# VERTICAL BLOCK HIDDEN $Z$-MATRICES AND THE GENERALIZED LINEAR COMPLEMENTARITY PROBLEM*

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**Abstract.** In this paper we introduce vertical block hidden $Z$-matrices and study their minimality and complementarity properties.

**Key words.** generalized linear complementarity problem, vertical block $Z$-matrices, vertical block hidden $Z$-matrices, VLCP, least element

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**1. Introduction.** Given a square matrix $M$ of order $n$ and a $q \in R^n$ the linear complementarity problem is to find $w \in R^n$ and $z \in R^n$ such that

$$w - Mz = q, \ w \geq 0, \ z \geq 0, \tag{1.1}$$

$$w^t z = 0. \tag{1.2}$$

The linear complementarity problem is well studied in the literature. For the latest books see Cottle, Pang, and Stone [2] and Murty [18]. In [14], Lemke proposes an algorithm which either computes a solution to the linear complementarity problem or shows that there is no solution to (1.1) and (1.2). We call a square matrix $M = ((m_{ij}))$ of order $n$ a $Z$-matrix or say that $M \in Z$ if $m_{ij} \leq 0, \ i \neq j, \ 1 \leq i \leq n, \ 1 \leq j \leq n$. The linear complementarity problem with a $Z$-matrix has a number of applications. See [2] and [26]. $Z$-matrices have a least element property related to their complementarity property which has been observed by Cottle and Veinott [3].

The generalized linear complementarity problem with a vertical block matrix of order $m \times k$ was introduced by Cottle and Dantzig [1]. Their statement of this problem is as follows: Given an $m \times k$ $(m \geq k)$ vertical block matrix $M$ of type $(m_1, m_2, \ldots, m_k)$ and $q \in R^m$ where $m = \sum_{j=1}^{k} m_j$, find $w \in R^m$ and $z \in R^k$ such that

$$w - Mz = q, \ w \geq 0, \ z \geq 0, \tag{1.3}$$

$$z_j \prod_{i=1}^{m_j} w_i^j = 0, \quad j = 1, 2, \ldots, k. \tag{1.4}$$

This problem is denoted as VLCP$(q, M)$.

Cottle and Dantzig [1] extended Lemke's algorithm to solve the above problem. They have also extended some of the properties of the square $P$-matrix to the vertical block $P$-matrix.

VLCP, or the vertical linear complementarity problem, has not been studied extensively until recently, although Lemke [15] as early as 1970 anticipated valuable applications of this problem. Recently, a number of applications of this problem have been noted in the literature. In [6], Ebiefung and Kostreva introduce a generalized Leontief input-output linear model and formulate it as a VLCP. This model can be effectively used for the problem of choosing a new technology and also for solving problems related to energy commodity demands, international trade, multinational army personnel assignment, and pollution control. In [12], Gowda and Sznajder introduce a generalized bimatrix game and formulate a special case of this as a VLCP. A slightly more general form of the VLCP also occurs in control theory [21], [22].

There have also been other generalizations of the linear complementarity problem motivated by certain other applications. The horizontal linear complementarity problem arises in nonlinear networks. See [8], [9], and [27]. Oh [19] has formulated a mixed lubrication problem as a generalized nonlinear complementarity problem.

The VLCP has been studied by Szanc [23]. A more general version in the setting of a finite dimensional lattice gives the generalized order linear complementarity problem studied by Gowda and Sznajder [11]. However, when specialized to $R^n$, the generalized order linear complementarity problem is seen to be equivalent to the VLCP. See Gowda and Sznajder [11]. Generalizations of $P_0$- and $Z$-matrices have been studied by Ebiefung and Kostreva [5] and Sznajder and Gowda [25]. See also [7] and [24]. The extended generalized order linear complementarity problem was considered by Goeleven [10], Gowda and Sznajder [11], and Isac and Goeleven [13].

Mangasarian [16] while studying the classes of linear complementarity problems solvable by a single linear program introduced a class of matrices which later came to be named as the class of hidden $Z$-matrices in [20].

