STUDIES IN MULTIPARTITE SELF-COMPLEMENTARY GRAPHS

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C O N T E N T S

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INTRODUCTION AND DEFINITIONS

The class of self-complementary graphs has been extensively studied by many people, among others by C.R.J. Clapham, S.B. Rao, G. Ringel and H. Sachs, and many problems have been solved for this class, such as the Hamiltonian problem and the characterisation of potentially and forcibly self-complementary degree sequences (see [1], [2], [12], [13], [14], [15]). Thus self-complementary graphs form an interesting class and this has been generalised by Hebbare [8] into the class of multipartite self-complementary graphs.

An r-partite self-complementary graph is an r-partite graph G which is isomorphic to its r-partite complement H where H has the same vertex set as G and uv is an edge of H iff u,v belong to different sets in the r-partition of G and uv is not an edge of G. A multipartite self-complementary graph is an r-partite self-complementary graph for some $r \geq 2$. In this thesis we study the properties of multipartite self-complementary graphs. Several well-known results on self-complementary graphs are obtained as corollaries.

The thesis is divided into two parts. In Part I, which consists of the first five chapters, we study the properties of r-partite self-complementary graphs for general r. In Part II, consisting of the last two chapters, we study the degree sequences of bipartite self-complementary graphs.

In Chapter 1, we study the properties of complementing permutations of r-partite self-complementary graphs. In particular we prove that any complementing permutation of a connected bipartite self-complementary graph permutes the partition sets as a whole and the square of any complementing permutation is an automorphism of the graph. We also deduce the following result of Ringel [17] and Sachs [18] on self-complementary graphs : if G is self-complementary and σ is a complementing permutation of G, then σ^2 is an automorphism of G and either (i) the length of every cycle of σ is a multiple of 4 or (ii) σ has a unique cycle of length one and the length of every other cycle of σ is a multiple of 4.

In Chapter 2 we characterise when certain simple graphs like trees, forests, unicyclic graphs and cacti are r-partite self-complementary.

In Chapter 3, we study the diameters of an r-partite graph and its r-partite complement. The range of diameters for r-partite self-complementary graphs is determined. In particular we deduce that the diameter of a self-complementary graph is either 2 or 3. Finally we solve completely a Nordhaus-Gaddum type problem in the class of bipartite graphs : the characterisation of all triplets (a,b,p) for which there exists a bipartite graph G on p vertices such that G has diameter a and the bipartite complement of G has diameter b.

In Chapter 4, we consider the problem of determining the maximum length of a path in r-partite self-complementary graphs on p vertices. The problem is completely solved for the class of connected bipartite self-complementary graphs with a complementing permutation σ such that v and $\sigma(v)$ belong to different sets of the bipartition for some vertex v. We also obtain sufficient conditions for the existence of a hamiltorian path in an r-partite self-complementary graph, when $r \neq 3$, and show that they are best possible in some sense. In particular we deduce the result, due to Clapham [1], that every self-complementary graph has a hamiltonian path.

In Chapter 5, we study disconnected r-partite self-complementary graphs. We determine when a disconnected r-partite graph without isolated vertices is r-partite self-complementary. It is also established that a disconnected bipartite self-complementary graph has a complementing permutation which maps each set of the bipartition to itself.

In Chapter 6, we characterise potentially bipartite self-complementary bipartitioned degree sequences, i.e. sequences of the type $(d_1,\dots,d_m|e_1,\dots,e_n)$ with a bipartite self-complementary realisation.

In Chapter 7, we characterise forcibly bipartite self-complementary bipartitioned degree sequences, i.e., graphic bipartitioned sequences π such that every realisation of π is bipartite self-complementary. This characterisation involves forcibly self-complementary degree sequences characterised by Rao [8] and unigraphic bipartitioned degree sequences characterised by Koren [5].

The results in Chapters 1-4, except Theorems 2.3, 2.4 and 3.8 are obtained jointly with S.P. Rao Hebbare.

