SOME MOMENTS OF MOMENT STATISTICS AND THEIR USE IN TESTS OF SIGNIFICANCE IN AUTO-CORRELATED SERIES

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1. Introduction

The moments and product moments of moment-statistics for samples of the finite and infinite populations have been obtained up to the order of 8 by Sukhatme (1943). For earlier work on this problem reference may be made to the papers by Karl Pearson (1898, 1903, 1913), Sheppard (1899), Isserlis (1916, 1919, 1931), Soper (1913), Student (1908), Tchouproff (1919), Church (1925, 1926), Neyman (1925), R. A. Fisher (1928), C. C. Craig (1928), N. St. Georgescu (1932) and Wishart (1930, 1933). These authors, in their investigations, have considered the variates to be independent.

In the analysis of economic time series and similar other problems one comes across variates that are autocorrelated and consequently the study of moments of moment-statistics of autocorrelated variables is of considerable importance. The present paper is a prelude to this study.

2. Notation

Let $x_1, x_2, \ldots, x_T$ and $y_1, y_2, \ldots, y_T$ represent two stationary time series with the following expectations.

$$E(x_t) = \mu_{10} \qquad \ldots \quad (1.a)$$

$$E(y_t) = \mu_{01} \qquad \ldots \quad (1.b)$$

$$E\{(x_t-\mu_{10})(x_{t+k}-\mu_{10})\} = \mu_{20}(k) \qquad \ldots \quad (1.c)$$

$$E\{(y_t-\mu_{01})(y_{t+k}-\mu_{01})\} = \mu_{02}(k) \qquad \ldots \quad (1.d)$$

$$E\{(x_t-\mu_{10})(y_{t+k}-\mu_{01})\} = \mu_{11}(k) \qquad \ldots \quad (1.e)$$

$$E\{(x_t-\mu_{10})(x_{t+i}-\mu_{10})(x_{t+j}-\mu_{10})(x_{t+k}-\mu_{10})\} = \mu_{40}(i,j,k) \qquad \ldots \quad (1.f)$$

$$E\{(x_t-\mu_{10})(x_{t+i}-\mu_{10})(x_{t+j}-\mu_{10})(y_{t+k}-\mu_{01})\} = \mu_{31}(i,j,k) \qquad \ldots \quad (1.g)$$

$$E\{(x_t-\mu_{10})(x_{t+i}-\mu_{10})(y_{t+j}-\mu_{01})(y_{t+k}-\mu_{01})\} = \mu_{22}(i,j,k) \qquad \ldots \quad (1.h)$$

$$E\{(x_t-\mu_{10})(y_{t+i}-\mu_{01})(y_{t+j}-\mu_{01})(y_{t+k}-\mu_{01})\} = \mu_{13}(i,j,k) \qquad \ldots \quad (1.i)$$

$$E\{(y_t-\mu_{01})(y_{t+i}-\mu_{01})(y_{t+j}-\mu_{01})(y_{t+k}-\mu_{01})\} = \mu_{04}(i,j,k) \qquad \ldots \quad (1.j)$$

$$\bar{x} = \frac{x_1+x_2+\ldots+x_T}{T} = m_{10} \qquad \ldots \quad (1.k)$$

$$\bar{y} = \frac{y_1+y_2+\ldots+y_T}{T} = m_{01} \qquad \ldots \quad (1.l)$$

* A. Khintchine defines as follows: Letting $(t) = (t_1, t_2, \ldots t_n)$ represent an arbitrary set of time points, and fixing arbitrarily a translation in time of this set, say $(T) = (t_1+t, t_2+t, \ldots t_n+t)$, a random process as defined by a set (F) of distribution functions is called stationary if the two functions belonging to the two sets (t) and (T) are identical. "A study in the analysis of time series" by Herman Wold (1938) Uppsala, p. 4.

Let also m_{ij} stand for sample values corresponding to the population values μ_{ij}.