A square matrix $M$ of order $n$ is called a hidden $Z$-matrix if there exist square matrices of order $n$, $X$ and $Y$, $X \in Z$, $Y \in Z$ such that (i) $MX = Y$ and (ii) there exist nonnegative vectors $r$, $s \in R^n$ such that $r^t X + s^t Y > 0$.

The class of hidden $Z$-matrices also possesses a least element property which is related to complementarity. For a study of this property see [2]. The least element theory for hidden $Z$-matrices was motivated by the observation of Mangasarian [16] that the linear complementarity problem with a hidden $Z$-matrix can be solved as a single linear programming problem. For related results see also [17].

Recently, Ebiefung and Kostreva [4] have studied the generalized linear complementarity problem with a vertical block $Z$-matrix. Complementarity and least element properties and a computational scheme using principal pivoting were studied in this paper.

The present work is motivated partly by a question which naturally arises from the work of Ebiefung and Kostreva [4] and Mangasarian [16, 17]: what is the largest class of vertical block matrices for which the associated VLCP has the least element property and hence can be solved as a single linear programming problem? This also has an implication for the class of VLCPs which has polynomial time complexity. Surprisingly, unlike in the generalization of other properties of square matrices, the required generalization of the hidden $Z$-property does not depend upon the representative submatrices. We introduce the class of vertical block hidden $Z$-matrices and study the associated minimality and complementarity properties.

In section 2, we present the required notations and definitions. In section 3, we study the least element and complementarity property possessed by vertical block hidden $Z$-matrices. In section 4, we present some characterization theorems for vertical block hidden $K$-matrices.

**2. Definitions and notation.** By writing $A \in R^{m\times n}$, we denote that $A$ is a matrix of real entries with $m$ rows and $n$ columns. For any matrix $A \in R^{m\times n}$, $a_{ij}$ denotes the $i$th row $j$th column entry and $\mathrm{Pos}(A)$ denotes the nonnegative cone generated by columns of $A$. If $A \in R^{m\times n}$ and $J \subseteq \{1,2,\ldots,m\}$, $A_J$ denotes the submatrix of $A$ consisting of the rows of $A$ whose indices are in $J$. $A_{\cdot i}$ denotes the $i$th column and $A_{i\cdot}$, the $i$th row of $A$. If $A \in R^{m\times n}$, $J_1 \subseteq \{1,2,\ldots,m\}$ and $J_2 \subseteq \{1,2,\ldots,n\}$, then $A_{J_1J_2}$ denotes the submatrix of $A$ consisting of only the rows and columns of $A$ whose indices are in $J_1$ and $J_2$, respectively. Any vector $x \in R^n$ is a column vector unless otherwise specified. $x^t$ denotes the transpose of $x$. For any two vectors $x, y \in R^n$, we define $\min(x,y)$ as the vector whose $i$th coordinate is $\min(x_i,y_i)$. Let $M$ be a vertical block matrix of order $m \times k$ and type $(m_1,\ldots,m_k)$ and $q \in R^m$ be given. The set $\mathrm{FEA}(q,M) = \{(w,z) \,|\, w \in R^m,\ z \in R^k,\ (w,z) \text{ satisfies } (1.3)\}$ is called the *feasible region* of VLCP$(q,M)$ and any vector in FEA$(q,M)$ is called a *feasible vector*.

Let $C$ be a convex cone. We say that $C$ is a *pointed* convex cone if $C$ does not contain any linear subspace except $\{0\}$. If $C$ is a pointed convex cone in $R^n$, $C$ induces a partial ordering of vectors in $R^n$ defined as follows : $x \preceq (C)\, y$ if $y - x \in C$. We call this partial ordering the *cone ordering* induced by $C$. In particular, in this paper we consider the cone ordering induced by C where $C = \mathrm{Pos}(X)$ for some nonsingular $X$.

A matrix $M \in R^{n\times n}$ is said to be a $P_0$-matrix ($P$-matrix) if all its principal minors are nonnegative (positive). Such a matrix is called a $K$-matrix if it is both a $Z$- and a $P$-matrix.

