alpha|^{-1/2}) \quad \text{as } |\alpha| \to \infty \text{ in } \tau > -\delta(\epsilon). \end{aligned} \tag{4.11}$$

Using (4.8) in Eqs. (4.9) and eliminating $D_2(\alpha)$ we obtain the following three-part Wiener–Hopf functional relation, for the determination of the three unknown functions $F_1(\alpha)$, $U_-(\alpha)$ and $V_+(\alpha)$, as given by

$$\frac{\gamma(\alpha) - b}{\gamma(\alpha) - a}F_1(\alpha) = U_-(\alpha) + \mathrm{e}^{\mathrm{i}\alpha l}V_+(\alpha) - \frac{b-a}{\mathrm{i}(\alpha+k)}\{\mathrm{e}^{\mathrm{i}(\alpha+k)l} - 1\} \tag{4.12}$$

valid in the strip $c < \tau < d$ where c (< 0) and d (> 0) satisfy inequality (3.10).

As in the case of BVP I, here also three cases arise according as $b > 0$, $b < 0$ and $|b| = \infty$, and we treat the three-part Wiener–Hopf relation (4.12) in three different manners as described below.

Case 1: $b > 0$. Using the same Wiener–Hopf decomposition (3.11) for $M(\alpha) = (\gamma + a)/(\gamma + b) = M_+(\alpha)M_-(\alpha)$, multiplying both sides of (4.12) by $\mathrm{e}^{-\mathrm{i}\alpha l}/M_+(\alpha)$ and rearranging, we obtain

$$\begin{aligned} &\frac{\alpha+k}{\alpha+\alpha_0}\frac{V_+(\alpha)}{M_+(\alpha)} - \frac{b-a}{\mathrm{i}(\alpha+\alpha_0)}\frac{\mathrm{e}^{\mathrm{i}kl}}{M_+(\alpha)} + \zeta_+(\alpha) + \eta_+(\alpha) \\ &= \frac{\alpha-\alpha_0}{\alpha-k}M_-(\alpha)\mathrm{e}^{-\mathrm{i}\alpha l}F_1(\alpha) - \zeta_-(\alpha) - \eta_-(\alpha), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\zeta_+(\alpha)+\zeta_-(\alpha)&=\frac{\mathrm{e}^{-\mathrm{i}\alpha l}(\alpha+k)U_-(\alpha)}{(\alpha+\alpha_0)M_+(\alpha)},\\ \eta_+(\alpha)+\eta_-(\alpha)&=\frac{(b-a)\mathrm{e}^{-\mathrm{i}\alpha l}}{\mathrm{i}(\alpha+\alpha_0)M_+(\alpha)}.\end{aligned}\tag{4.14}$$

In (4.14), $\zeta_+(\alpha)$, $\eta_+(\alpha)$ are analytic in $\tau > c$ and $\zeta_-(\alpha)$, $\eta_-(\alpha)$ are analytic in $\tau < d$, and their explicit forms can be obtained by employing the additive decomposition theorem (see [5, p. 13]). Similarly, multiplying both sides of (4.12) by $1/(M_-(\alpha))$ and rearranging we obtain

$$\begin{aligned}&\frac{\alpha-k}{\alpha-\alpha_0}\frac{U_-(\alpha)}{M_-(\alpha)}+R_-(\alpha)-S_-(\alpha)+\frac{b-a}{\mathrm{i}(\alpha-\alpha_0)}\\ &\quad\times\left[\frac{\alpha-k}{\alpha+k}\left\{\frac{1}{M_-(\alpha)}-\frac{1}{M_-(-k)}\right\}+\frac{\alpha_0-k}{\alpha_0+k}\frac{1}{M_-(-k)}\right]\\ &=\frac{\alpha+\alpha_0}{\alpha+k}M_+(\alpha)F_1(\alpha)-R_+(\alpha)+S_+(\alpha)-\frac{2(b-a)k}{\mathrm{i}(\alpha_0+k)M_-(-k)}\frac{1}{\alpha+k},\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}R_+(\alpha)+R_-(\alpha)&=\frac{\mathrm{e}^{\mathrm{i}\alpha l}(\alpha-k)V_+(\alpha)}{(\alpha-\alpha_0)M_-(\alpha)},\\ S_+(\alpha)+S_-(\alpha)&=\frac{(b-a)(\alpha-k)\mathrm{e}^{\mathrm{i}(\alpha+k)l}}{\mathrm{i}(\alpha+k)(\alpha-\alpha_0)M_-(\alpha)},\end{aligned}\tag{4.16}$$

$R_+(\alpha)$, $S_+(\alpha)$ being analytic in $\tau > c$ and $R_-(\alpha)$, $S_-(\alpha)$ in $\tau < d$ and their explicit forms being obtained by employing the additive decomposition theorem mentioned above.

The left-hand side of (4.13) and the right-hand side of (4.15) are analytic in $\tau > c$ while the other sides are analytic in $\tau < d$. Using (4.11), it is seen that each side of (4.13) and (4.15) tends to zero as $|\alpha| \to \infty$ in the appropriate half planes having a common region $c < \tau < d$, so that by Liouville's theorem, each side is identically zero. We are interested in the left-hand sides of (4.13) and (4.15).

For brevity, we introduce the notation

$$\Psi_-^*(\alpha)=U_-(\alpha)+\frac{b-a}{\mathrm{i}(\alpha+k)},\qquad \Psi_+(\alpha)=V_+(\alpha)-\frac{b-a}{\mathrm{i}(\alpha+k)}\mathrm{e}^{\mathrm{i}lk}\tag{4.17}$$

where the superscript star is used to indicate that $\Psi_-^*(\alpha)$ has a pole at $\alpha = -k$ but apart from this, it is analytic in $\tau < d$, and $\Psi_+(\alpha)$ is analytic in $\tau > c$. On equating the left-hand sides of (4.13) and (4.15), and introducing the explicit expressions for $\zeta_+(\alpha)$, $\eta_+(\alpha)$, $R_-(\alpha)$, $S_-(\alpha)$ and using the notations (4.17), we obtain