3. SOME FORMULAE

We then have

$$E(\bar{x}) = E\left\{\frac{1}{T}\sum_{t=1}^{T} x_t\right\} = \mu_{10} \qquad \text{(2.a)}$$

$$E(\bar{y}) = E\left\{\frac{1}{T}\sum_{t=1}^{T} y_t\right\} = \mu_{01} \qquad \ldots \quad \text{(2.b)}$$

$$V(\bar{x}) = E\left\{\left(\frac{1}{T}\sum_{t=1}^{T} x_t\right)^2\right\} - E^2\left\{\frac{1}{T}\sum_{t=1}^{T} x_t\right\}$$

$$= \frac{1}{T}\,\mu_{20}(0) + \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{20}(k) \qquad \ldots \quad \text{(3.a)}$$

$$V(\bar{y}) = E\left\{\left(\frac{1}{T}\sum_{t=1}^{T} y_t\right)^2\right\} - E^2\left\{\frac{1}{T}\sum_{t=1}^{T} y_t\right\}$$

$$= \frac{1}{T}\,\mu_{02}(0) + \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{02}(k) \qquad \ldots \quad \text{(3.b)}$$

$$E(m_{20}) = E\left\{\frac{1}{T}\sum_{t=1}^{T}(x_t-\bar{x})^2\right\}$$

$$= \left(1-\frac{1}{T}\right)\mu_{20}(0) - \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{20}(k) \qquad \ldots \quad \text{(4.a)}$$

$$E(m_{02}) = E\left\{\frac{1}{T}\sum_{t=1}^{T}(y_t-\bar{y})^2\right\}$$

$$= \left(1-\frac{1}{T}\right)\mu_{02}(0) - \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{02}(k) \qquad \ldots \quad \text{(4.b)}$$

$$E(m_{11}) = E\left\{\frac{1}{T}\sum_{t=1}^{T}(x_t-\bar{x})(y_t-\bar{y})\right\}$$

$$= \left(1-\frac{1}{T}\right)\mu_{11}(0) - \frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{11}(k)$$

$$-\frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)\mu_{11}(-k). \qquad \ldots \quad \text{(5)}$$

Now if we define

$$\left.\begin{aligned} \mu_{20}(0) &= \sigma_x^2 \\ \mu_{02}(0) &= \sigma_y^2 \\ \mu_{11}(0) &= \sigma_x\sigma_y\rho_0 \end{aligned}\right\} \qquad \ldots \quad \text{(6)}$$

and

$$\left.\begin{aligned} \mu_{20}(k) &= \sigma_x^2\xi_k \\ \mu_{02}(k) &= \sigma_y^2\eta_k \\ \mu_{11}(k) &= \sigma_x\sigma_y\rho_k \end{aligned}\right\} \qquad \ldots \quad (7)$$

then (3) to (5) become

$$\left.\begin{aligned} V(\bar{x}) &= \frac{1}{T}\sigma_x^2+\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x^2\xi_k \\ V(\bar{y}) &= \frac{1}{T}\sigma_y^2+\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_y^2\eta_k \\ E(m_{20}) &= \left(1-\frac{1}{T}\right)\sigma_x^2-\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x^2\xi_k \\ E(m_{02}) &= \left(1-\frac{1}{T}\right)\sigma_y^2-\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_y^2\eta_k \\ E(m_{11}) &= \left(1-\frac{1}{T}\right)\sigma_x\sigma_y\rho_0-\left\{\frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x\sigma_y\rho_k \right. \\ &\qquad \left. +\frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x\sigma_y\rho_{-k}\right\} \end{aligned}\right\} \qquad \ldots \quad (8)$$

If $x_1, x_2, \ldots, x_T$ and $y_1, y_2, \ldots, y_T$ are considered independent (*i.e.* not autocorrelated) between themselves then the second term in each of the above formulae vanishes and we obtain $V(\bar{x})$ etc. in the usual forms already known.

Moments of some of the higher order statistics have also been derived by the author and they are expected to be published in a separate paper. If, however, $x_1, x_2, \ldots, x_T$ and $y_1, y_2, \ldots, y_T$ are distributed normally, we have