DEFINITION 2.1. *Consider a rectangular matrix $M \in R^{m\times k}$ with $m \geq k$. Suppose $M$ is partitioned row-wise into $k$ blocks in the form*

$$M = \begin{bmatrix} M^1 \\ M^2 \\ \vdots \\ M^k \end{bmatrix},$$

*where each $M^j = ((m^j_{rs})) \in R^{m_j\times k}$ with $\sum_{j=1}^k m_j = m$. Then $M$ is called a* vertical block matrix of type $(m_1, m_2, \ldots, m_k)$.

DEFINITION 2.2. *A submatrix of size $k$ of $M$ is called a* representative submatrix *if its $j$th row is drawn from the $j$th block $M^j$ of $M$.*

*Remark* 2.1. If $m_j = 1$, $j = 1,\ldots,k$, then $M$ is a square matrix. Thus, a vertical block matrix is a natural generalization of a square matrix. Clearly, a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$ has at most $\prod_{j=1}^k m_j$ distinct representative submatrices.

Let $J_1 = \{1,2,\ldots,m_1\}$ and let $J_i = \{\sum_{j=1}^{i-1} m_j + 1, \sum_{j=1}^{i-1} m_j + 2, \ldots, \sum_{j=1}^{i} m_j\}$, $2 \leq i \leq k$.

The vectors $q, w \in R^m$ in (1.3) are decomposed to conform to the partition of $M$ into blocks of $M^j$, $1 \leq j \leq k$, i.e.,

$$q = \begin{bmatrix} q^1 \\ q^2 \\ \vdots \\ q^k \end{bmatrix} \text{ and } w = \begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^k \end{bmatrix},$$

where $q^j = (q_i^j)$ and $w^j = (w_i^j)$ are $m_j \times 1$ column vectors.

DEFINITION 2.3. *A vertical block matrix $M$ of type $(m_1, m_2, \ldots, m_k)$ is called a* vertical block $Z$-matrix *if all its representative submatrices are $Z$-matrices. Vertical block $P_0\,(P)$-matrices are also defined in a similar manner.*

DEFINITION 2.4. *Let $M \in R^{m\times k}$ be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. $M$ is called a* vertical block hidden $Z$-matrix *if there exists a $Z$-matrix $X = ((x_{ij})) \in R^{k\times k}$ and a vertical block $Z$-matrix $Y = ((y_{ij})) \in R^{m\times k}$ of the same type as $M$ and nonnegative vectors $r \in R^k$, $s \in R^m$ such that*

(i) $MX = Y$,

(ii) $r^tX + s^tY > 0$.

LEMMA 2.1. *Let $M$ be a vertical block hidden $Z$-matrix. Let $X \in R^{k\times k}$ be any $Z$-matrix and $Y \in R^{m\times k}$ be a vertical block $Z$-matrix of the same type as $M$ satisfying the conditions of Definition 2.4. Then $X$ is nonsingular and there exists an index set $\alpha \subseteq \{1, 2, \ldots, k\}$ such that the matrix*

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

*is in $K$, where $V$ is a representative submatrix of $Y$ corresponding to a representative submatrix $G$ of $M$.*

*Proof.* Let $r$, $s$ be as in Definition 2.4. Let $p = X^tr + Y^ts > 0$. Hence, $Ax = p$ where $A = [X^t, Y^t] \in R^{k\times(m+k)}$, $x \geq 0$ has a solution $x = \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$.

We now proceed as in the proof of Theorem 3.11.17 of Cottle, Pang, and Stone [2, p. 207] to conclude the proof of the lemma. ∎

We also observe the following result.

PROPOSITION 2.1. *Let $M$ be a vertical block hidden $Z$-matrix with $X$ and $Y$ as any matrices satisfying the conditions of Definition 2.4. Then there is at least one representative submatrix of $M$ which is hidden $Z$ with respect to $X$ and the corresponding representative submatrix of $Y$.*

*Proof.* This result follows from Lemma 2.1. By Lemma 2.1, we have an index set $\alpha \subseteq \{1, 2, \ldots, k\}$ and a representative submatrix $V$ of $Y$ such that