We now list the general definitions from Graph Theory which will be used in this thesis. More specialised definitions and terminology will be given at the beginning of each part.

By a graph we mean a finite undirected graph without loops and multiple edges. Thus a graph G consists of a finite non-empty set $V(G)$ of vertices and a prescribed set $E(G)$ of unordered pairs of distinct vertices. Each pair $e = (u,v)$ of vertices in $E(G)$ is called an edge of G and e is said to join u and v. We then write $e = uv$ and say that u and v are adjacent vertices ; vertex u and edge e are incident with each other, as are v and e. A graph G is called trivial if $|V(G)| = 1$ and non-trivial otherwise.

A graph in which every pair of distinct vertices is joined by an edge is called a complete graph. The complete graph on p vertices is denoted by K_p and sometimes by K when the number of vertices is not of interest. Similarly a graph on p vertices and with no edge is denoted by $\bar{K}_p$ and sometimes simply by $\bar{K}$.

Two graphs G and H are said to be isomorphic (written $G \simeq H$) if there is a bijection from $V(G)$ onto $V(H)$ which preserves adjacency. Such a bijection is called an isomorphism of G onto H. An automorphism of G is an isomorphism of G onto itself. The class of all automorphisms of G forms a group and is called the automorphism group of G, denoted by $Aut(G)$.

Let G be graph. By a subgraph of G we mean a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$. Then by the subgraph $G[S]$ induced by S in G we mean the subgraph of G, whose vertex set is S and whose edge set consists of all those edges of G which join vertices in S. If $G[S] = K$ then S is said to be complete and if $G[S] = \bar{K}$, then S is said to be independent. We denote by $G - S$ the subgraph of G whose vertex set is $V(G)-S$ and whose edge set consists of all those edges of G which are not incident with any element of S.

A $v_o - v_n$ *path* of a graph G is a sequence $v_o\, v_1 \ldots v_n$ of distinct vertices such that for each i, $v_i\, v_{i+1}$ is an edge of G. The vertices v_o and v_n are called the *terminal vertices* of the path and the path is said to *connect* v_o and v_n. The *length* of a path is the number of edges in it. An *n-path* is a path of length n. A *cycle* is a path, with length at least 3, whose terminal vertices coincide. Let $|V(G)| = p$. Then a cycle of length p is called a *hamiltonian cycle* and a $(p-1)$-path a *hamiltonian path*. If G has a hamiltonian cycle, then G is called *hamiltonian*.

A graph G is said to be *connected* if any two vertices of G are connected by a path, and *disconnected* otherwise. A maximal connected subgraph of G is called a *connected component* or simply a *component* of G. Thus a disconnected graph has at least two components. A *cut-vertex* of G is a vertex whose removal increases the number of components of G. A connected non-trivial graph without cut-vertices is called a *non-separable* graph. A *block* of a graph is a maximal non-separable subgraph.

The *distance* $d_G(u,v)$ between two vertices u and v in the graph G is the minimum length of a u-v path if any ; otherwise $d_G(u,v) = \infty$. We note that $d_G(u,u)$ is zero. The *diameter* of a graph G is the maximum distance between two vertices of G.

A graph is acyclic if it has no cycles. An acyclic graph is also called a forest. A connected acyclic graph is called a tree. A graph with exactly one cycle is called a unicyclic graph. A connected graph whose only blocks are K_2's or cycles is called a cactus.

Let G be a graph. The neighbourhood $N_G(v)$ of a vertex v in G is the set of all vertices adjacent to v in G, and the degree $d_G(v)$ of v is the cardinality of $N_G(v)$. A vertex v is called isolated if $d_G(v) = 0$ and an end-vertex if $d_G(v) = 1$. Let $V(G) = \{v_1,\ldots,v_p\}$. Then the sequence of non-negative integers $\pi(G) = (d_1,\ldots,d_p)$ where $d_i = d_G(v_i)$ is called the degree sequence of G. Conversely a sequence π of non-negative integers is said to be graphic if there is a graph G such that $\pi(G) = \pi$. In this case G is called a realisation of π.