$$\left.\begin{aligned} \rho_{-k} &= \rho_k \\ \mu_{40}(0,0,0) &= 3\sigma_x^4 \\ \mu_{04}(0,0,0) &= 3\sigma_y^4 \\ \mu_{22}(0,0,0) &= (1+2\rho_0^2)\sigma_x^2\sigma_y^2 \\ \mu_{31}(0,0,0) &= 3\rho_0\sigma_x^3\sigma_y \\ \mu_{13}(0,0,0) &= 3\rho_0\sigma_x\sigma_y^3 \\ \mu_{40}(0,k,k) &= (1+2\xi_k^2)\sigma_x^4 \\ \mu_{04}(0,k,k) &= (1+2\eta_k^2)\sigma_y^4 \\ \mu_{22}(k,0,k) &= (\xi_k\eta_k+\rho_k^2+\rho_0^2)\sigma_x^2\sigma_y^2 \\ \mu_{22}(0,k,k) &= \mu_{22}(0,-k,-k) = (1+2\rho_k^2)\sigma_x^2\sigma_y^2 \\ \mu_{31}(k,k,0) &= \mu_{31}(0,k,k) = (\rho_0+2\xi_k\rho_k)\sigma_x^3\sigma_y \\ \mu_{13}(0,k,k) &= \mu_{13}(0,-k,-k) = (\rho_0+2\eta_k\rho_k)\sigma_x\sigma_y^3 \end{aligned}\right\} \qquad \ldots \quad (9)$$

and up to the order of $\frac{1}{T}$,

$$
\left.
\begin{aligned}
V(m_{20}) &= \frac{1}{T}\,\sigma_x^4\{3+4\sum_{k=1}^{T-1}\xi_k(1+\xi_k)\} \\
V(m_{02}) &= \frac{1}{T}\,\sigma_y^4\{3+4\sum_{k=1}^{T-1}\eta_k(1+\eta_k)\} \\
V(m_{11}) &= \frac{1}{T}\,\sigma_x^2\sigma_y^2\{1+2\rho_0^2+2\sum_{k=1}^{T-1}(\xi_k\eta_k+\rho_k^2+2\rho_0\rho_k)\} \\
\text{cov}\,(m_{20}, m_{02}) &= \frac{1}{T}\,\sigma_x^2\sigma_y^2\{1+2\rho_0^2+4\sum_{k=1}^{T-1}(\rho_k+\rho_k^2)\} \\
\text{cov}\,(m_{11}, m_{20}) &= \frac{1}{T}\,\sigma_x^3\sigma_y\{3\rho_0+2\sum_{k=1}^{T-1}(2\xi_k\rho_k+\rho_k+\xi_k)\} \\
\text{cov}\,(m_{11}, m_{02}) &= \frac{1}{T}\,\sigma_x\sigma_y^3\{3\rho_0+2\sum_{k=1}^{T-1}(2\eta_k\rho_k+\rho_k+\eta_k)\}
\end{aligned}
\right\} \quad \dots \quad (10)
$$

Now if $x_1, x_2, \dots, x_T$ and $y_1, y_2, \dots, y_T$ are considered to be non-autocorrelated then the last term in each of the above curly brackets vanishes and we have the values of $V(m_{20})$ etc. already known.

4. Correlation coefficient

The correlation coefficient is defined as

$$E\,(r) = E\left\{\frac{m_{11}}{\sqrt{m_{20}.m_{02}}}\right\} = \frac{E(m_{11})}{\sqrt{E(m_{20}).E(m_{02})}} \text{ up to the order of } \frac{1}{T}.$$

Thus in large samples

$$E(r) = \frac{\left(1-\frac{1}{T}\right)\rho_0 - \frac{2}{T}\sum_{k=1}^{T-1}\rho_k}{\sqrt{\left(1-\frac{1}{T}\right)^2+\frac{4}{T^2}\sum_{k=1}^{T-1}\xi_k\eta_k-\frac{2}{T}\left(1-\frac{1}{T}\right)\sum_{k=1}^{T-1}(\xi_k+\eta_k)}} \quad \dots \quad (11)$$

when x's and y's are non-autocorrelated this reduces to ρ_0.

Since

$$\frac{1}{r^2}V(r) = \frac{V(m_{11})}{E^2(m_{11})} + \frac{1}{4}\frac{V(m_{20})}{E^2(m_{20})} + \frac{1}{4}\frac{V(m_{02})}{E^2(m_{02})} + \frac{\text{cov}\,(m_{02}, m_{20})}{2\{E(m_{02})\}\{E(m_{20})\}}$$

$$-\frac{\text{cov}\,(m_{11}, m_{20})}{\{E(m_{11})\}\{E(m_{20})\}} - \frac{\text{cov}\,(m_{11}, m_{02})}{\{E(m_{11})\}\{E(m_{02})\}}$$

upto the order of $\frac{1}{T}$.