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

is a $K$-matrix. Let $G$ be the corresponding representative submatrix of $M$. Since $W \in K$, it follows that $W^t \in K$ and there is a $v \in R^k$, $v \geq 0$ such that $v^tW > 0$. Let $v = (v_\alpha, v_{\bar{\alpha}})$. Take $r(G)^t = (v_\alpha^t, 0)$ and $s(G)^t = (0, v_{\bar{\alpha}}^t)$. It is easy to verify that $GX = V$ and $r(G)^tX + s(G)^tV = v^tW > 0$. This shows that $G$ is a hidden $Z$-matrix, and this completes the proof of the proposition. ∎

*Remark* 2.2. The above proposition implies in particular that if $M$ is a vertical block hidden $Z$-matrix with $X$ and $Y$ as any matrices satisfying the conditions of Definition 2.4 then there exists a nonnegative matrix $U \in R^{k\times m}$ of the form

$$U = \begin{bmatrix} u^1 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & u^k \end{bmatrix},$$

where $u^r = (u_1^r, \ldots, u_{m_r}^r)$ is a nonnegative row vector of order $m_r$ such that $UM$ is a square hidden $Z$-matrix.

*Remark* 2.3. It is easy to see that if $X$ is a $K$-matrix then $UM$ is a hidden $Z$-matrix for any nonnegative $U$ of the above form. For similar results on vertical block $P$-matrices, see [1]; for vertical block $P_0$- and $Z$-matrices, see [5].

**3. Least element property.** In this section, we consider the least element property of vertical block hidden $Z$-matrices.

DEFINITION 3.1. *Let $S \subseteq R^n$ be a polyhedral set. We say that $x \in S$ is the* least element of $S$ with respect to the cone ordering induced by a convex cone $C$ *if $y - x \in C$ for any $y \in S$.*

DEFINITION 3.2. *$S \subset R^n$ is called a* meet semisublattice (under the componentwise ordering of $R^n$) *if for any two vectors $x, y \in S$ their meet $z = \min(x, y) \in S$.*

In what follows, let $M$ be a vertical block hidden $Z$-matrix with $X$ and $Y$ as any matrices satisfying the conditions of Definition 2.4. Note that by Lemma 2.1, $X$ is nonsingular. Let $S = \{v \in R^n : Xv \geq 0,\ q + Yv \geq 0\}$.

LEMMA 3.1. *A vector $z \in FEA(q, M)$ iff $v = X^{-1}z \in S$. Also $S$ is a meet semisublattice.*

*Proof.* To show this, note that $MX = Y$ and $w = q + Mz \geq 0$ as $z \in \mathrm{FEA}(q, M)$. Let $v = X^{-1}z$. So, $z = Xv \geq 0$. Note that $q + Mz = q + MXv = q + Yv \geq 0$. Hence $v \in S$.

Now given $v \in S$, take $z = Xv$. Note that $z \geq 0$. We have $q + Yv = q + MXv = q + Mz \geq 0$. Hence $z = Xv \in \mathrm{FEA}(q, M)$.

Now, we have to show that $S$ is a meet semisublattice. Let $v^*, \bar{v} \in S$ and let $\hat{v}$ be a vector whose $i$th coordinate is defined by $\hat{v}_i = \min(v_i^*, \bar{v}_i)$.

Suppose $s \in J_i$, the set of indices of rows of $M$ in the $i$th block. Note that

$$\begin{aligned} q_s + (Y\hat{v})_s &= q_s + \sum_{j=1}^{k} y_{sj}\,\hat{v}_j \\ &= q_s + y_{si}\,\hat{v}_i + \sum_{j \neq i} y_{sj}\,\hat{v}_j \\ &= q_s + y_{si}\,v_i^* + \sum_{j \neq i} y_{sj}\,\hat{v}_j, \text{ assuming (without loss of generality) } \hat{v}_i = v_i^*, \\ &\geq q_s + y_{si}\,v_i^* + \sum_{j \neq i} y_{sj}\,v_j^*, \text{ since } y_{sj} \leq 0 \text{ for } j \neq i, \\ &= q_s + \sum y_{sj}\,v_j^* \geq 0, \text{ since } v^* \in S. \end{aligned}$$

Similarly, we can show that $z = X\hat{v} \geq 0$. Thus $S$ is a meet semisublattice. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *S contains a least element.*

*Proof.* It is sufficient to verify that $S$ is bounded below as $S$ is a meet semisublattice.