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{\Psi_+(\alpha)}{M_+(\alpha)}+\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}c_1-\infty}^{\mathrm{i}c_1+\infty}\frac{\mathrm{e}^{-\mathrm{i}l\xi}(\xi+k)\Psi_-^*(\xi)}{M_+(\xi)(\xi+\alpha_0)(\xi-\alpha)}\,\mathrm{d}\xi=0,\quad \tau>c\tag{4.18}$$

and

$$\begin{aligned}&\frac{\alpha-k}{\alpha-\alpha_0}\frac{\Psi_-^*(\alpha)}{M_-(\alpha)}-\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}d_1-\infty}^{\mathrm{i}d_1+\infty}\frac{\mathrm{e}^{\mathrm{i}l\xi}(\xi-k)\Psi_+(\xi)}{M_-(\xi)(\xi-\alpha_0)(\xi-\alpha)}\,\mathrm{d}\xi\\ &\quad-\frac{2(b-a)k}{\mathrm{i}M_-(-k)(\alpha_0+k)}\frac{1}{\alpha+k}=0,\quad \tau<d,\end{aligned}\tag{4.19}$$

where $c < c_1 < 0 < d_1 < d$. We choose $c_1 = -h$, $d_1 = h$ where h is positive, then replace ξ by $-\xi$ in (4.18) and α by $-\alpha$ in (4.19). Noting that $M_+(-\alpha) = M_-(\alpha)$, this produces

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{\Psi_+(\alpha)}{M_+(\alpha)} - \frac{1}{2\pi \mathrm{i}}\int_{ih-\infty}^{ih+\infty} \frac{\mathrm{e}^{il\xi}(\xi-k)\Psi_-^*(-\xi)}{M_-(\xi)(\xi-\alpha_0)(\xi+\alpha)}\,\mathrm{d}\xi = 0 \tag{4.20}$$

and

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{\Psi_-^*(-\alpha)}{M_+(\alpha)} - \frac{1}{2\pi \mathrm{i}}\int_{ih-\infty}^{ih+\infty} \frac{\mathrm{e}^{il\xi}(\xi-k)\Psi_+(\xi)}{M_-(\xi)(\xi-\alpha_0)(\xi+\alpha)}\,\mathrm{d}\xi + \frac{2(b-a)k}{\mathrm{i}M_-(-k)(\alpha_0+k)}\frac{1}{\alpha-k} = 0 \tag{4.21}$$

where now $\tau > -h$ in both the equations (4.20) and (4.21). We define

$$S_+^*(\alpha) = \Psi_+(\alpha) + \Psi_-^*(-\alpha), \qquad D_+^*(\alpha) = \Psi_+(\alpha) - \Psi_-^*(-\alpha), \tag{4.22}$$

where in this case the star denotes that the expressions are analytic in $\tau > c$ except for simple pole at $\alpha = k$. Then addition and subtraction of Eqs. (4.20) and (4.21) produce

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{S_+^*(\alpha)}{M_+(\alpha)} - \frac{1}{2\pi \mathrm{i}}\int_{ih-\infty}^{ih+\infty} \frac{\mathrm{e}^{il\xi}(\xi-k)S_+^*(\xi)}{M_-(\xi)(\xi-\alpha_0)(\xi+\alpha)}\,\mathrm{d}\xi + \frac{2(b-a)k}{\mathrm{i}M_+(k)(\alpha_0+k)}\frac{1}{\alpha-k} = 0, \quad \tau > -h \tag{4.23}$$

and

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{D_+^*(\alpha)}{M_+(\alpha)} + \frac{1}{2\pi \mathrm{i}}\int_{ih-\infty}^{ih+\infty} \frac{\mathrm{e}^{il\xi}(\xi-k)D_+^*(\xi)}{M_-(\xi)(\xi-\alpha_0)(\xi+\alpha)}\,\mathrm{d}\xi - \frac{2(b-a)k}{\mathrm{i}M_+(k)(\alpha_0+k)}\frac{1}{\alpha-k} = 0, \quad \tau > -h. \tag{4.24}$$

Eqs. (4.23) and (4.24) are of the same type and can be treated for approximate solution for large l. We write them in a compact form, as given by

$$\frac{\alpha+k}{\alpha+\alpha_0}\frac{F_{1+}^*(\alpha;\lambda)}{M_+(\alpha)} + \frac{\lambda}{2\pi \mathrm{i}}\int_{ih-\infty}^{ih+\infty} \frac{\mathrm{e}^{il\xi}(\xi-k)F_{1+}^*(\xi;\lambda)}{M_-(\xi)(\xi-\alpha_0)(\xi+\alpha)}\,\mathrm{d}\xi = \frac{2\lambda(b-a)k}{\mathrm{i}M_+(k)(\alpha_0+k)}\frac{1}{\alpha-k}, \quad \tau > -h, \tag{4.25}$$

where $F_{1+}^*(\alpha;\lambda)$ is $S_+^*(\alpha)$ or $D_+^*(\alpha)$ for $\lambda = -1$ or $+\,1$, so that from (4.17) and (4.22) to (4.24) we find that $F_{1+}^*(\alpha;\lambda)$ has the form

$$F_{1+}^*(\alpha;\lambda) = F_{1+}(\alpha;\lambda) - \frac{(b-a)\mathrm{e}^{ikl}}{\mathrm{i}(\alpha+k)} + \frac{\lambda(b-a)}{\mathrm{i}(\alpha-k)}, \tag{4.26}$$

where $F_{1+}(\alpha;\lambda)$ is analytic in $\tau > c$, it being understood that $F_{1+}(\alpha;1) = V_+(\alpha) - U_-(-\alpha)$ and $F_{1+}(\alpha;-1) = V_+(\alpha) + U_-(-\alpha)$.

Now writing

$$\frac{1}{M_-(\xi)} = M_+(\xi)\frac{\gamma(\xi)+b}{\gamma(\xi)+a} = M_+(\xi)\left[1 - \frac{b-a}{\xi^2-k^2}\{a-(\xi^2-\epsilon^2)^{1/2}\}\right] \tag{4.27}$$

in the integrand of the integral in the left-hand side of relation (4.25) we note that the integrand consists of two types of terms. The first type involves simple poles while the second type involves branch points at $\xi = \pm\epsilon$ in the complex ξ-plane. The integrals involving the first type of terms are evaluated by using the residue theorem after completing the contour by a semi-circle of large radius in the upper half. To evaluate the integrals involving the branch points, only one branch point, viz. $\xi = \epsilon$ needs to be considered and as such a branch cut is taken parallel to the positive imaginary axis from $\xi = \epsilon$ to infinity. Then the contour is deformed into the two sides of the branch cut and contributions from the poles, if any, are taken into account. The contributions from the two sides of the branch cut involve integrals of the form

$$\int_0^\infty \psi(u)u^{1/2}\mathrm{e}^{-ul}\,\mathrm{d}u, \tag{4.28}$$

where $\psi(u)$ is an analytic function. These are evaluated asymptotically for large l. If $I(l)$ denotes the integral (4.28), then

$$I(l) = \sum_{j=0}^{\infty}\beta_j(l)\frac{\psi^j(0)}{j!},$$

where

$$\beta_j(l) = \int_0^\infty z^{j+1/2}\mathrm{e}^{-zl}\,\mathrm{d}z \quad = \left(\frac{1}{l}\right)^{j+3/2}\Gamma\left(j+\frac{3}{2}\right).$$

