A UNIFIED APPROACH TO PROBLEMS OF SCATTERING OF SURFACE WATER WAVES BY VERTICAL BARRIERS

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(Received 20 October 1994; revised 1 December 1995)

Abstract

A unified analysis involving the solution of multiple integral equations *via* a simple singular integral equation with a Cauchy type kernel is presented to handle problems of surface water wave scattering by vertical barriers. Some well known results are produced in a simple and systematic manner.

1. Introduction

The problems of scattering of time-harmonic, two-dimensional surface water waves by vertical barriers (see Chakrabarti and Vijaya Bharathi [2], Evans [5], Mandal and Kundu [8], Ursell [12], etc.) give rise to an interesting class of mixed boundary-value problems involving the two-dimensional Laplace's equation, satisfied by the total velocity potential of the irrotational motion of an incompressible and inviscid fluid occupying a semi-infinite region, in which there exist some fixed obstacles in the form of plane rigid vertical barriers.

The first three such boundary-value problems, referred here as problems I, II and III, are the ones involving: (I) a fully submerged vertical barrier, (II) a partially immersed vertical barrier and (III) a fully submerged vertical plate. These three problems have been analysed by various workers (see Chakrabarti and Vijaya Bharathi [2], Vijaya Bharathi *et al.* [14], Evans [4], Porter [11] and Ursell [12], for example) and complete solution of these problems have been determined by employing different kinds of mathematical analysis.

The general problem, involving an infinite vertical barrier, with a finite number of gaps in it, extending from the surface of deep water to the bottom, which is at an infinite distance away from the surface, has been analysed by Mei [9], in certain specific circumstances.

In the present paper, we have made an attempt to present an unified approach to solve the first three problems I, II, and III, mentioned above, completely by first reducing them to multiple integral equations, involving a linear combination of the cosine and sine functions as their kernels, with the aid of the theory of Havelock's expansion (see Ursell [12]), and then converting each of these multiple integral equations to a single Cauchy-type singular integral equation of the first kind whose solutions are well known.

First we obtain the solutions of these three problems in their known forms. The general problem involving an infinite vertical barrier with a finite number of gaps in it is then treated.

Because of the fact that the final forms of the solution of the problems I, II and III are already known in the literature, we have tried to present these results in as brief a manner as possible, avoiding repetitions, highlighting mainly our unified approach which is applicable even to the general problem as mentioned above.

In Section 2, we have presented the mathematical formulation of all the problems handled in the present paper. In Section 3, we have reduced each of the boundary value problems considered here to a problem of solving Cauchy-type singular integral equations. In Section 4, we have presented solutions of problems I, II and III completely. In Section 5, the general problem involving a finite number of gaps in an infinite vertical barrier is treated.

2. Mathematical formulation

The problems (see Evans [4], Porter [11], Ursell [12], Mandal and Kundu [8], etc.) of the scattering of surface water waves by vertical barriers, in the linearised theory, are those of solving mixed two-dimensional boundary-value problems for Laplace's equation.

$$\frac{\partial^2\phi_j}{\partial x^2} + \frac{\partial^2\phi_j}{\partial y^2} = 0, \quad -\infty < x < \infty,\ y > 0, \tag{2.1}$$

with

$$\frac{\partial\phi_j}{\partial y} + \lambda\phi_j = 0, \quad \text{on } y = 0,\ (\lambda = \frac{\sigma^2}{g} > 0,\ \text{a constant}) \tag{2.2}$$

$$\frac{\partial\phi_j}{\partial x} = 0, \quad \text{on } x = 0,\ y \in L_j \tag{2.3}$$

for $j = 1, 2, 3$ and 4, where L_j represents the vertical barrier occupying the interval $a_j < y < b_j$. We note that for $j = 1, 2$ and 3 we obtain the three problems I, II, III respectively discussed in the introduction. We use $j = 1$ for problem I, with $a_1 = a, b_1 = \infty$, $j = 2$ for problem II with $a_2 = 0, b_2 = b$ and $j = 3$ with $a_3 = c, b_3 = d$ for problem III, where a, b, c, d represent known positive constants. For the general problem under consideration we use $j = 4$, with $L_4 = (0, a_1)\cup(b_1, a_2)\cup(b_2, a_3)\cdots\cup(b_n, \infty)$, such that $0 < a_1 < b_1 < a_2 < \cdots < b_n < \infty$.

