AN EXAMPLE OF NON-EXISTENCE OF A MINIMUM VARIANCE ESTIMATOR

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An unbiased estimator t of a population characteristic θ is said to be a minimum variance (or efficient) estimator (m.v.e.) of θ if in the class of all unbiassed estimators of θ the variance of t is uniformly the lowest for all θ. In this note we construct a simple example to show that minimum variance estimators do not exist in many situations.

Let $x=(x_1,x_2,\ldots x_n)$ be a random sample from a population with probability density

$$p(u)du=\text{const}\quad e^{-(u-\theta)^4}du \qquad \ldots \quad (1)$$

$$-\infty<u<\infty \quad -\infty<\theta<\infty$$

The joint probability density of the sample is

$$P(x|\theta)dx = \text{const}\; e^{-\Sigma(x_i-\theta)^4}dx$$

$$= \text{const}\; e^{-\Sigma x_i^4+4\theta\Sigma x_i^3-6\theta^2\Sigma x_i^2+4\theta^3\Sigma x_i-n\theta^4}dx$$

so that Σx_i, Σx_i^2 and Σx_i^3 are shared sufficient statistics for θ. This, therefore, is not a situation (Rao 1952) where we may expect the existence of a m.v.e. for all estimable functions $\tau(\theta)$. As a matter of fact we now give an indirect proof of the non-existence of a m.v.e. for

$$\tau(\theta)=Ex_i^3=\theta^3+3\mu_2\theta \qquad \ldots \quad (2)$$

where μ_2 is the central second moment of x_i.

Let
$$t^0 = \left.\frac{P'_\theta(x|\theta)}{4nP(x|\theta)}\right|_{\theta=0} = \frac{1}{n}\Sigma x_i^3$$

clearly
$$E\,t^0=\theta^3+3\mu_2\theta.$$

Since at the point $\theta=0$ the estimator t^0 has zero covariance with all statistics z such that

$$E(z|\theta)=0 \text{ and } V(z|\theta)<\infty \text{ for all } \theta \qquad \ldots \quad (3)$$

it follows from Rao (1952) that t^0 has minimum variance at $\theta=0$. Hence if there exist a m.v.e for (2) then that must be the same (excepting possibly for a set of Lebesgue measure zero) as t^0 (Rao 1952).

Now, since θ is a location parameter here, it can be easily seen that if $t(x_1,x_2,\ldots x_n)$ be a m.v.e. for $\tau(\theta)$ then $t(x_1+\lambda,\ldots x_n+\lambda)$ is a m.v.e. for $\tau(\theta+\lambda)$ for all real λ.

Also it is easily proved that if $t_1, t_2, \ldots t_k$ are m.v.o.'s for $\tau_1, \tau_2, \ldots \tau_k$ respectively then any linear function of the t's is the m.v.o. for the corresponding function of the τ's.

Thus if t^0 be the m.v.o. for $\theta^3+3\mu_2\theta$ then $t^{-1} = \frac{1}{n}\Sigma(x_i-1)^3$ is the m.v.o. for $(\theta-1)^3+3\mu_2(\theta-1)$ and $t^1 = \frac{1}{n}\Sigma(x_i+1)^3$ is the m.v.o. for $(\theta+1)^3+3\mu_2(\theta+1)$

$$\bar{x} = \frac{1}{6}(t^{-1}-2t^0+t^1) \text{ is the m.v.o. for } \theta.$$

We now show that $\bar{x}$ is not the m.v.o. for θ and thus prove that there cannot exist an m.v.o. for (2).

Consider the statistic

$$t = \frac{1}{2n(n-1)\mu_2}[n\Sigma x_i^3 - \Sigma x_i \Sigma x_i^2]$$

$$= \frac{1}{2n(n-1)\mu_2}[(n-1)\Sigma x_i^3 - \Sigma x_i x_j^2]$$

Now $$Ex_i^2 = \theta^2+\mu_2 \text{ and } Ex_i^3 = \theta^3+3\mu_2\theta$$

$$\therefore E(t\,|\,\theta) = \frac{1}{2n(n-1)\mu_2}[n(n-1)(\theta^3+3\mu_2\theta)-n(n-1)\theta(\theta^2+\mu_2)]$$
$$= \theta$$

Let us compute the variance of t at $\theta=0$

$$V(t|\theta=0) = \frac{1}{n.4\mu_2^2}\left\{\mu_6+\frac{\mu_2\mu_4}{n-1}+\frac{n-2}{n-1}\mu_2^3-2\mu_2\mu_4\right\}$$

Now $$V(\bar{x}|\theta) = \frac{\mu_2}{n}.$$

$\therefore$ the variance of t at $\theta=0$ will be smaller than that of $\bar{x}$ if

$$\mu_6+\frac{\mu_2\mu_4}{n-1}+\frac{n-2}{n-1}\mu_2^3-2\mu_2\mu_4 < 4\mu_2^3 \qquad \ldots \quad (4)$$

Now $$\mu_2 = \frac{\Gamma(3/4)}{\Gamma(1/4)} = .338,\ \mu_4 = \frac{\Gamma(5/4)}{\Gamma(1/4)} = 1/4,$$

$$\mu_6 = \frac{\Gamma(7/4)}{\Gamma(1/4)} = 3/4.\mu_2$$

$\therefore$ the inequality (4) will hold if

$$\frac{1}{n-1}(1/4-\mu_2^2) < 3\mu_2^2-1/4$$

or if $$\frac{1}{n-1} < .68$$

which is so if $n>2$.

Thus $V(t)<V(\bar{x})$ at $\theta=0$ for all $n>2$ and thus our proposition is proved.

REFERENCE

RAO, C. R. (1952): Some theorems on minimum variance estimation. *Sankhyā*, 12, Parts 1 & 2.