Thus

$$\begin{aligned} I(l) &= \beta_0(l)\left[\psi(0) + \frac{\beta_1}{\beta_0}\psi'(0) + \frac{\beta_2}{2!\beta_0}\psi''(0) + \cdots\right] \\ &\approx \beta_0\psi\left(\frac{\beta_1}{\beta_0}\right) + \mathrm{O}(l^{-7/2}) \\ &= \frac{\pi^{1/2}}{2}\left(\frac{1}{l}\right)^{3/2}\psi\left(\frac{3}{2l}\right) + \mathrm{O}(l^{-7/2}). \end{aligned} \tag{4.29}$$

Incorporating the aforesaid method we find that for large l

$$\int_{\mathrm{i}h-\infty}^{\mathrm{i}h+\infty} \frac{\mathrm{e}^{\mathrm{i}l\xi}(\xi-k)F_{1+}(\xi;\lambda)}{M_-(\xi)(\xi+\alpha)(\xi-\alpha_0)}\,\mathrm{d}\xi \approx 2\pi\mathrm{i}[T(\alpha)F_{1+}(\epsilon';\lambda) + T_1(\alpha)F_{1+}(\alpha_0;\lambda)], \tag{4.30}$$

where

$$T(\alpha) = \frac{(b-a)E_0}{2\pi\mathrm{i}(\epsilon'-\alpha_0)(\epsilon'+k)}\frac{1}{\epsilon'+\alpha}$$

with

$$E_0 = -\frac{\pi^{1/2}}{l^{3/2}} e^{i(\epsilon l + (3/4)\pi)} M_+(\epsilon')(\epsilon' + \epsilon)^{1/2}, \qquad \epsilon' = \epsilon + \frac{3i}{2l}$$

and

$$T_1(\alpha) = \frac{2b(b-a)M_+(\alpha_0)e^{i\alpha_0 l}}{\alpha_0 + k}\frac{1}{\alpha_0 + \alpha}, \tag{4.31}$$

$$\int_{ih-\infty}^{ih+\infty} \frac{e^{il\xi}}{M_-(\xi)(\xi+\alpha)(\xi-\alpha_0)}\, d\xi \approx 2\pi i\left[R_1(\alpha) + \frac{T_1(\alpha)}{\alpha_0 - k}\right], \tag{4.32}$$

$$\int_{ih-\infty}^{ih+\infty} \frac{e^{il\xi}(\xi - k)}{M_-(\xi)(\xi+\alpha)(\xi-\alpha_0)}\, d\xi \approx 2\pi i\left[R_2(\alpha) + \frac{T_1(\alpha)}{\alpha_0 + k}\right], \tag{4.33}$$

where

$$R_1(\alpha) = \frac{(b-a)E_0}{2\pi i(\epsilon' - \alpha_0)(\epsilon'^2 - k^2)}\frac{1}{\epsilon' + \alpha}, \qquad R_2(\alpha) = \frac{(b-a)E_0}{2\pi i(\epsilon' - \alpha_0)(\epsilon' + k)^2}\frac{1}{\epsilon' + \alpha}. \tag{4.34}$$

Using results (4.30), (4.32) and (4.33) for large l in (4.25) we obtain an approximate relation between the function $F_{1+}(\alpha;\lambda)$ and the unknown quantities $F_{1+}(\alpha_0;\lambda)$, $F_{1+}(\epsilon';\lambda)$. Setting $\alpha = \alpha_0$ and $\alpha = \epsilon'$ in this we get two equations involving these two unknowns, which, when solved, produce

$$F_{1+}(\alpha_0;\lambda) = -\frac{b-a}{i(A^\lambda D^\lambda - B^\lambda C^\lambda)}\{e^{ikl}(S^\lambda B^\lambda - Q^\lambda D^\lambda) + T^\lambda B^\lambda - R^\lambda D^\lambda\}, \tag{4.35}$$

$$F_{1+}(\epsilon';\lambda) = \frac{b-a}{i(A^\lambda D^\lambda - B^\lambda C^\lambda)}\{e^{ikl}(S^\lambda A^\lambda - Q^\lambda C^\lambda) + T^\lambda A^\lambda - R^\lambda C^\lambda\}, \tag{4.36}$$

where

$$\begin{aligned}
A^\lambda &= \frac{\alpha_0 + k}{2\alpha_0 M_+(\alpha_0)} + \lambda T_1(\alpha_0),\\
B^\lambda &= \lambda T(\alpha_0),\\
Q^\lambda &= \lambda\left\{R_2(\alpha_0) + \frac{T_1(\alpha_0)}{\alpha_0 + k}\right\} + \frac{1}{2\alpha_0 M_+(\alpha_0)},\\
R^\lambda &= -R_1(\alpha_0) - \frac{T_1(\alpha_0)}{\alpha_0 - k} + \lambda\left\{\frac{2k}{M_+(k)(\alpha_0^2 - k^2)} - \frac{\alpha_0 + k}{2M_+(\alpha_0)\alpha_0(\alpha_0 - k)}\right\},\\
C^\lambda &= \lambda T_1(\epsilon'),\\
D^\lambda &= \frac{\epsilon' + k}{M_+(\epsilon')(\epsilon' + \alpha_0)} + \lambda T(\epsilon'),\\
S^\lambda &= \frac{1}{M_+(\epsilon')(\epsilon' + \alpha_0)} + \lambda\left\{R_2(\epsilon') + \frac{T_1(\epsilon')}{\alpha_0 + k}\right\},\\
T^\lambda &= -R_1(\epsilon') - \frac{T_1(\epsilon')}{\alpha_0 - k} + \lambda\left\{\frac{2k}{M_+(k)(\alpha_0 + k)(\epsilon' - k)} - \frac{\epsilon' + k}{M_+(\epsilon')(\epsilon' + \alpha_0)(\epsilon' - k)}\right\}.
\end{aligned} \tag{4.37}$$