Along with the above equation and conditions, we shall also have to allow the following requirements for the functions ϕ_j ($j = 1, 2, 3, 4$):

$$\begin{aligned} \phi_j &\sim T_j e^{i\lambda x - \lambda y}, && \text{as } x \to \infty, \\ \phi_j &\sim e^{i\lambda x - \lambda y} + R_j e^{-i\lambda x - \lambda y}, && \text{as } x \to -\infty, \end{aligned}$$

where T_j, R_j are unknown complex constants with $j = 1, 2, 3, 4$, $i^2 = -1$ and

$$\operatorname{grad}\phi_1 = \mathrm{O}((r-a)^{-1/2}), \quad \text{as } r = (x^2+y^2)^{1/2} \to a,\ x \to 0, \qquad (2.5(i))$$

for problem I. For problem II we have

$$\operatorname{grad}\phi_2 = \begin{cases} \mathrm{O}((r-b)^{-1/2}), & \text{as } r \to b,\ x \to 0, \\ \text{bounded}, & \text{as } r \to 0, \end{cases} \qquad (2.5(ii))$$

and for problem III

$$\operatorname{grad}\phi_3 = \begin{cases} \mathrm{O}((r-c)^{-1/2}), & \text{as } r \to c,\ x \to 0 \\ \mathrm{O}((r-d)^{-1/2}), & \text{as } r \to d,\ x \to 0. \end{cases} \qquad (2.5(iii))$$

For the general problem, for which $j = 4$, we require that

$$\operatorname{grad}\phi_4 = \begin{cases} \text{bounded}, & \text{as } r \to 0, \\ \mathrm{O}((r-e_\alpha)^{-1/2}), & \text{as } r \to e_\alpha,\ x \to 0, \end{cases} \qquad (2.5(iv))$$

with e_α representing the edges $e_1 = a_1, e_2 = b_1, e_3 = a_2, \cdots, e_{2n-1} = a_n$ and $e_{2n} = b_n$. Also, in all these problems the conditions to be met with as $y \to \infty$ are that

$$\operatorname{grad}\phi_j \to 0 \quad \text{as } y \to \infty. \qquad (2.6)$$

The constants T_j and R_j, occuring in (2.4) are unknown along with the functions ϕ_j, for each j and this aspect of the boundary-value problems under consideration has created considerable interest in the study of scattering of water waves (see Chakrabarti and Vijaya Bharathi [2], Evans [5], Porter [11], etc.).

The functions ϕ_j ($j = 1, 2, 3, 4$) appearing here represent the total velocity potentials in respect of the irrotational fluid motion corresponding to the various scattering problems. The function $e^{i\lambda x - \lambda y}$ (dropping the time dependent factor $e^{-i\sigma t}$

throughout, where σ represents the circular frequency) stands for the incident plane wave propagating from negative infinity in the x-direction and hitting the vertical barriers L_j ($j = 1, 2, 3, 4$), ($\lambda = \frac{\sigma^2}{g}$ represents the wave number, g being the acceleration due to gravity) occupying the y-axis, where the usual two-dimensional Cartesian co-ordinate system has been employed. The complex constants R_j and T_j ($j = 1, 2, 3, 4$) represent the "reflection" and "transmission" coefficients associated with the problems.

Without going into any more details of the physical problems leading to the above three boundary-value problems, for which the reader is referred to the original work of Ursell [12], as well as the article by Evans [5], we shall present in the next section a unified approach involving solutions of multiple-integral equations to solve these problems.

3. Reduction to multiple-integral equations and singular-integral equations

We start by representing the solutions of the p.d.e. (2.1) satisfying the conditions (2.2), (2.4) and (2.6) in the forms

$$\phi_j = \begin{cases} T_j e^{i\lambda x - \lambda y} + \int_0^\infty A_j(k) L(k, y) e^{-kx}\, dk, & x > 0, \\ e^{i\lambda x - \lambda y} + R_j e^{-i\lambda x - \lambda y} + \int_0^\infty B_j(k) L(k, y) e^{kx}\, dk, & x < 0,\ j = 1, 2, 3, \end{cases} \tag{3.1}$$

with

$$L(k, y) = k \cos ky - \lambda \sin ky, \tag{3.2}$$

where A_j and B_j are certain unknown functions to be determined as described below.