Let $v \in S$ and $\tilde{q} = \left[\begin{smallmatrix} 0 \\ q_{\bar{\alpha}} \end{smallmatrix}\right]$, where $\bar{\alpha}$ is as in Lemma 2.1. Let $W$ be as in Lemma 2.1. Note that $W^{-1} \geq 0$ and by the definition of $S$, we have $Xv \geq 0$ and $q + Yv \geq 0$. Hence $\tilde{q} + Wv \geq 0$. Let $u = \tilde{q} + Wv$. Then $W^{-1}u = W^{-1}\tilde{q} + v \geq 0$ as $u \geq 0$ and $W^{-1} \geq 0$. Hence $v \geq -W^{-1}\tilde{q}$. This concludes the proof. □

THEOREM 3.1. *Suppose that $M \in R^{m \times k}$ is a vertical block hidden $Z$-matrix of type $(m_1, m_2, \ldots, m_k)$. Then there exists a simplicial cone $C$ in $R^n$ such that $\forall\ q \in Pos(I, -M)$, $FEA(q, M)$ contains a least element $\bar{z}$ with respect to the cone ordering induced by $C$ and $\bar{z}$ satisfies $\bar{z}_i \prod_{s=1}^{m_i}(q_s^i + (M^i\bar{z})_s) = 0\ \forall\, i = 1, 2, \ldots, k$.*

*Proof.* By Lemma 3.2, $S$ has a least element $\bar{v}$ with respect to Pos($I$). Let $\bar{z} = X\bar{v}$. Note that by Lemma 3.1, $\bar{z} \in$ FEA$(q, M)$, and it follows that it is a least element of FEA$(q, M)$ with respect to the cone ordering induced by Pos($X$). Now it remains to verify that $\bar{z}_i \prod_{s=1}^{m_i} (q_s^i + (M^i \bar{z})_s) = 0$. To see this, we first show that if $(X\bar{v})_i > 0$ then $\exists$ an $s \in J_i$ such that

$$q_s + (Y\bar{v})_s = 0.$$

Suppose $\forall \; s \in J_i$,

$$q_s + (Y\bar{v})_s > 0.$$

Now consider a $v^*(\epsilon)$ whose coordinates are defined as follows:

$$v_j^*(\epsilon) = \bar{v}_j, \;\; j \neq i,$$

$$v_i^*(\epsilon) = \bar{v}_i - \epsilon.$$

Note that as $X$ is a $Z$-matrix, for $\epsilon$ sufficiently small, $X v^*(\epsilon) \geq 0$. Also, it is easy to verify using the fact that $Y$ is a vertical block $Z$-matrix that

$$q_s + (Y v^*(\epsilon))_s \geq 0, \;\; \forall \;\; s.$$

This, however, contradicts the minimality of $\bar{v}$ and completes the proof. □

We shall now prove the converse of Theorem 3.1.

THEOREM 3.2. *Suppose $X$ is a $k \times k$ nonsingular matrix. Let $C = Pos(X)$. Suppose $M$ is a given vertical block matrix. If $FEA(q, M) \neq \phi$ implies that $FEA(q, M)$ has a least element with respect to the ordering induced by $C$, which is also a solution to the VLCP$(q, M)$, then $M$ is a vertical block hidden $Z$-matrix.*

*Proof.* Let $\tilde{e}^j$ be an $m \times 1$ vector whose $i$th coordinate $(\tilde{e}^j)_i = 1 \;\; \forall \, i \in J_j$ and 0 otherwise. Also, let $e_j^*$ be the unit vector in $R^k$ with $(e_j^*)_j = 1$ and $(e_j^*)_i = 0$ for $i \neq j$. Now let $q^j = \tilde{e}^j - M e_j^*$. Clearly, $e_j^* \in$ FEA$(q^j, M)$ and hence FEA$(q^j, M) \neq \phi$. Therefore, by our hypothesis it has a least element $\bar{z}^j$ which satisfies VLCP condition (1.4). Clearly, $e_j^*$ does not satisfy this condition. Hence $\bar{z}^j \neq e_j^*$ and, by the minimality of $\bar{z}^j$, we have $X^{-1}(\bar{z}^j) \leq X^{-1}(e_j^*)$.