Thus $F_{1+}(\alpha;\lambda)$ is obtained for large l and is given by

$$F_{1+}(\alpha;\lambda)=\frac{(\alpha+\alpha_0)M_+(\alpha)}{\alpha+k}\left[\frac{b-a}{i}\left\{e^{ikl}\left(\frac{1}{(\alpha+\alpha_0)M_+(\alpha)}+\frac{\lambda T_1(\alpha)}{\alpha_0+k}+\lambda R_2(\alpha)\right)\right.\right.$$
$$\left.-\lambda^2\left(R_1(\alpha)+\frac{T_1(\alpha)}{\alpha_0-k}\right)-\frac{\lambda(\alpha+k)}{M_+(\alpha)(\alpha-k)(\alpha+\alpha_0)}+\frac{2\lambda k}{M_+(k)(\alpha_0+k)(\alpha-k)}\right\}$$
$$\left.-\lambda\{T(\alpha)F_{1+}(\epsilon';\lambda)+T_1(\alpha)F_{1+}(\alpha_0;\lambda)\}\right] \tag{4.38}$$

where $F_{1+}(\alpha_0;\lambda)$ and $F_{1+}(\epsilon';\lambda)$ are given in (4.35) and (4.36), respectively. Putting $\lambda=-1$ and 1 in (4.38) we obtain two equations for $V_+(\alpha)+U_-(-\alpha)$ and $V_+(\alpha)-U_-(\alpha)$. By addition and subtraction we find $V_+(\alpha)$ and $U_-(-\alpha)$ for large l. Replacing α by $-\alpha$ we obtain $U_-(\alpha)$. Thus $V_+(\alpha)$ and $U_-(\alpha)$ are obtained for large l. Now the use of (4.8) in the second equation of (4.9) produces $D_2(\alpha)$. Thus we obtain $D_2(\alpha)$. Using this in (4.8) we obtain $\phi_2(x,y)$ for large l after taking Fourier inversion, as given by

$$\phi_2(x,y)=\phi_1(x,y)-\frac{b-a}{2\pi i}\int_C\left[\frac{(\alpha+\alpha_0)M_+(\alpha)e^{i\alpha l}}{\alpha+k}\right.$$
$$\times\left\{\frac{i(C_1T(\alpha)+C_2T_1(\alpha))}{b-a}-R_1(\alpha)-\frac{T_1(\alpha)}{\alpha_0-k}\right\}$$
$$+\frac{(\alpha-\alpha_0)M_-(\alpha)}{\alpha-k}\left\{\frac{i(C_3T(-\alpha)+C_4T_1(-\alpha))}{b-a}\right.$$
$$\left.\left.-e^{ikl}\left(R_2(-\alpha)+\frac{T_1(-\alpha)}{\alpha_0+k}\right)\right\}\right]\frac{e^{-i\alpha x-ry}}{r-b}\,d\alpha \tag{4.39}$$

where $\phi_1(x,y)$ is the same expression as given in (3.14) and

$$\begin{aligned}C_1&=\tfrac{1}{2}\{F_{1+}(\epsilon;-1)-F_{1+}(\epsilon;1)\},\\ C_2&=\tfrac{1}{2}\{F_{1+}(\alpha_0;-1)-F_{1+}(\alpha_0;1)\},\\ C_3&=\tfrac{1}{2}\{F_{1+}(\epsilon;-1)+F_{1+}(\epsilon;1)\},\\ C_4&=\tfrac{1}{2}\{F_{1+}(\alpha_0;-1)+F_{1+}(\alpha_0;1)\}.\end{aligned} \tag{4.40}$$

The second term in (4.39) may be regarded as due to the presence of the second discontinuity at $(l,0)$ for large l.

Case 2: $b<0$. Using the same Wiener–Hopf decomposition (3.15) for $L(\alpha)=(\gamma+a)(\gamma+|b|)$ and proceeding as in Case 1, we obtain in place of (4.25)

$$\frac{(\alpha+k)F^*_{1+}(\alpha;\lambda)}{L_+(\alpha)}-\frac{\lambda}{2\pi i}\int_{ih-\infty}^{ih+\infty}\frac{e^{il\xi}(\xi-k)F^*_{1+}(\xi,\lambda)}{L_-(\xi)(\xi+\alpha)}\,d\xi=-\frac{2\lambda(a-b)k}{iL_+(k)}\frac{1}{\alpha-k},\quad \tau>-h, \tag{4.41}$$

where $F^*_{1+}(\alpha;\lambda)$ has the same meaning as given in Eq. (4.26). We write

$$\frac{1}{L^-(\xi)}=\frac{L_+(\xi)}{(a+b)}\left[\frac{a}{\xi^2-k^2}+\frac{b(\xi^2-k^2)+(\alpha_0^2-k^2)(\xi^2-\epsilon^2)^{1/2}}{(\xi^2-k^2)(\xi^2-\alpha_0^2)}\right] \tag{4.42}$$

in the integrand of the integral in the left-hand side of (4.41). Following similar arguments as described earlier, we find that, for large l,

$$\int_{ih-\infty}^{ih+\infty} \frac{e^{il\xi}(\xi - k)F_{1+}(\xi;\lambda)}{(\xi + k)L_-(\xi)}\, d\xi \approx 2\pi i T^1(\alpha) F_{1+}(\epsilon';\lambda)$$

where

$$T^1(\alpha) = \frac{b-a}{2\pi i}\frac{E_0^1}{(\epsilon'^2 - \alpha_0^2)(\epsilon' + k)}\frac{1}{\epsilon' + \alpha},$$
$$E_0^1 = -\frac{\pi^{1/2}}{l^{3/2}} e^{i(\epsilon l + 3\pi/4)} L_+(\epsilon')(\epsilon' + \epsilon)^{1/2}, \tag{4.43}$$

$$\int_{ih-\infty}^{ih+\infty} \frac{e^{il\xi}}{(\xi + \alpha)L_-(\xi)}\, d\xi \approx 2\pi i R_1^1(\alpha)$$

where

$$R_1^1(\alpha) = \frac{b-a}{2\pi i}\frac{E_0}{(\epsilon'^2 - k^2)(\epsilon'^2 - \alpha_0^2)}\frac{1}{\epsilon' + \alpha}, \tag{4.44}$$

and

$$\int_{ih-\infty}^{ih+\infty} \frac{(\xi - k)e^{i\xi l}}{(\xi + \alpha)(\xi + k)L_-(\xi)}\, d\xi \approx 2\pi i R_2^1(\alpha),$$

where

$$R_2^1(\alpha) = \frac{b-a}{2\pi i}\frac{E_0}{(\epsilon' + k)^2(\epsilon'^2 - \alpha_0^2)}\frac{1}{\epsilon' + \alpha}, \tag{4.45}$$

E^o being the same as given in (4.31).