Continuity of $\dfrac{\partial \phi_j}{\partial x}$ across $x = 0$ provides, by using Havelock's expansion theorem (see Ursell [12]), that

$$A_j(k) = -B_j(k) \quad \text{and} \quad T_j + R_j = 1 \quad \text{for } j = 1, 2, 3, 4. \tag{3.3}$$

Now if we assume that the functions ϕ_j are continuous across the gaps $G_j =: (0, \infty) - L_j$, on the line $x = 0$, we find, by using the solutions (3.1) along with (3.3) that we must have

$$\int_0^\infty A_j(k) L(k, y)\, dk = R_j e^{-\lambda y}, \quad y \in G_j,\ j = 1, 2, 3, 4. \tag{3.4}$$

Also using the boundary conditions (2.3) once again, we obtain the relations

$$\int_0^\infty k A_j(k) L(k, y)\, dk = i\lambda(1 - R_j) e^{-\lambda y}, \quad y \in L_j,\ j = 1, 2, 3, 4. \tag{3.5}$$

Equations (3.4) and (3.5) can be simplified by applying the operator $\frac{d}{dy} + \lambda$ formally to both sides (see Ursell [12]), to the following multiple-integral equations for the determination of the functions A_j:

$$\left.\begin{aligned} &\int_0^\infty A_j(k)(k^2+\lambda^2)\sin ky\,dk = 0, \quad y \in G_j \\ \text{and} \quad &\int_0^\infty A_j(k)k(k^2+\lambda^2)\sin ky\,dk = 0, \quad y \in L_j, \end{aligned}\right\} \quad (j = 1, 2, 3, 4). \tag{3.6}$$

If we next set

$$F_j(k) = (k^2+\lambda^2)A_j(k), \tag{3.7}$$

we can express the relations (3.6) in the equivalent form

$$\left.\begin{aligned} &\int_0^\infty F_j(k)\sin ky\,dk = 0, \quad y \in G_j \\ \text{and} \quad &\frac{d}{dy}\int_0^\infty \frac{F_j(k)}{k}\sin ky\,dk = C_j, \quad y \in L_j, \end{aligned}\right\} \quad j = 1, 2, 3, 4, \tag{3.8}$$

where the C_js are arbitrary constants of integration. We find for problems I, II and III, corresponding to the values $j = 1, 2$ and 3, we must have that

$$C_1 = 0 \text{ with } C_2 \text{ and } C_3 \text{ arbitrary.} \tag{3.9}$$

In the case of the general problem, for which $j = 4$, we will have to interpret the number C_4 as representing different constants for different positions of the barrier L_4, that is, we must use the fact that

$$C_4 = C_4^{(k)}, \quad \text{for } y \in L_4^{(k)},\ k = 1, 2, \cdots, n+1,$$

with the understanding that

$$L_4^{(1)} = 0 < y < a_1, \quad L_4^{(2)} = b_1 < y < a_2, \quad L_4^{(3)} = b_2 < y < a_3, \cdots$$
$$L_4^{(n-1)} = b_{n-2} < y < a_{n-1}, \quad L_4^{(n)} = b_{n-1} < y < a_n \quad \text{and} \quad L_4^{(n+1)} = b_n < y < \infty.$$

We must also bear in mind that $C_4^{(n+1)} = 0$.

Then (3.8) constitute the desired multiple-integral equations for the boundary value problems under consideration. Setting

$$h_j(y) = \int_0^\infty F_j(k)\sin ky\,dk, \quad y \in (0, \infty),\ j = 1, 2, 3 \tag{3.10}$$

and using the Fourier sine inversion formula on (3.10) and the first of the relations (3.8), we obtain that

$$F_j(k) = \frac{2}{\pi} \int_{L_j} h_j(y) \sin ky \, dy. \tag{3.11}$$

Then, substituting (3.11) into the second of the relations (3.8) and using the well-known result

$$\int_0^\infty \frac{\sin kt \sin ky}{k} \, dk = \ln \left| \frac{y+t}{y-t} \right|, \tag{3.12}$$

we derive the following singular-integral equations for the determination of the functions $h_j(y)$:

$$\int_{L_j} \frac{2t h_j(t) \, dt}{t^2 - y^2} = \frac{\pi}{2} C_j, \quad y \in L_j \ (j = 1, 2, 3, 4). \tag{3.13}$$

In order to solve the singular-integral equations (3.13) by using standard results available in Muskhelishvili [10] and Gakhov [6], we must have definite ideas about the behaviour of the functions $h_j(t)$ at the end points of the segments L_j under consideration and, for that purpose, we observe the following fact.