Thus

$$V(r) = \Big[\frac{2}{\sigma_x^2\sigma_y^2}(2+\rho_0^2)\,\mu_{22}(0,0,0) + \frac{8}{T\sigma_x^2\sigma_y^2}\sum_{k=1}^{T-1}(T-k)\mu_{22}(k,0,k)$$

$$+\rho_0^2\Big\{\frac{1}{\sigma_x^4}\mu_{40}(0,0,0) + \frac{1}{\sigma_y^4}\mu_{04}(0,0,0)\Big\}$$

$$+\frac{2\rho_0^2}{T}\sum_{k=1}^{T-1}(T-k)\Big\{\frac{1}{\sigma_x^4}\mu_{40}(0,k,k)\,\frac{1}{\sigma_y^4}\mu_{04}(0,k,k)$$

$$+\frac{1}{\sigma_x^2\sigma_y^2}\mu_{22}(0,k,k) + \frac{1}{\sigma_x^2\sigma_y^2}\mu_{22}(0,-k,-k)\Big\}$$

$$-4\rho_0\Big\{\frac{1}{\sigma_x^3\sigma_y}\mu_{31}(0,0,0) + \frac{1}{\sigma_x\sigma_y^3}\mu_{13}(0,0,0)\Big\}$$

$$-\frac{4\rho_0}{T}\sum_{k=1}^{T-1}(T-k)\Big\{\frac{1}{\sigma_x^3\sigma_y}\mu_{31}(k,k,0) + \frac{1}{\sigma_x^3\sigma_y}\mu_{31}(0,k,k)$$

$$+\frac{1}{\sigma_x\sigma_y^3}\mu_{13}(0,k,k) + \frac{1}{\sigma_x\sigma_y^3}\mu_{13}(0,-k,-k)\Big\}\Big]\Big/\Big\{4T-8-8\sum_{k=1}^{T-1}(\xi_k+\eta_k)\Big\}\ldots \quad (12)$$

If x's and y's are normally distributed then

$$V(r) = \frac{(1-\rho_0^2)-\rho_0^2\sum_{k=1}^{T-1}\xi_k^2+\eta_k^2+2\rho_k^2)-4\rho_0\sum_{k=1}^{T-1}\rho_k(\xi_k+\eta_k)+2\sum_{k=1}^{T-1}(\xi_k\eta_k+\rho_k^2)}{T-2-2\sum_{k=1}^{T-1}(\xi_k+\eta_k)} \quad \ldots \quad (13)$$

Further in addition to the above if x's and y's are non-autocorrelated then

$V(r) = \dfrac{(1-\rho_0^2)^2}{T-2}$, a well known result.

5. Regression

Let $\qquad E(x) = \alpha_1+\beta_1 y$

and $\qquad E(y) = \alpha_2+\beta_2 x$

be regression equations of x on y and y on x respectively. The parameters β_1 and β_2 are estimated by the statistics $b_1 = \frac{m_{11}}{m_{02}}$ and $b_2 = \frac{m_{11}}{m_{20}}$ by the method of least squares. $E(b_1)$, $E(b_2)$, $V(b_1)$ and $V(b_2)$ are already known for large samples when x's and y's are non-autocorrelated. If now we consider them to be autocorrelated and stationary then in large samples,

$$E(b_1) = \frac{E(m_{11})}{E(m_{02})} = \frac{\sigma_x\left\{\left(1-\frac{1}{T}\right)\rho_0 - \frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)(\rho_k+\rho_{-k})\right\}}{\sigma_y\left\{\left(1-\frac{1}{T}\right) - \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\eta_k\right\}} \quad \cdots \quad (14)$$

$$E(b_2) = \frac{E(m_{11})}{E(m_{20})} = \frac{\sigma_y\left\{\left(1-\frac{1}{T}\right)\rho_0 - \frac{1}{T^2}\sum_{k=1}^{T-1}(T-k)(\rho_k+\rho_{-k})\right\}}{\sigma_x\left\{\left(1-\frac{1}{T}\right) - \frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\xi_k\right\}} \quad \cdots \quad (15)$$