Let $v^j = X^{-1}(e_j^* - \bar{z}^j)$. Note that $0 \neq v^j \geq 0$. Now for $i \in \{1, 2, \ldots, k\} \setminus \{j\}$, we have $X_{i.} v^j = (e_j^* - \bar{z}^j)_i \leq 0$. Let $Y = MX$. Note that $Y$ is a vertical block matrix. Now consider $Y_{s.} v^j$:

$$\begin{aligned} Y_{s.} v^j &= (Y v^j)_s \\ &= (M X v^j)_s \\ &= [M(e_j^* - \bar{z}^j)]_s \\ &= (\tilde{e}^j - q^j - M\bar{z}^j)_s \\ &= -(q^j + M\bar{z}^j)_s \text{ for } s \notin J_j. \end{aligned}$$

Therefore, noting that $(q^j + M\bar{z}^j) \geq 0$, we have $Y_{s.} v^j \leq 0$ for $s \notin J_j$.

Let $W = (v^1, v^2, \ldots, v^k)$. Then it follows that $\tilde{X} = XW$ is a $Z$-matrix and $\tilde{Y} = YW$ is a vertical block $Z$-matrix.

We now have to show the existence of nonnegative vectors $r$ and $s$ satisfying condition (ii) of Definition 2.4. To do this consider the linear programming problem

$$\text{Minimize } e^t u$$

subject to

$$X u \geq 0,$$

$$Y u \geq 0,$$

where $e$ is a $k$-vector of 1.

Note that $u$ is feasible to the above problem if and only if $X u \in \text{FEA}(0, M)$. As $0 \in \text{FEA}(0, M)$ it follows that $\text{FEA}(0, M) \neq \phi$, and hence it has a least element under the cone ordering induced by $\text{Pos}(X)$, which is also a solution to the VLCP$(0, M)$. Therefore, the above problem has an optimal solution. By the duality theorem, there exist nonnegative vectors $r$ and $s$ such that $X^t r + Y^t s = e$.

As $W \geq 0$ and no column of $W$ is 0, we have

$$\tilde{X}^t r + \tilde{Y}^t s = W^t (X^t r + Y^t s) = W^t e > 0.$$

This completes the proof. □

*Remark* 3.1. In view of Theorem 3.1, the VLCP$(q, M)$ with a vertical block hidden $Z$-matrix with respect to $X$ and $Y$ can be formulated as the linear programming problem

$$\text{Minimize} \sum_{i=1}^{k} p_i z_i,$$

$$w - Mz = q,$$

$$w \geq 0, \quad z \geq 0,$$

where $p = (p_1, p_2, \ldots, p_k)$ is any vector such that $p^t X > 0$.

*Remark* 3.2. Thus the remarks of Cottle, Pang, and Stone [2, p. 212] in the context of hidden $Z$-matrices also apply to the vertical block hidden $Z$-matrices. Thus, given an arbitrary vertical block matrix $M$ it is not in general easy to test whether or not it is vertical block hidden $Z$.

## 4. Vertical block hidden $K$-matrices.

DEFINITION 4.1. *Let $M$ be a vertical block hidden $Z$-matrix. We say that $M$ is a* vertical block hidden $K$-matrix *if every representative submatrix of $M$ is a $P$-matrix.*

In the example below we exhibit the blocks by separating them from one another using blank space.

*Example* 4.1. Let $M$ be the following vertical block matrix:

$$\begin{bmatrix} 1.76 & 0.36 & 0.16 \\ 1 & 0 & 0 \\ 0.80 & -0.20 & -0.20 \\ & & \\ 0.32 & 1.52 & 0.12 \\ 0.44 & 1.84 & 0.04 \\ & & \\ -1.56 & -1.16 & 0.04 \\ -0.60 & -0.60 & 0.40 \end{bmatrix},$$

where $m_1 = 3$, $m_2 = 2$, and $m_3 = 2$.

It is easy to verify that $M$ is a vertical block hidden $K$-matrix with respect to $X, Y$,

$$\text{where } X = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 7 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -2 \\ & & \\ -1 & 4 & -1 \\ -1 & 5 & -2 \\ & & \\ -2 & -2 & 3 \\ -1 & -2 & 4 \end{bmatrix}.$$

We take $r^t = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ and $s^t = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

The following theorem characterizes a vertical block hidden $K$-matrix $M$ assuming that it is a vertical block hidden $Z$-matrix.