Proceeding as in Case 1 we finally obtain $\phi_2(x, y)$ for large l in this case, as given by

$$\phi_2(x,y) = \phi_1(x,y) + \frac{a-b}{2\pi i}\int_C \left[\frac{L_+(\alpha)}{\alpha + k} e^{i\alpha l}\{R_1^1(\alpha) + C_5 T^1(\alpha)\}\right.$$
$$\left. - \frac{L_-(\alpha)}{\alpha - k}\{e^{ikl} R_2^1(-\alpha) + C_6 T^1(-\alpha)\}\right] \frac{e^{-i\alpha x - ry}}{\gamma - b}\, d\alpha, \tag{4.46}$$

where now $\phi_1(x, y)$ is the same expression as given in (3.17) and

$$C_5 = \frac{L_+(\epsilon')}{(\epsilon' + k)^2 - \{T^1(\epsilon')L_+(\epsilon')\}^2}\{(\epsilon' + k)G_2^1(\epsilon') - T^1(\epsilon')L_+(\epsilon')G_1^1(\epsilon')\}$$

with

$$G_1^1(\alpha) = -\frac{e^{ikl}}{L_+(\alpha)} - R_1^1(\alpha), \qquad G_2^1(\alpha) = \frac{2k}{L_+(k)}\frac{1}{\alpha - k} - \frac{\alpha + k}{L_+(\alpha)(\alpha - k)} - e^{ikl} R_2^1(\alpha), \tag{4.47}$$

and C_6 is the same as C_5 with G_1^1 and G_2^1 interchanged, $R_1^1(\alpha)$ and $R_2^1(\alpha)$ being given by (4.44) and (4.45) respectively. As before, the second term in (4.46) may be regarded as due to the presence of second discontinuity at $(l, 0)$ for large l.

Case 3: $|b| = \infty$. In this case condition (2.16) assumes the form

$$\phi = -e^{ikx} \quad \text{on } y = 0 \ \text{ for } 0 < x < l,$$

so that the modification of relation (4.12) is

$$\frac{\gamma(\alpha) + a}{\alpha^2 - k^2} F_1(\alpha) = -U_-(\alpha) - e^{i\alpha l} V_+(\alpha) + \frac{1}{i(\alpha + k)}\{e^{i(\alpha+k)l} - 1\}, \quad c < \tau < d. \tag{4.48}$$

Here $P(\alpha) = \gamma + a$ is factorised as $P(\alpha) = P_+(\alpha)P_-(\alpha)$ where $P_-(\alpha) = P_-(-\alpha)$ and $P_-(\alpha)$ is given by (3.20). Proceeding as in Case 1, we obtain in place of (4.25)

$$\frac{(\alpha + k)F^*_{2+}(\alpha; \lambda)}{P_+(\alpha)} - \frac{\lambda}{2\pi i} \int\limits_{ih-\infty}^{ih+\infty} \frac{e^{il\xi}(\xi - k)F^*_{2+}(\xi; \lambda)}{(\xi + \alpha)P_-(\xi)} d\xi = \frac{2\lambda k}{iP_+(k)} \frac{1}{\alpha - k}, \quad \tau > -h \tag{4.49}$$

where now

$$F^*_{2+}(\alpha; \lambda) = F_{2-}(\alpha; \lambda) - \frac{e^{ikl}}{i(\alpha + k)} + \frac{\lambda}{i(\alpha - k)} \tag{4.50}$$

with

$$F_{2+}(\alpha; 1) = V_+(\alpha) - U_-(-\alpha), \qquad F_{2+}(\alpha; -1) = V_+(\alpha) + U_-(-\alpha). \tag{4.51}$$

We write

$$\frac{1}{P_-(\xi)} = \frac{-a + (\xi^2 - \epsilon^2)^{1/2}}{\xi^2 - k^2} P_+(\xi) \tag{4.52}$$

in the integrand of the integral in the left-hand side of (4.49) and proceeding in a manner similar to Cases 1 and 2 we find that, for large l, $\phi_2(x, y)$ is given by

$$\begin{aligned}\phi_2(x, y) = \phi_1(x, y) + \frac{1}{2\pi i} \int\limits_C \Bigg[&\frac{P_+(\alpha)}{\alpha + k} e^{i\alpha l}\{R_1^2(\alpha) + C_7 T^2(\alpha)\} \\ &- \frac{P_-(\alpha)}{\alpha - k}\{e^{ikl} R_2^2(-\alpha) + C_8 T^2(-\alpha)\}\Bigg] e^{-i\alpha x - ry} d\alpha,\end{aligned} \tag{4.53}$$

where now $\phi_1(x, y)$ is given by (3.21) and

$$R_1^2(\alpha) = \frac{E_0^2}{2\pi i} \frac{1}{\epsilon'^2 - k^2} \frac{1}{\epsilon' + \alpha}, \qquad R_2^2(\alpha) = \frac{E_0^2}{2\pi i} \frac{1}{(\epsilon' + k)^2} \frac{1}{\epsilon' + \alpha},$$

$$T^2(\alpha) = \frac{E_0^2}{2\pi i} \frac{1}{\epsilon' + k} \frac{1}{\epsilon' + \alpha}$$

with

$$E_0^2 = -\frac{\pi^{1/2}}{l^{3/2}} e^{i(\epsilon l + 3\pi/4)} P_+(\epsilon')(\epsilon' + \epsilon)^{1/2},$$

$$C_7 = \frac{P_+(\epsilon')}{(\epsilon' + k)^2 - \{T^2(\epsilon')P_+(\epsilon')\}^2}\{(\epsilon' + k)G_2^2(\epsilon') - T^2(\epsilon')P_+(\epsilon')G_1^2(\epsilon')\},$$

where

$$G_1^2(\alpha) = -\frac{e^{ikl}}{P_+(\alpha)} - R_1^2(\alpha), \qquad G_2^2(\alpha) = \frac{2k}{(\alpha-k)P_+(k)} - \frac{\alpha+k}{(\alpha-k)P_+(\alpha)} - R_2^2(\alpha)e^{ikl}, \tag{4.54}$$

and C_8 is the same as C_7 with G_1^2 and G_2^2 interchanged. The second term in (4.53) is due to the presence of the second discontinuity at $(l, 0)$ for large l.