Since
$$\frac{V(b_1)}{E^2(b_1)} = \frac{V(m_{11})}{E^2(m_{11})} + \frac{V(m_{02})}{E^2(m_{02})} - \frac{2\,\mathrm{cov}\,(m_{11}, m_{02})}{\{E(m_{11})\}\{E(m_{02})\}}$$

$$\frac{V(b_2)}{E^2(b_2)} = \frac{V(m_{11})}{E^2(m_{11})} + \frac{V(m_{20})}{E^2(m_{20})} - \frac{2\,\mathrm{cov}\,(m_{11}, m_{20})}{\{E(m_{11})\}\{E(m_{20})\}}$$

$$\begin{aligned} V(b_1) = \frac{1}{\sigma_y^4}\Big[& \mu_{22}(0,0,0) + \frac{2}{T}\sum_{k=1}^{T-1}(T-k)\,\mu_{22}(k,0,k) + \frac{\sigma_x^2\rho_0^2}{\sigma_y^2}\,\mu_{04}(0,0,0) \\ & + \frac{2\rho_0^2\sigma_x^2}{T\sigma_y^2}\sum_{k=1}^{T-1}(T-k)\mu_{04}(0,k,k) - \frac{2\rho_0\sigma_x}{\sigma_y}\,\mu_{13}(0,0,0) \\ & - \frac{2\rho_0\sigma_x}{T\sigma_y}\sum_{k=1}^{T-1}(T-k)\{\mu_{13}(0,k,k)+\mu_{13}(0,-k,-k)\}\Big] \Big/ \left(T-2-4\sum_{k=1}^{T-1}\eta_k\right) \quad \cdots \quad (16) \end{aligned}$$

$$\begin{aligned} V(b_2) = \frac{1}{\sigma_x^4}\Big[& \mu_{22}(0,0,0) + \frac{2}{T}\sum_{k=1}^{T-1}(T-k)\,\mu_{22}(k,0,k) + \frac{\sigma_y^2\rho_0^2}{\sigma_x^2}\,\mu_{40}(0,0,0) \\ & + \frac{2\rho_0\sigma_y^2}{T\sigma_x^2}\sum_{k=1}^{T-1}(T-k)\mu_{40}(0,k,k) - \frac{2\rho_0\sigma_y}{\sigma_x}\,\mu_{31}(0,0,0) \\ & - \frac{2\rho_0\sigma_y}{T\sigma_x}\sum_{k=1}^{T-1}(T-k)\{\mu_{31}(0,k,k)+\mu_{31}(0,-k,-k)\}\Big] \Big/ \left(T-2-4\sum_{k=1}^{T-1}\xi_k\right) \quad \cdots \quad (17) \end{aligned}$$

If x's and y's are normally distributed then

$$V(b_1) = \frac{\sigma_x^2}{\sigma_y^2}\left\{\frac{(1-\rho_0^2)+4\rho_0^2\sum_{k=1}^{T-1}\eta_k-8\rho_0\sum_{k=1}^{T-1}\rho_k\eta_k+2\sum_{k=1}^{T-1}(\rho_k^2+\xi_k\eta_k)}{T-2-4\sum_{k=1}^{T-1}\eta_k}\right\} \quad \dots \quad (18)$$

$$V(b_2) = \frac{\sigma_y^2}{\sigma_x^2}\left\{\frac{(1-\rho_0)^2+4\rho_0^2\sum_{k=1}^{T-1}\xi_k-8\rho_0\sum_{k=1}^{T-1}\rho_k\xi_k+2\sum_{k=1}^{T-1}(\rho_k^2+\xi_k\eta_k)}{T-2-4\sum_{k=1}^{T-1}\xi_k}\right\}$$

Further if x's and y's are considered to be non-autocorrelated then we have the well known results in large samples

$$V(b_1) = \frac{\sigma_x^2(1-\rho_0^2)}{\sigma_y^2(T-2)} \quad \text{and} \quad V(b_2) = \frac{\sigma_y^2(1-\rho_0^2)}{\sigma_x^2(T-2)}$$

6. Use of t-statistics when the variables are autocorrelated

When $x_1, x_2, \dots, x_T$ are independent $t = \sqrt{\frac{(\bar{x}-\mu_{10})^2}{V(\bar{x})}}$ is the statistic for testing the hypothesis concerning the mean. When x's are autocorrelated and stationary then from (8)

$$V(\bar{x}) = \frac{1}{T}\sigma_x^2+\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x^2\xi_k$$

$$E(m_{20}) = \left(1-\frac{1}{T}\right)\sigma_x^2-\frac{2}{T^2}\sum_{k=1}^{T-1}(T-k)\sigma_x^2\xi_k$$

