Distributions Characterized Through Conditional Expectations

K. BALASUBRAMANIAN AND ALOKE DEY

Indian Statistical Institute, Stat/Math Unit, 7, S.J.S. Sansanwal Marg, New Delhi 110016, India

Abstract: Using conditional expectations, we present results that lead to the characterization of several distributions. Both absolutely continuous random variables and discrete random variables are considered. In the case of absolutely continuous random variables, the results lead to the characterization of a family of distributions while in the case of discrete random variables, the distribution is almost uniquely determined under the stated conditions.

Key Words: Characterization, Order statistics

1 Introduction

Characterizations of distributions using conditional expectations have been studied by several authors. Let $X_1, X_2, \ldots, X_n$ be a random sample of a random variable X with a continuous distribution function F which is strictly increasing over (a, b), $-\infty \leq a < b \leq \infty$, the support of F, and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, the corresponding order statistics. Ferguson (1967) characterized distributions using the condition $E(X_{i:n}|X_{i+1:n} = x) = \alpha x - \beta$, while Beg and Kirmani (1974) used the condition $E(X_i|X_{n:n} = x) = \alpha x - \beta$ to characterize distributions. Here α and β are some constants. Related results can be found in Galambos and Kotz (1978), Azlarov and Volodin (1986), Beg and Balasubramanian (1990) and Balasubramanian and Beg (1992). Let g be a nonconstant continuous function over (a, b) with finite $g(a+)$ and finite expectation, $E[g(X)]$. By a suitable choice of g, Berg and Balasurbramanian (1990) were able to characterize all distributions for which the explicit form of the distribution function is known, continuous and strictly increasing in its support (a, b), through the property,

$$E\left\{\frac{1}{s-1}\Sigma g(X_{i:n})|\, X_{s:n} = x\right\} = \frac{g(x) + g(a+)}{2} \qquad \forall x \in (a, b) \ .$$

Balasubramanian and Beg (1992) subsequently characterized distributions by conditioning on a pair of order statistics. In the last two papers the characterizations were through the arithmetic means of the function at the end points.

In this communication, we present certain results based on conditional expectations that lead to characterizations of several distributions. We use harmonic and geometric means of the functions at the end points to arrive at our results. Specifically, we prove the following main results:

(i) If X is a random variable with cumulative distribution function (cdf) $F(x)$ and density function $f(x)$, which is continuous in the support of the density, then

$$E[g(X)|x \le X \le y] = \frac{2g(x)g(y)}{g(x)+g(y)}, \qquad \forall x, y, x \le y$$

if and only if $g(x) = \sqrt{\left(\frac{b}{F(x)+a}\right)}$, where g is a positive function with continuous derivative and a and b are constants.

(ii) *If* X and g are as in (i) above, then

$$E[g(X)|x \le X \le y] = \sqrt{\{g(x)g(y)\}}, \qquad \forall x, y, x \le y$$

if and only if

$$g(x) = \frac{b^2}{(F(x)-a)^2}.$$

(iii) If X is a discrete random variable taking nonnegative integral values over the interval $[u, v]$ and g be a positive function with $g(i) \neq g(i+1)$ $\forall i \in [u, v]$, then

$$E[g(X)|a \le X \le b] = \frac{2g(a)g(b)}{g(a)+g(b)}, \qquad \forall a, b, a \le b$$

if and only if $\Pr(X = i) = p_i = \alpha + \beta i$ for $u \le i \le v$, $g(i) = c/p_i$ $\forall i \in [u, v]$ and c is some constant.

(iv) with X and g as in (iii) above,

$$E[g(X)|a \le X \le b] = \sqrt{\{g(a)g(b)\}}, \qquad \forall a, b, a \le b$$

if and only if $\Pr(X = i) = p_i = \alpha r^{i-1}$ and $g(i) = c/p_i^2$, where $r = p_{i+1}/p_i$, $i \in [u, v]$, the support of X.

The above results lead to characterizations of several continuous and discrete distributions. The proofs of the above stated results are given in Section 2. In Section 3, we give some characterizations.

2 Proofs of the Results

Theorem 1: Let X be a random variable with c.d.f. $F(x)$ and density function $f(x)$ and let $f(x)$ be continuous in the support of the density. Then

$$E[g(X)|x \leq X \leq y] = \frac{2g(x)g(y)}{g(x)+g(y)}, \qquad \forall x, y\ x \leq y, \tag{2.1}$$

if and only if $g(x) = \sqrt{\left(\frac{b}{F(x)+a}\right)}$, where $g(\cdot)$ is a positive function with continuous derivative and a and b are constants.