Denoting by $f_j(y)$ the function defined by

$$f_j(y) = \phi_j(+0, y) - \phi_j(-0, y), \quad (j = 1, 2, 3, 4), \tag{3.14}$$

we find, on using (3.1), in conjunction with (3.3), that

$$\int_0^\infty A_j(k) L(k, y) \, dk = \frac{1}{2} f_j(y) + R_j e^{-\lambda y}, \quad \text{for } y \in L_j, \tag{3.15}$$

whilst $f_j(y) = 0$ for $y \in G_j$, because of the continuity of ϕ_j across G_j.

Applying the operator $\dfrac{d}{dy} + \lambda$ to both sides of (3.15), we then find that

$$\int_0^\infty A_j(k)(k^2 + \lambda^2) \sin ky \, dk = \frac{1}{2} \left[\frac{df_j}{dy} + \lambda f_j \right], \quad \text{for } y \in L_j \tag{3.16}$$

and this, along with (3.7) and (3.10), suggest that

$$2h_j(y) = \frac{df_j}{dy} + \lambda f_j. \tag{3.17}$$

Thus, if the conditions (2.5(i), (ii), (iii), (iv)) and (2.6) are utilized, we observe that we must have

$$
h_1(t) = \begin{cases} O(|t-a|^{-1/2}), & \text{as } t \to a, \\ \to 0, & \text{as } t \to \infty, \end{cases}
$$
$$
h_2(t) = \begin{cases} O(|t-b|^{-1/2}), & \text{as } t \to b, \\ \text{bounded}, & \text{as } t \to 0, \end{cases}
$$
$$
h_3(t) = \begin{cases} O(|t-c|^{-1/2}), & \text{as } t \to c, \\ O(|t-d|^{-1/2}), & \text{as } t \to d, \end{cases} \tag{3.18}
$$
$$
h_4(t) = \begin{cases} O(|t-a_1|^{-1/2}), & \text{as } t \to a_1, \\ \text{bounded}, & \text{as } t \to 0, \\ O(|t-a_k|^{-1/2}), & \text{as } t \to a_k, k = 1, 2, \cdots, n, \\ O(|t-b_k|^{-1/2}), & \text{as } t \to b_k, k = 1, 2, \cdots, n, \\ 0, & \text{as } t \to \infty. \end{cases}
$$

These conditions (3.18) then settle the end conditions to be met with by the solutions of the singular-integral equations (3.13), which can be determined by using the results available in Muskhelishvili [10].

In the next two sections, we shall present the solutions of (3.13) and determine serially the complete solutions of all the boundary-value problems of concern here.

4. The full solutions of problems I, II, and III

The full solutions of problems I, II, III can be determined once the solutions of the singular-integral equations (3.13) are obtained, for $j = 1, 2$ and 3 respectively (see Muskhelishvili [10]).

We find that

$$
h_1(t) = \frac{D_1}{(t^2 - a^2)^{1/2}}, \quad \text{for } t \in L_1 \ (D_1 \text{ an arbitrary constant}), \tag{4.1}
$$

$$
h_2(t) = \frac{C_2 t}{\pi (b^2 - t^2)^{1/2}}, \quad \text{for } t \in L_2 \tag{4.2}
$$

and

$$
h_3(t) = -\frac{1}{\pi((t^2 - c^2)(d^2 - t^2))^{1/2}} \left[D_3 - C_3 \int_c^d \frac{v((v^2 - c^2)(d^2 - v^2))^{1/2}}{v^2 - t^2} \, dv \right]
$$
$$
\text{for } t \in L_3. \tag{4.3}
$$

Here C_3, D_3 are arbitrary constants, with C_3 being the same as before and D_3 is a new constant.

In order to derive the complete solution for problem III, for $j = 3$, we first recast the function $h_3(t)$ given by (4.3) into the equivalent form

$$h_3(t) = \frac{1}{((t^2 - c^2)(d^2 - t^2))^{1/2}} \left[-\frac{C_3}{2} t^2 - \frac{1}{\pi} \left(D_3 - \frac{C_3 \pi}{4} (c^2 + d^2) \right) \right] \quad (4.4)$$

by the standard evaluation of singular integrals. That is,

$$h_3(t) = \frac{C_3}{2} \frac{(d_0^2 - t^2)}{X(t)}, \quad (4.5)$$

with

$$d_0^2 = \frac{2}{\pi C_3} \left[\frac{C_3 \pi}{4} (d^2 + c^2) - D_3 \right], \quad (4.6)$$

where

$$X(t) = [(t^2 - c^2)(d^2 - t^2)]^{1/2}. \quad (4.7)$$

When the expressions (4.1), (4.2) for the functions h_j ($j = 1, 2$) and (4.5) for $h_3(t)$ are utilised in (3.11) and (3.7) along with the results of certain standard integrals available in Gradshteyn and Rhyzik [7], we derive that

$$\begin{aligned} A_1(k) &= \frac{D_1 J_0(ka)}{k^2 + \lambda^2}, \\ A_2(k) &= -\frac{C_2 b J_1(bk)}{2(k^2 + \lambda^2)} \\ \text{and} \quad A_3(k) &= \frac{C_3}{\pi(k^2 + \lambda^2)} \int_c^d \frac{d_0^2 - y^2}{X(y)} \sin ky \, dy. \end{aligned} \quad (4.8)$$