Thus

$$V(\bar{x})+E(m_{20}) = \sigma_x^2$$

$$(T-1)V(\bar{x})-E(m_{20}) = \frac{2}{T}\cdot\sum_{k=1}^{T-1}(T-k)\sigma_x^2\xi_k$$

$$\therefore \quad \frac{(T-1)V(\bar{x})-E(m_{20})}{V(\bar{x})+E(m_{20})} = \frac{2}{T}\sum_{k=1}^{T-1}(T-k)\xi_k$$

or

$$V(\bar{x}) = \frac{1+\frac{2}{T}\sum_{k=1}^{T-1}(T-k)\xi_k}{(T-1)-\frac{2}{T}\sum_{k=1}^{T-1}(T-k)\xi_k}E(m_{20})$$

Thus

$$= \frac{\sqrt{\left((T-1)-\frac{2}{T}\sum_{k=1}^{T-1}(T-k)\xi_k\right)T(\bar{x}-\mu_{10}')^2}}{\sqrt{\left(1+\frac{2}{T}\sum_{k=1}^{T-1}(T-k)\xi_k\right)S^2}} \quad \dots \quad (19)$$

where $$S^2 = \sum_{t=1}^{T}(x_t-\bar{x})^2$$

We now consider five experimental models

(A) $x_t = \eta_t + \eta_{t-1}$
(B) $x_t = \eta_t + \frac{1}{2}\eta_{t-1}$
(C) $x_t = \eta_t + \eta_{t-1} + \eta_{t-2} + \eta_{t-3}$
(D) $x_t = .8x_{t-1} + \eta_t$
(E) $x_t = .7x_{t-1} - .0125x_{t-2} + \eta_t$

Here η_t's are random deviates with zero mean and unit standard deviation.

Table 1 shows how 25 samples of each scheme with a sample size of 35 behave in respect of their means when tested by the usual 't' and corrected 't' as given in (19). Owing to the existence of autocorrelation between x's of each sample of the above schemes $V(\bar{x})$ is greater than that for the independent set of variables when the x's are positively correlated and less when the x's are negatively correlated. Thus in the above two circumstances the usual 't' respectively overestimates and underestimates the significance of the mean. The behaviour of the uncorrected t (i.e. usual 't') as seen from the tables 1 and 2, shows how unreliable the test can be when not much is known about the exact distribution of x's of the stationary time series.

The above results also show that proper corrections can be obtained for 't' by using the theoretical values of the autocorrelations. In actual practice these have to be substituted by estimated values. The theoretical consequences of such a procedure require further study.

7. DEGREES OF FREEDOM OF

We shall in this section investigate whether the degrees of freedom of t can be taken to be (T−1) as when the variables are independent or an improved substitute is available.

From the procedure adopted in arriving at the t statistic above, it is seen that the variance μ_{20} is estimated from the corresponding sample value m_{20}. $E(m_{20})$ has $(T-1)$ degrees of freedom when the variables are independent.

In a similar way the degrees of freedom of $E(m_{20})$ when the variables are autocorrelated, may be modified into the form :

$$\left\{(T-1)-\frac{2}{T}\sum_{k=1}^{T-1}(T-k)\,\xi_k\right\} \quad \dots \quad (20)$$

The appropriateness of (20) as giving the degree of freedom to be used has been verified by experiments using the same models A, B, C, D and E considered before. For each model, 125 samples (for model C only 100 samples) of size 7 were taken.

The results of the tests with these models are summarised in Table 3, giving the frequencies of values of t arising from tests of three types

α) t of the usual type given by $t = \frac{\bar{x}-\mu_{10}}{s\sqrt{T}}$ with $d.f. = T-1$

β) t given by (19) but with $d.f. = T-1$

γ) t given by (19) but $d.f.$ given by (20).

The expected frequency distribution is rectangular and it is seen that test (γ) is the most satisfactory as confirmed by the χ^2 values also given in Table 3.

The author finally wishes to thank Prof. Herman Wold and Dr. C. R. Rao for guiding him in the foregoing investigations.

References

Sukhatme, P. V. (1943): Moments and product moments of moment-statistics for samples of finite and infinite populations. Sankhyā 6, 363-382.

Wold, Herman (1938): A study in the analysis of stationary time-series. *Dissertation, Stockholm*. Uppsala.