Making $\epsilon \to 0$ in the solutions (4.39), (4.46) and (4.53) of the generalised BVP II for $b > 0$, $b < 0$ and $|b| = \infty$, respectively, we obtain the solutions of the original BVP II for $b > 0$, $b < 0$ and $|b| = \infty$. The second terms in (4.39), (4.46) and (4.53), after making $\epsilon \to 0$, involve $e^{-i\alpha(x-l)}$ and $e^{-i\alpha x}$ in the integrands. The integrals involving $e^{-i\alpha x}$ can be evaluated for $x < 0$ ($x > 0$) by deforming the contour along the bisectors of the first and second (third and fourth) quadrants of the complex α-plane as has been done for $\phi_1(x, y)$. The integrals involving $e^{-i\alpha(x-l)}$ can be evaluated similarly for $x < l$ and $x > l$. However, considerable effort is needed in the evaluations of these integrals. Since we are interested only in the wave terms, we need to find only the wave terms of $\phi_2(x, y)$ for different cases and different regions. If we denote

$$\phi_2^t(x, y) = e^{-ay+iax} + \phi_2(x, y), \tag{4.55}$$

then the asymptotic expressions valid for large r, for the wave terms of $\phi_2^t(x, y)$ are obtained, as given below.

For $x < 0$

$$\phi_2^t(x,y) \approx \begin{cases} e^{-ay+iax} - \dfrac{2a(b-a)}{\{(a+b)M_+^0(a)\}^2} e^{-ay-iax} \\ \quad + \left[\dfrac{4ab(T_1^0(b))^2 M_-^0(a)}{(b^2-a^2)M_+^0(a)} + i\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{a(b-a)^2 M_-^0(a) e^{ibl}}{b^2(a+b)M_+^0(a)(la)^2} \right] \\ \quad \times \dfrac{e^{-ay-iax}}{\{M_-^0(b)\}^2 - \{T_1^0(b)\}^2} + O\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b > 0, \\ e^{-ay+iax} + \dfrac{2a(b-a)e^{-ay-iax}}{\{L_+^0(a)\}^2} + O\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b < 0, \\ e^{-ay+iax} - \dfrac{2ae^{-ay-iax}}{\{P_+^0(a)\}^2} + O\left(\dfrac{1}{(la)^3}\right) \quad \text{for } |b| = \infty; \end{cases} \tag{4.56}$$

for $0 < x < l$

$$\phi_2^t(x,y) \approx \begin{cases} \dfrac{4abM_+^0(b)}{(a+b)^2 M_+^0(a)} e^{-by+ibx} - i\left(\dfrac{6}{\pi}\right)^{1/2} \\ \quad \times \dfrac{a^2(b-a)^2 M_+^0(b) e^{ibl} e^{-by+ibx}}{b^2(a+b)^2 M_+^0(a)\{(M_-^0(b))^2 - (T_1^0(b))^2\}} \\ \quad + i\left(\dfrac{6}{\pi}\right)^{1/2} \dfrac{1}{(la)^2} \dfrac{a(b-a)^2}{b(a+b)^2} \dfrac{M_+^0(b)}{M_+^0(a)} \\ \quad \times \left[1 - \dfrac{2aT_1^0(b)}{(b-a)\{(M_-^0(b))^2 - (T_1^0(b))^2\}}\right] e^{-by-ib(x-l)} \\ \quad + O\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b > 0, \\ 0 \quad \text{for } b < 0, \\ 0 \quad \text{for } |b| = \infty \end{cases} \tag{4.57}$$

and for $x > l$

$$\phi_2^t(x,y) \approx \begin{cases} \Bigg[-\dfrac{2aM_-^0(a)\mathrm{e}^{ibl}}{(a+b)M_+^0(a)\{(M_-^0(b))^2 - (T_1^0(b))^2\}} \\ \quad +\mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{1}{(la)^2} \dfrac{(b-a)^2 M_-^0(a)}{b(a+b)M_+^0(a)} \\ \quad \times \left\{ 1 - \dfrac{2a}{(b-a)} \dfrac{T_1^0(b)}{(M_-^0(b))^2 - (T_1^0(b))^2} \right\} \Bigg] \\ \quad \times \mathrm{e}^{-ay+\mathrm{i}a(x-l)} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b > 0, \\ -\mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{1}{(la)^2} \dfrac{a-b}{b} \dfrac{L_-^0(a)}{L_+^0(a)} \mathrm{e}^{-ay+\mathrm{i}a(x-l)} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b < 0, \\ \mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{1}{(la)^2} \dfrac{P_-^0(a)}{P_+^0(a)} \mathrm{e}^{-ay+\mathrm{i}a(x-l)} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } |b| = \infty, \end{cases} \tag{4.58}$$

where $T_1^0(b) = \lim_{\epsilon\to 0} T_1(\alpha_0) = ((b-a)M_+^0(b)\mathrm{e}^{ibl})/(a+b)$.

Comparing (2.6a) with (4.57) and (4.58) we find that the unknown complex constants A_2, B_2, B_3 and B_4 of (2.6a) are now obtained approximately for large l and are given by

$$A_2 = \begin{cases} -\dfrac{2a(b-a)}{\{(a+b)M_+^0(a)\}^2} + \Bigg[\dfrac{4ab\{T_1^0(b)\}^2 M_-^0(a)}{(b^2-a^2)M_+^0(a)} \\ \quad +\mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{a(b-a)^2 M_-^0(a)\mathrm{e}^{ibl}}{b^2(a+b)M_+^0(a)} \dfrac{1}{(la)^2} \Bigg] \\ \quad \times \dfrac{1}{\{M_-^0(b)\}^2 - \{T_1^0(b)\}^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b > 0, \\ \dfrac{2a(b-a)}{\{L_+^0(a)\}^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } b < 0, \\ \dfrac{2a}{\{P_+^0(a)\}^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) \quad \text{for } |b| = \infty; \end{cases} \tag{4.59}$$

$$B_2 = \frac{4abM_+^0(b)}{(a+b)^2 M_+^0(a)} - \mathrm{i}\left(\frac{6}{\pi}\right)^{1/2} \frac{a^2(b-a)^2}{b^2(a+b)^2} \frac{M_+^0(b)\mathrm{e}^{ibl}}{\{M_+^0(b)\}^2 - \{T_1^0(b)\}^2} \frac{1}{(la)^2} + \mathrm{O}\left(\frac{1}{(la)^3}\right), \tag{4.60}$$