Proof: Observe that if $g(\cdot)$ is a positive function with continuous derivative then

$$E[g(X)|x \leq X \leq y] = \int_x^y \frac{g(t)f(t)\,dt}{F(y)-F(x)}. \tag{2.2}$$

Suppose (2.1) holds. Then equating (2.1) and (2.2), we have

$$\int_x^y g(t)f(t)\,dt = \{F(y)-F(x)\}\frac{2g(x)g(y)}{g(x)+g(y)}.$$

Differentiating both sides w.r.t. y, we have

$$g(y)f(y) = \frac{\partial}{\partial y}\left[\{F(y)-F(x)\}\frac{2g(x)g(y)}{g(x)+g(y)}\right]$$

$$= \frac{2\{F(y)-F(x)\}}{\{1+g(y)/g(x)\}^2}g'(y) + \frac{2f(y)}{1/g(x)+1/g(y)},$$

which yields

$$\frac{f(y)}{F(y)-F(x)}=\frac{2\{g(x)\}^2 g'(y)}{g(y)[\{g(y)\}^2-\{g(x)\}^2]}. \tag{2.3}$$

Note that the l.h.s. of (2.3) is the partial derivative w.r.t. y of $\ln[F(y)-F(x)]$ and the r.h.s., that of $\ln[1-(g(x)/g(y))^2]$. Consequently,

$$F(y)-F(x)=k\left(1-\left(\frac{g(x)}{g(y)}\right)^2\right),$$

k being an arbitrary constant. This gives

$$g(y)=\sqrt{\left(\frac{b}{F(y)+a}\right)} \tag{2.4}$$

where a, b are constants such that $b/\{F(y)+a\}>0$.

Conversely, suppose (2.4) holds. Then

$$\int_x^y \frac{g(t)f(t)\,dt}{F(y)-F(x)}=\frac{1}{F(y)-F(x)}\int_x^y \sqrt{\left(\frac{b}{F(t)+a}\right)}f(t)\,dt$$

$$=\frac{2g(x)g(y)}{g(x)+g(y)}.$$ ■

Suppose $X_1, X_2, \ldots, X_n$ is a sample from the random variable X, defined in Theorem 1 and let $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ be the order statistics. Then, it is known that the conditional distribution of $X_{r+1:n}, X_{r+2:n}, \ldots, X_{s-1:n}$ given $X_{r:n}=x, X_{s:n}=y\,(r<s)$ is the same as the distribution of $Y_{1:s-r-1}, Y_{2:s-r-1}, \ldots, Y_{s-r-1:s-r-1}$ where $Y_{i:s-r-1}$ is the ith order statistic based on a sample of size $(s-r-1)$ from the truncated distribution $\{X\,|\,x\le X\le y\}$. Using this result, we have the following corollary to Theorem 1.

Corollary 1: Let X and $g(\cdot)$ be as in Theorem 1. Then

$$E\left[\sum_{i=r+1}^{s-1} g(X_{i:n})/(s-r-1)\,|\,X_{r:n}, X_{s:n}\right]=\frac{2g(X_{r:n})g(X_{s:n})}{g(X_{r:n})+g(X_{s:n})}$$

if and only if $g(x)=\sqrt{\left(\dfrac{b}{F(x)+a}\right)}$.

On similar lines, one can prove the following results.

Theorem 2: Let X and $g(\cdot)$ be as in Theorem 1. Then

$$E[g(X)|x \leq X < y] = \sqrt{\{g(x)g(y)\}} \tag{2.5}$$

if and only if

$$g(x) = \frac{b^2}{(F(x) - a)^2} \tag{2.6}$$

where a and b are constants.

Corollary 2: Let X and $g(\cdot)$ be as in Theorem 1. Then

$$E\left(\sum_{i=r+1}^{s-1} g(X_{i:n})/(s-r-1)|X_{r:n}, X_{s:n}\right) = \sqrt{g(X_{r:n})g(X_{s:n})\}} \tag{2.7}$$

if and only if (2.6) holds.

Remark: The result in Corollary 1 characterizes several distributions. This result can be used to test the hypothesis that the sample is from a known population. For example, if $F_0(x) = b/\{g(x)\}^2 - a$, then the hypothesis that the observations are from the distribution $F_0(x)$ can be tested by considering the difference

$$T = \sum_{i=r+1}^{s-1} g(X_{i:n})/(s-r-1) - \frac{2g(X_{r:n})g(X_{s:n})}{g(X_{r:n}) + g(X_{s:n})} .$$

Under the hypothesis, $E(T) = 0$. Hence large values of $|T|$ lead to the rejection of the hypothesis. As this test is based on a characterization property, it is likely to have large power. This inferential aspect will be dealt in a separate paper. Corollary 2 can also be used similarly. This inferential aspect makes the characterizations very useful. ■

Now suppose that X is a discrete random variable taking nonnegative integral values in the interval $[u, v]$, $v < \infty$. Suppose $g(\cdot)$ is a positive function such that $g(i) \neq g(i+1)$ $\forall i \in [u, v]$. Then, we have

Theorem 3: With X and $g(\cdot)$ as above,

$$E[g(X)|a \leq X < b] = \frac{2g(a)g(b)}{g(a) + g(b)}, \qquad \forall a, b \; a \leq b , \tag{2.8}$$

if and only if $\Pr(X = i) = p_i = \alpha + \beta i$ and $g(i) = c/p_i$ $\forall i \in [u, v]$, where c is some constant.