TABLE 1. VALUES OF USUAL AND CORRECTED 't's (d.f. 34)

model :–	A		B		C		D		E	
sample no.	usual	corrected	usual	corrected	usual	corrected	usual	corrected	usual	corrected
1	−2.431*	−1.082	+1.130	+0.844	+1.293	+0.633	+4.075***	+1.298	−0.200	−0.259
2	−1.309	−0.908	+0.283	+0.210	+0.488	+0.239	+4.628***	+1.443	+0.680	+0.853
3	+1.513	+1.046	+1.525	+1.129	−2.125*	−1.040	+1.338	+0.426	+0.982	+1.232
4	−0.951	−0.630	−3.284**	−2.432*	−1.060	−0.519	+0.209	+0.067	+0.687	+0.862
5	+1.824	+1.261	−2.545	−1.885	−2.658	−1.301	+0.710	+0.226	+1.155	+1.449
6	+0.745	+0.532	+0.184	+0.136	−1.901	−0.930	−2.648*	−0.844	+0.338	+0.424
7	−0.979	−0.607	+3.204**	+2.373*	−1.541	−0.754	−2.810	−0.895	−0.747	−0.937
8	−0.570	−0.395	+1.378	+1.021	−0.112	−0.055	+1.059	+0.529	+1.050	+1.317
9	−0.488	−0.340	−0.842	−0.609	−4.620***	−2.204*	+0.990	+0.315	+1.163	+1.460
10	+0.603	+0.416	+0.156	+0.118	−1.364	−0.667	+1.508***	+3.348	+1.534	+1.924
11	+1.382	+0.956	−1.281	−0.949	+2.478*	+1.213	−4.795***	−1.528	−0.762	−0.957
12	−0.155	−0.110	−0.725	−0.537	+1.172	+0.574	+0.487	+0.155	+1.146	+1.437
13	+1.744	+1.206	−0.177	−0.131	−3.096**	−1.515	−1.413	−0.450	−0.058	−0.073
14	−1.853	−1.282	+2.986**	+2.211*	−0.577	−0.282	+0.788	+0.251	−0.817	−1.025
15	−0.905	−0.626	−0.849	−0.629	+2.143*	+1.049	−3.407**	−1.105	−0.532	−0.668
16	+1.091	+0.755	+0.448	+0.332	−2.723*	−1.333	+2.904**	+0.925	+1.160	+1.455
17	+1.585	+1.093	−0.385	−0.285	+1.594	+0.780	−0.441	−0.141	+0.180	+0.234
18	−1.046	−0.724	+0.529	+0.392	−0.286	+0.140	−1.076	−0.343	−0.954	−1.197
19	−2.901**	−2.016	−0.367	−0.272	−2.820**	−1.380	−1.532	−0.488	−0.468	−0.587
20	+0.402	+0.277	−4.957***	−3.671***	+4.715***	+2.307*	−2.000	−0.637	+0.445	+0.558
21	−2.066*	−1.429	−0.929	−0.688	−3.067**	−1.501	+4.419***	+1.408	+0.190	+0.239
22	+0.217	+0.151	−0.403	−0.305	+0.620	+0.303	+4.789***	+1.526	+0.491	+0.615
23	+0.515	+0.356	+0.005	+0.004	+0.852	+0.417	+2.873**	+0.915	+0.448	+0.562
24	+2.736**	+1.886	−0.220	−0.170	−0.611	−0.299	−3.601**	−1.147	−1.113	−1.396
25	−0.739	−0.510	−1.020	−0.756	+0.476	+0.233	−1.864	−0.594	+2.552*	+3.256*

* Significant at 5% level for 34 degrees of freedom
** " 1% "
*** " 0.1% "

TABLE 2. FREQUENCY DISTRIBUTIONS OF t VALUES—USUAL AND CORRECTED (d.f. 34)