$$B_3 = \mathrm{i}\left(\frac{6}{\pi}\right)^{1/2} \frac{a(b-a)^2}{b(a+b)^2} \frac{M_+^0(b)}{M_+^0(a)} \left[1 - \frac{2a}{(b-a)} \frac{T_1^0(b)}{\{M_-^0(b)\}^2 - \{T_1^0(b)\}^2} \right] \frac{1}{(la)^2} + \mathrm{O}\left(\frac{1}{(la)^3}\right), \tag{4.61}$$

$$B_4 = \begin{cases} -\dfrac{2aM_-^0(a)}{(a+b)M_+^0(a)} \dfrac{\mathrm{e}^{\mathrm{i}bl}}{\{M_-^0(b)\}^2 - \{T_1^0(b)\}^2} \\ +\mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{(b-a)^2 M_-^0(a)}{b(a+b)M_+^0(a)} \\ \times\left[1 - \dfrac{2a}{(b-a)} \dfrac{T_1^0(b)}{\{M_-^0(b)\}^2 - \{T_1^0(b)\}^2}\right] \dfrac{1}{(la)^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) & \text{for } b > 0, \\ -\mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{a-b}{b} \dfrac{L_-^0(a)}{L_+^0(a)} \dfrac{1}{(la)^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) & \text{for } b < 0, \\ \mathrm{i}\left(\dfrac{3}{2\pi}\right)^{1/2} \dfrac{P_-^0(a)}{P_+^0(a)} \dfrac{1}{(la)^2} + \mathrm{O}\left(\dfrac{1}{(la)^3}\right) & \text{for } |b| = \infty. \end{cases} \quad (4.62)$$

In the expressions for A_2 and B_i $(i = 2, 3, 4)$, $M_\pm^0(a)$, $M_\pm^0(b)$, $P_\pm^0(a)$ are given by (3.29).

It may be noted from (2.6a) that B_2 and B_3 exist only for $b > 0$. It is observed from (4.57) that in the region $0 < x < l$, progressive waves exist only for $b > 0$, and these consist of transmitted and reflected waves. In this case the incident wave from the region $x < 0$ undergoes partial transmission below the edge at $x = 0$ which then is partially reflected by the edge at $x = l$. For $b < 0$ or $|b| = \infty$, there is no progressive waves in this region. (4.58) shows that in the region $x > l$, there exist progressive waves due to transmission of the incident wave field through the region below the inertial surface even though there may not be any progressive wave in the region $0 < x < l$.

5. Reflection coefficient in the region $x < 0$

The quantities a and b are related and for deep water the relationship can be expressed as

$$b = \frac{a}{1 - l_0 a},$$

where l_0 $(= \sigma/\rho)$ can be interpreted as the height of a vertical cylinder containing the fluid whose mass is the same as that of the floating matter distributed over the cross-sectional area of the cylinder at the inertial surface. In order that there exist time-harmonic progressive waves at the inertial surface, b must be positive, as is mentioned in Section 1. Hence if the frequency w of the incident wave in the region $x < 0$ is kept fixed, then $b > 0$ implies that $\sigma < \rho g/w^2$, which is usually interpreted as the inertial surface to be *light*. However, if $b < 0$ or $|b| = \infty$, then $\sigma \geq \rho g/w^2$ and the inertial surface is then interpreted as *heavy* since it does not allow propagation of time harmonic waves, as was pointed out earlier. Again, $b \lessgtr 0$ also implies $w \lessgtr w_0$ where $w_0 = (\rho g/\sigma)^{1/2}$. This means that w_0 represents a kind of threshold frequency, since if the frequency of the incident wave train exceeds this frequency, then the inertial surface does not allow propagation of any time-harmonic wave. This phenomenon is well known in the literature (cf. [1,2,4]).

When the inertial surface is in the form of a semi-infinite plane $(y = 0, x \geq 0, -\infty < z < \infty)$ as in BVP I, the incident wave field undergoes reflection into the region $x < 0$ by the edge $x = 0$ and transmission into the region $x > 0$ provided the incident wave frequency w is less than the threshold frequency w_0. However, when $w \geq w_0$ the incident wave field from the region $x < 0$ is totally reflected back into the region $x < 0$ by the edge $x = 0$. Of course, there are local excitations by the edge $x = 0$ and these do not propagate as waves and die out away from the edge.

When the inertial surface is in the form of a strip $(y = 0, 0 \leq x \leq l, -\infty < z < \infty)$ as in BVP II, there are now two edges, one at $x = 0$ and another at $x = l$. Expressions (4.56) show that the incident wave is reflected back

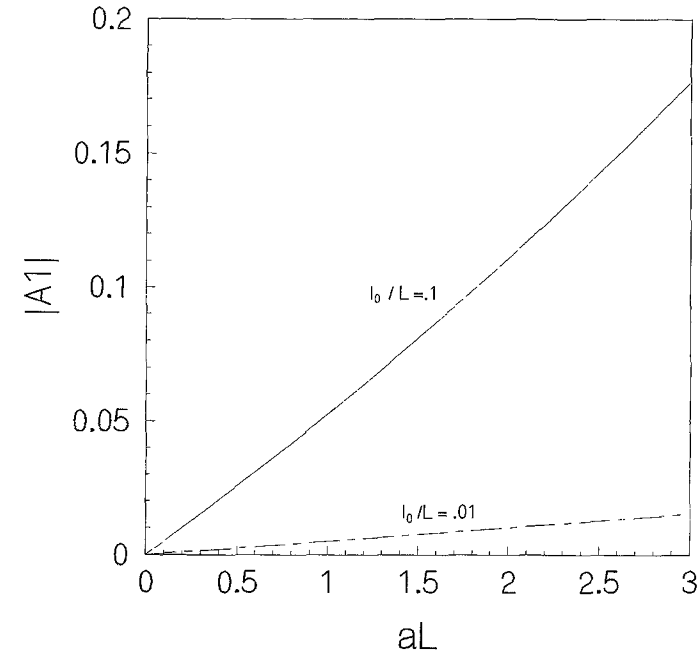

Fig. 1. Reflection coefficient due to a semi infinite inertial surface.

into the region $x < 0$ by the left edge of the strip. Expressions (4.57) show that for $b > 0$, there is a transmitted wave and a reflected wave inside the strip, the transmission being through the edge $x = 0$ and the reflection being by the edge $x = l$. For $b < 0$ or $|b| = \infty$, no wave propagates inside the strip apart from some local excitations by the two edges. Finally, (4.58) show that in the region right of the strip, progressive waves exist which are due to transmission of the incident wave field through the region below the inertial surface.