Substituting for A_j for $j = 1, 2, 3$ from (4.8) into (3.4) and (3.5), we derive that

$$R_1 = D_1 K_0(\lambda a), \quad \text{with } D_1 = \frac{1}{K_0(\lambda a) + i\pi I_0(\lambda a)}, \quad (4.9)$$

$$R_2 = \frac{C_2 b}{2} \pi I_1(\lambda b), \quad \text{with } C_2 b = \frac{2}{\pi I_1(\lambda b) + i K_1(\lambda b)}, \quad (4.10)$$

and

$$\begin{aligned} R_3 &= -C_3 \gamma_0, \\ \text{with} \quad \gamma_0 &= \int_c^d \frac{d_0^2 - u^2}{X(u)} e^{-\lambda u} \, du, \\ d_0^2 &= \int_c^d \frac{u^2 e^{\lambda u} \, du}{X(u)} \Big/ \int_c^d \frac{e^{\lambda u} \, du}{X(u)}, \\ \text{and} \quad C_3 &= \frac{i}{\alpha_0 - \beta_0 - i\gamma_0}, \end{aligned} \quad (4.11)$$

where

$$\begin{aligned}\alpha_0 &= \int_{-c}^{c} \frac{(d_0^2 - u^2)e^{-\lambda u}\, du}{\sqrt{(c^2 - u^2)(d^2 - u^2)}}, \\ \beta_0 &= \int_{d}^{\infty} \frac{(d_0^2 - u^2)e^{-\lambda u}\, du}{\sqrt{(u^2 - c^2)(u^2 - d^2)}}.\end{aligned} \tag{4.12}$$

This completes the description of the unified method to solve problems I, II and III.

5. The general problem

In the case of the general problem the singular-integral equation to be solved is given by (3.13), to be satisfied by the function $h_4(t)$, along with the edge requirements (3.18).

Borrowing results available in Banerjea [1] we may write down the solution $h_4(t)$ in the form

$$h_4(t) = \frac{(-1)^{n+1-r}}{Q(t)} \sum_{k=1}^{n} \left[p_{k-1} t^{2k-1} - \frac{2t}{\pi} C_4^{(k)} F_k(a_k, b_{k-1}, t) \right] \tag{5.1}$$

$$\text{for } t \in B_r \equiv (b_{r-1}, a_r),\ r = 1, 2, \cdots, n+1,\ (b_0 = 0),$$

where

$$F_k(a_k, b_{k-1}, t) = (-1)^{n+1-k} \int_{b_{k-1}}^{a_k} \frac{Q(x)\, dx}{x^2 - t^2}, \tag{5.2}$$

with $p_0, p_1, \cdots, p_{n-1}, C_4^{(1)}, C_4^{(2)}, \cdots, C_4^{(n)}$ $2n$ unknown constants and

$$Q(t) = \prod_{j=1}^{n} \sqrt{(t^2 - a_j^2)(t^2 - b_j^2)}. \tag{5.3}$$

From (3.7) and (3.11) we derive that

$$A_4(k) = \frac{2}{\pi} \frac{1}{k^2 + K^2} \sum_{j=1}^{n} \int_{B_j} h_4(t) \sin(kt)\, dt, \quad \text{for } k > 0. \tag{5.4}$$

Substituting for $A_4(k)$ in (3.4) for $j = 4$ and assuming $y \in (a_r, b_r)$, $r = 1, 2, \cdots, n$, we derive that

$$\begin{aligned}R_4 = &\sum_{j=1}^{n+1} (-1)^{n+1-j} \int_{b_{j-1}}^{a_j} \psi(t) e^{-\lambda t}\, dt \\ &- \sum_{j=1}^{r} (-1)^{n+1-j} \int_{b_{j-1}}^{a_j} \psi(t) e^{\lambda t}\, dt, \quad r = 1, 2, \cdots, n,\end{aligned} \tag{5.5}$$

where

$$\psi(t) = \sum_{k=1}^{n} \left[p_{k-1} t^{2k-1} - \frac{2t}{\pi} C_4^{(k)} F_k(a_k, b_{k-1}, t) \right] \frac{1}{Q(t)}. \tag{5.6}$$