model :–	A		B		C		D		E		total	
probability level of t	usual	corrected	usual	corrected	usual	corrected	usual	corrected	usual	corrected	usual	corrected
.00–	4	1	3	3	6	1	7	—	—	—	20	5
.05–	1	2	—	—	3	4	2	—	—	1	6	7
.10–	—	1	1	—	1	1	1	2	1	1	4	5
.15–	4	1	2	1	—	2	—	1	1	3	7	8
.20–	1	2	3	2	—	1	—	1	2	—	6	6
.25–	1	2	—	3	2	1	—	2	2	2	5	10
.30–	1	2	1	—	—	1	1	2	1	—	4	5
.35–	—	1	2	3	1	2	—	1	—	—	3	7
.40–	1	—	2	2	—	1	—	1	1	1	4	5
.45–	—	1	—	—	1	1	—	—	1	1	2	3
.50–	—	—	1	2	—	—	—	1	—	—	1	3
.55–	1	1	2	2	1	2	1	2	2	2	7	9
.60–	—	1	1	1	—	1	—	2	1	—	2	5
.65–	2	2	2	1	2	1	1	2	3	1	10	7
.70–	1	1	—	—	2	2	—	1	1	3	4	7
.75–	1	1	—	1	2	1	2	—	1	—	6	3
.80–	—	2	—	1	1	1	1	1	2	2	4	7
.85–	1	3	1	1	2	1	—	1	4	2	8	8
.90–	3	—	2	—	—	—	2	4	1	4	8	8
.95–	3	1	2	2	1	1	7	1	1	2	14	7
total	25	25	25	25	25	25	25	25	25	25	125	125

Note: The usual 't' over-estimates for the first four schemes and under-estimates for the last scheme.

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TABLE 3: FREQUENCY DISTRIBUTION OF t-VALUES ARISING FROM TESTS OF DIFFERENT TYPES (SAMPLE SIZE 7)

model	A			B			C			D			E			total			χ^2		
probability level	(α)	(β)	(γ)	(α)	(β)	(γ)	(α)	(β)	(γ)	(α)	(β)	(γ)	(α)	(β)	(γ)	(α)	(β)	(γ)	(α)	(β)	(γ)
.00–	17	7	6	19	11	11	31	7	7	36	7	5	5	7	7	108	39	36	202.80	2.70	1.20
.05–	6	8	9	5	10	9	9	9	7	7	7	7	7	5	5	34	30	37	0.53	2.70	1.63
.10–	5	2	2	3	3	4	1	6	7	—	7	8	1	6	6	10	24	27	13.33	1.20	0.30
.15–	9	5	5	11	4	4	3	7	6	3	5	3	8	11	11	34	32	29	0.53	0.13	0.03
.20–	7	5	4	5	10	10	—	8	9	2	7	8	11	8	8	25	38	39	0.83	2.13	2.70
.25–	6	11	12	4	6	6	3	3	4	2	8	7	6	4	4	21	32	33	2.70	0.13	0.03
.30–	7	7	6	3	4	4	3	2	2	1	4	6	6	3	3	20	20	21	3.33	3.33	2.70
.35–	3	10	11	11	5	5	1	4	2	3	3	3	—	3	3	18	25	24	4.80	0.83	1.20
.40–	1	5	4	5	12	12	2	5	7	3	5	4	6	6	6	19	33	33	4.03	0.30	0.30
.45–	3	3	3	3	4	4	—	2	2	1	7	7	9	8	8	16	24	24	6.53	1.20	1.20
.50–	2	4	4	8	8	8	5	8	8	—	5	6	7	5	5	22	30	30	2.13	—	—
.55–	4	6	6	4	4	4	3	3	3	—	10	11	3	5	5	14	28	29	8.53	0.13	0.03
.60–	4	5	5	2	4	4	1	6	6	5	6	5	8	5	5	20	26	25	3.33	0.53	0.83
.65–	2	5	5	4	2	2	2	3	3	2	6	8	4	4	4	14	20	22	8.53	3.33	2.13
.70–	4	8	10	1	5	5	1	—	1	4	7	6	7	7	7	17	27	20	5.63	0.30	0.03
.75–	5	5	4	5	5	6	5	3	2	5	3	5	6	8	7	20	24	23	0.53	1.20	1.63
.80–	7	9	9	5	6	6	2	2	4	1	7	8	5	4	5	20	28	32	3.33	0.13	0.13
.85–	5	5	5	7	11	11	1	5	5	4	5	5	6	5	6	23	31	31	1.63	0.03	0.03
.90–	10	6	8	10	5	5	3	9	8	8	6	10	8	7	7	30	33	38	2.70	0.30	2.13
.95–	18	9	7	10	6	6	24	8	7	38	10	4	10	14	14	100	47	38	163.33	9.63	2.13
total	125	125	125	125	125	125	100	100	100	125	125	125	125	125	125	600	600	000	439.08	30.23	20.30