The reflection coefficient $|A_1|$ for $b > 0$ is depicted graphically in Fig. 1 against the wave number aL where as mentioned earlier, L is a characteristic length used to nondimensionalise a, b (> 0) and σ/ρ ($= l_0$) for deep water, choosing $l_0/L = 0.01, 0.1$. It is observed from this figure that for fixed l_0/L, $|A_1|$ increases uniformly with the wave number aL.

This is expected since, as the wave number increases, the incident wave remains confined within a thin layer below the free surface in the region $x < 0$ and as it encounters the edge $x = 0$, reflection by the edge becomes more. It is also observed that for fixed wave number, $|A_1|$ increases as l_0/L increases, i.e., as the surface density of the material of the inertial surface increases. This means that as the inertial surface becomes heavier, more energy is reflected by its edge, provided of course the inertial surface is 'light' enough to allow progressive waves to propagate on it.

When the inertial surface is in the form of a strip of finite but large breadth l, the reflection coefficient $|A_2|$ for $b > 0$ in the region $x < 0$ is depicted graphically against the wave number aL in Fig. 2 taking $l/L = 10$ and $l_0/L = 0.01$ and 0.1 and in Fig. 3 taking $l_0/L = 0.01$ and $l/L = 10, 20$. It has been observed from Fig. 1 that in the presence of the semi-infinite inertial surface, the reflection coefficient $|A_1|$ steadily increases with the wave number. However, when the inertial surface is in the form of a strip, this qualitative behaviour of the reflection coefficient ($|A_2|$) is lost. In this case, the behaviour of $|A_2|$ changes significantly against the wave number. Each graph of the reflection coefficient has the same basic feature, consisting of a series of concave curves which meet the wave number axis at their ends, so that total transmission occurs for a sequence of discrete values of the wave number. It is observed that larger values of l_0/L leads to higher maxima in $|A2|$ (cf. Fig. 2) and larger values of l/L leads to more number of zeros of $|A_2|$ (cf. Fig. 3). The oscillatory behaviour of $|A_2|$ against the wave number may

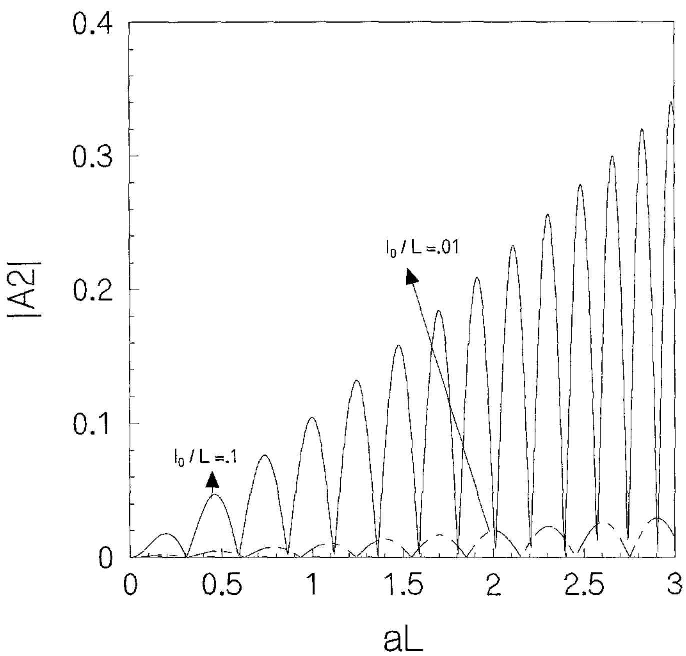

Fig. 2. Reflection coefficient due to a strip like inertial surface, $l/L = 10$.

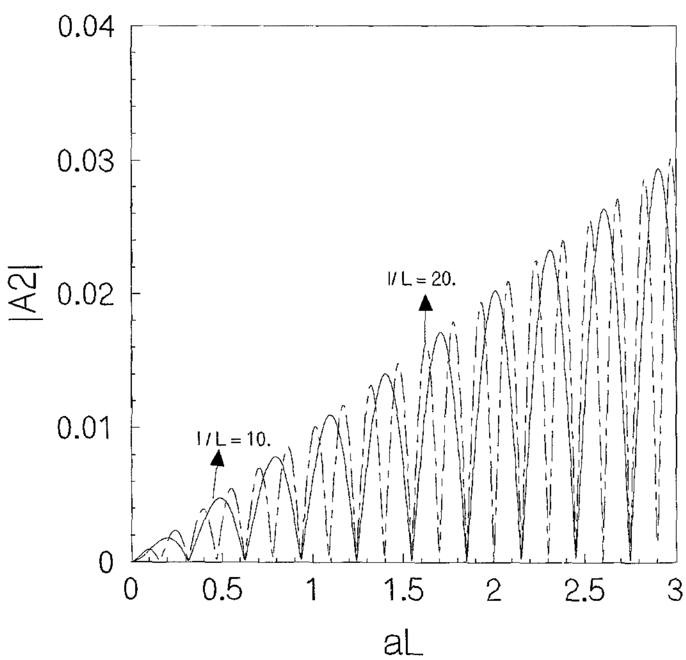

Fig. 3. Reflection coefficient due to a strip like inertial surface, $l_0/L = 0.01$.

be attributed due to multiple reflections of the wave by two edges of the strip.

6. Discussion

Two mixed boundary value problems arising in the linearised theory of water waves have been solved by using Wiener–Hopf technique. The first problem involves water wave scattering by a discontinuity on the surface arising due to the presence of a semi-infinite inertial surface while the second problem is a generalisation of the first after introducing a second discontinuity at a distance l away from the first, the two discontinuities arising due to the presence of an inertial surface in the form of a strip of breadth l instead of the semi-infinite inertial surface. The BVP I is reduced to a two-part Wiener–Hopf problem whose solution is obtained in closed form. The reflection and transmission coefficients for this BVP are then obtained in closed form also. The BVP II reduces to a three-part Wiener–Hopf problem whose solution is obtained asymptotically for large l. This produces approximate analytical expressions for the reflection and transmission coefficients in the appropriate regions whereever they exist.

The behaviour of the reflection coefficient in the region of the free surface for *light* inertial surface is depicted graphically against the wave number in a number of figures and appropriate conclusions are drawn.

It may be noted that by increasing the breadth of the strip for the BVP II indefinitely, the results of BVP I cannot be recovered. This is because the two BVPs are basically different in the sense that BVP I involves only one edge while the BVP II involves two edges, and by increasing the distance between the two edges, the second edge can never be eliminated.

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