Further substituting for $A_4(k)$ in (3.5), assuming that $y \in (b_{r-1}, a_r)$ we derive that

$$\begin{aligned} i(1 - R_4) = {} & \sum_{j=1}^{n} (-1)^{n+1-j} \int_{a_j}^{b_j} \psi(t) e^{-\lambda t} + \sum_{j=1}^{n-1} \frac{C_4^{(j+1)}}{\lambda} e^{-\lambda b_j} - \sum_{j=1}^{n} \frac{C_4^{(j)}}{\lambda} e^{-\lambda a_j} \\ & + \sum_{j=1}^{r-1} (-1)^{n+1-j} \int_{a_j}^{b_j} \psi(t) e^{\lambda t}\, dt - \sum_{j=1}^{r-1} \frac{C_4^{(j+1)}}{\lambda} e^{\lambda b_j} + \sum_{j=1}^{r-1} \frac{C_4^{(j)}}{\lambda} e^{\lambda a_j}. \end{aligned} \tag{5.7}$$

Again, substituting for $A_4(k)$ in (3.5) and assuming $y \in (b_r, a_{r+1})$, we obtain that

$$\begin{aligned} i(1 - R_4) = {} & \sum_{j=1}^{n} (-1)^{n+1-j} \int_{a_j}^{b_j} \psi(t) e^{-\lambda t} + \sum_{j=1}^{n-1} \frac{C_4^{(j+1)}}{\lambda} e^{-\lambda b_j} - \sum_{j=1}^{n} \frac{C_4^{(j)}}{\lambda} e^{-\lambda a_j} \\ & + \sum_{j=1}^{r} (-1)^{n+1-j} \int_{a_j}^{b_j} \psi(t) e^{\lambda t}\, dt - \sum_{j=1}^{r} \frac{C_4^{(j+1)}}{\lambda} e^{\lambda b_j} + \sum_{j=1}^{r} \frac{C_4^{(j)}}{\lambda} e^{\lambda a_j}. \end{aligned} \tag{5.8}$$

From (5.7) and (5.8) we obtain that

$$(-1)^{n+1-r} \int_{a_r}^{b_r} \psi(t) e^{\lambda t}\, dt - \frac{C_4^{(r+1)}}{\lambda} e^{\lambda b_r} + \frac{C_4^{(r)}}{\lambda} e^{\lambda a_r} = 0, \quad r = 1, 2, \cdots, n. \tag{5.9}$$

Thus, from (5.8), we obtain another expression for R_4, namely

$$\begin{aligned} R_4 = 1 + i & \sum_{j=1}^{n} (-1)^{n+1-j} \int_{a_j}^{b_j} \psi(t) e^{-\lambda t}\, dt \\ & - i \sum_{j=1}^{n} \frac{C_4^{(j)}}{\lambda} e^{-\lambda a_j} + \sum_{j=1}^{n-1} \frac{C_4^{(j+1)}}{\lambda} e^{-\lambda b_j}. \end{aligned} \tag{5.10}$$

We have in total $(2n+1)$ unknown constants namely $p_0, p_1, \cdots, p_{n-1}, C_4^{(1)}, C_4^{(2)}, \cdots, C_4^{(n)}$ and R_4. (5.9) gives a system of n equations. Further (5.5) and (5.10) together give rise to $(n+1)$ equations. Thus we have a system of $2n+1$ equations for the determination of the $2n+1$ constants which can be solved by standard methods to obtain the full solution.

6. Conclusion

The theme of the present paper has been to utilize a unified approach involving the solution of dual-, triple- and multiple-integral equations to solve the four problems considered.

Acknowledgement

The authors acknowledge the referees for their comments and suggestions to bring the paper to this present form. S. Banerjea acknowledges the support received from the IISc-TIFR Mathematics programme for her visit to Indian Institute of Science, Bangalore, during the summer of 1994 during which this work was carried out. S. Banerjea also acknowledges partial support from DST under young scientist scheme. B. N. Mandal acknowledges the support of IISc, for his visit there during the summer of 1994. T. Sahoo acknowledges the University Grants Commission, New Delhi, for receiving financial support as a research student of Indian Institute of Science, Bangalore.

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