THEOREM 4.1. *Let $M$ be a vertical block hidden $Z$-matrix of type $(m_1, m_2, \ldots, m_k)$. Let $X$ and $Y$ be as in Definition 2.4. The following are equivalent:*

(a) *$M$ is a vertical block hidden $K$-matrix.*

(b) *There exists an $x \in R^k$, $x > 0$ such that $Mx > 0$.*

(c) *There exists a vector $v \in R^k$, $v > 0$ such that for any given index set $\alpha \subseteq \{1, 2, \ldots, k\}$, $Wv > 0$, where*

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

*and $V$ is the representative submatrix of $Y$ corresponding to any given representative submatrix $G$ of $M$. Furthermore, $W \in K$.*

(d) *Every representative submatrix $G$ of $M$ is completely hidden $K$; i.e., for every index set $\beta \subseteq \{1, 2, \ldots, k\}$, $G_{\beta\beta}$ is hidden $K$.*

*Proof.* (a) $\Rightarrow$ (b). Suppose $M$ is a vertical block hidden $K$-matrix. In particular, by definition $M$ is a vertical block $P$-matrix. Now from Theorem 6 of Cottle and Dantzig [1, p. 89] it follows that there is an $x \in R^k$, $x > 0$ such that $Mx > 0$.

(b) $\Rightarrow$ (c). Let $x > 0$, $x \in R^k$ be such that $Mx > 0$. Let $v = X^{-1}x$. We have $Xv > 0$, $Yv = MXv = Mx > 0$. By Lemma 2.1, there exists a representative submatrix $V$ and an index set $\alpha_0 \subseteq \{1, 2, \ldots, k\}$ such that

$$W_0 = \begin{bmatrix} X_{\alpha_0\alpha_0} & X_{\alpha_0\bar{\alpha}_0} \\ V_{\bar{\alpha}_0,\alpha_0} & V_{\bar{\alpha}_0\bar{\alpha}_0} \end{bmatrix}$$

is a $K$-matrix. As $Xv > 0$ and $Yv > 0$, it follows that $W_0 v > 0$. This implies that $v > 0$.

Now let $G$ be any representative submatrix of $M$ and let $H$ be the corresponding representative submatrix of $Y$. Let $\alpha \subseteq \{1, 2, \ldots, k\}$ be any index set. Consider the matrix

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ H_{\bar{\alpha}\alpha} & H_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

As $Xv > 0$, $Yv > 0$, it follows that $Wv > 0$.

Since $W \in Z$ and $v > 0$, it follows that $W \in K$.

(c) ⇒ (d). Let $G$ be any given representative submatrix of $M$. Let $\beta \subseteq \{1, 2, \ldots, k\}$ be given. By (c) the matrix

$$W = \begin{bmatrix} X_{\beta\beta} & X_{\beta\bar{\beta}} \\ V_{\bar{\beta}\beta} & V_{\bar{\beta}\bar{\beta}} \end{bmatrix}$$

is a $K$-matrix, where $V$ is the representative submatrix of $Y$ corresponding to $G$. We now proceed as in Theorem 3.11.19 of Cottle, Pang, and Stone [2, pp. 211–212] to conclude that every representative submatrix is completely hidden $K$.

(d) ⇒ (a). Note that we have $MX = Y$ with $X \in Z$, $Y \in$ vertical block $Z$ and $r^t X + s^t Y > 0$. Since every representative submatrix is a hidden $K$-matrix, it follows that every representative submatrix is a $P$-matrix. Hence by definition, statement (a) follows.

*Remark* 4.1. In relation to Remark 3.2 if we know that $M$ is a vertical block $P$-matrix and wish to test its membership in vertical block hidden $K$ then it is possible to do so by solving two linear programs: one to determine if there exists a $y > 0$ such that $My > 0$ and another to determine if the required $X$ exists. Also the corresponding VLCP is solvable in polynomial time once we have determined the required $X$ in polynomial time.

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