Proof: To begin with, let us fix $a = i$, $b = i + 1$ for some $i \in [u, v]$. Then,

$$E[g(X)|i \le X \le i + 1] = \frac{g(i)p_i + g(i+1)p_{i+1}}{p_i + p_{i+1}}$$

$$= \frac{2g(i)g(i+1)}{g(i) + g(i+1)}, \qquad \text{from (2.8)} .$$

This implies that $g(i)p_i = g(i+1)p_{i+1}$, provided $g(i) \neq g(i+1)$.

$$\Rightarrow g(i) = c/p_i . \tag{2.9}$$

Again, let $a = i$, $b = i + 2$, for some $i \in [u, v]$. Then

$$E[g(X)|i \le X \le i + 2] = \frac{g(i)p_i + g(i+1)p_{i-1} + g(i+2)p_{i+2}}{p_i + p_{i+1} + p_{i+2}}$$

$$= \frac{2g(i)g(i+2)}{g(i) + g(i+2)}, \qquad \text{from (2.8)} .$$

From (2.9), we get, therefore

$$p_{i+1} = \frac{p_i + p_{i+2}}{2} \tag{2.10}$$

which implies that p_i's are in arithmetic progression, say $p_i = \alpha + \beta i$ $\forall i \in [u, v]$.
It is now easy to verify that for $g(i)$ given by (2.9) and p_i's satisfying (2.10),

$$E(g(X)|a \le X \le b] = \frac{2g(a)g(b)}{g(a) + g(b)} . \qquad \blacksquare$$

On similar lines, one can prove the following:

Theorem 4: Suppose X is a discrete random variable taking nonnegative integral values in the interval $[u, v]$, $v < \infty$ and $g(\cdot)$ is a positive function such that

$g(i) \neq g(i+1)\ \forall i \in [u, v]$. Then

$$E[g(X) | a \leq X \leq b] = \sqrt{\{g(a)g(b)\}}\ , \qquad \forall a, b\ a \leq b$$

if and only if $\Pr(X = i) = p_i = \alpha r^{i-1}$ and $g(x) = c/p_i^2\ \forall i \in [u, v]$, where α and c are constants and $r = p_i/p_{i+1}$.

3 Characterization of Distributions

We give two examples to illustrate how the results of the previous section help in characterization of distributions.

1. Distributions with c.d.f. of the form

$$F(x) = Ax^k + B \tag{3.1}$$

are characterized by any one of the following two conditions:-

(a) $E\left\{\frac{1}{(s-r-1)} \sum_{i=r+1}^{s-1} \frac{1}{X_{i:n}^{k/2}} \middle| X_{r:n}, X_{s:n}\right\} = \frac{2}{X_{r:n}^{k/2} + X_{s:n}^{k/2}};$

(b) $E\left\{\frac{1}{(s-r-1)} \sum_{i=r+1}^{s-1} \frac{1}{X_{i:n}^{2k}} \middle| X_{r:n}, X_{s:n}\right\} = \frac{1}{X_{r:n}^{k} + X_{s:n}^{k}}.$

Note that the family of distributions given by (3.1) includes power function distributions, Pareto distribution and Rectangular distribution.

2. Distributions with c.d.f. of the form

$$F(x) = Ae^{-\theta x^\eta} + B \tag{3.2}$$

are characterized by any one of the following conditions:-

(a) $E\left\{\frac{1}{(s-r-1)} \sum_{i=r+1}^{s-1} e^{\theta X_{i:n}^\eta/2} | X_{r:n}, X_{s:n}\right\} = \frac{2e^{\theta(X_{r:n}^\eta + X_{s:n}^\eta)/2}}{e^{\theta X_{r:n}^\eta/2} + e^{X_{s:n}^\eta/2}}$

(b) $E\left\{\frac{1}{(s-r-1)} \sum_{i=r+1}^{s-1} e^{2\theta X_{i:n}^\eta} | X_{r:n}, X_{s:n}\right\} = e^{\theta(X_{r:n}^\eta + X_{s:n}^\eta)}\ .$

The family of distribution given by (3.2) includes exponential and the Weibull distribution.

The above characterizations are for absolutely continuous distribution. But the discrete case is totally different. For discrete random variables our results do not characterize a family of distributions. Under the stated conditions, g and F are almost unique.

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