The complement $\overline{G}$ of a graph G is the graph defined by

$$V(\overline{G}) = V(G)$$

$$E(\overline{G}) = \{uv \mid v,v \in V(G),\ u \neq v \text{ and } uv \notin E(G)\}.$$

G is said to be self-complementary if $G \simeq \overline{G}$. If G is self-complementary then an isomorphism of G onto $\overline{G}$ is called a complementing permutation of G. We denote by $\mathcal{C}(G)$ the class of all complementing permutation of the graph G. Note that $\mathcal{C}(G) = \emptyset$ if G is not self-complementary.

A graphic sequence of non-negative integers π is said to be <u>potentially self-complementary</u> if there is a self-complementary realisation of π. A sequence of non-negative integers π is said to be <u>forcibly self-complementary</u> if π is graphic and every realisation of π is self-complementary.

A <u>directed graph</u> (or <u>digraph</u>) D consists of a finite non-empty set V(D) of <u>vertices</u> and a prescribed set A(D) of ordered pairs of vertices (not necessarily distinct). The elements of A(D) are called <u>arcs</u> of D. The <u>outdegree</u> (resp. <u>indegree</u>) of a vertex v in D is the number of outgoing (resp. incoming) arcs incident at v.

PART I

MULTIPARTITE SELF-COMPLEMENTARY GRAPHS

In Part I, we study some properties of r-partite self-complementary graphs for general r. The case $r = 2$ is of special interest since it often yields stronger theorems as well as simpler proofs. We also obtain several well-known results on self-complementary graphs as corollaries.

A graph G is said to be <u>r-partite</u> if there exist r sets $A_1, A_2, \ldots, A_r$ such that $\bigcup_{i=1}^{r} A_i = V(G), A_i \cap A_j = \emptyset$ if $i \neq j$ and each A_i is independent. Such a partition $\{A_1, \ldots, A_r\}$ is called an <u>r-partition</u> of G. An <u>r-partitioned</u> graph is a pair (G,P) where G is an r-partite graph and P is an r-partition of G. A <u>complete</u> <u>r-partite</u> graph is an r-partitioned graph (G,P) in which each vertex in A_i is adjacent to all vertices in A_j, for all i,j, $i \neq j$, $1 \leq i,j \leq r$. for $r=2$, a complete 2-partite graph is also called a <u>complete bipartite</u> graph.

<u>Throughout Part I</u>, (G,P) <u>denotes an r-partitioned graph and the sets of</u> P <u>are denoted by</u> $A_1, A_2, \ldots, A_r$ <u>with their respective cardinalities</u> $n_1, n_2, \ldots, n_r$ <u>in non-decreasing order. The only exception to this rule will occur in Theorems</u> 5.2-5.4, <u>where</u> (G,P) <u>will denote a bipartitioned graph and the sets of</u> P <u>will be denoted by</u> A <u>and</u> B (<u>instead of</u> A_1 <u>and</u> A_2 <u>respectively</u>).

We will sometimes say "(G, P) has property X" to mean "G has property X", where X is an invariant prop[erty] of graphs.

Given an r-partitioned graph (G, P), we define its r-partite complement to be the r-partitioned graph $(\overline{G}(P), [P)]$ where

$$V(\overline{G}(P)) = V(G)$$

$$E(\overline{G}(P)) = \{uv \mid u, v \text{ belong to different sets of } P \text{ and } uv \notin E(G)\}.$$

An r-partitioned graph (G, P) is said to be r-parti[te] self-complementary (abbreviated r-psc) if $G \simeq \overline{G}(P)$. A 2-partite self-complementary graph is also called biparti[te] self-complementary (abbreviated bipsc).

It is easily seen that if (G, P) is r-psc and each s[et] of P is a singleton, then G is self-complementary in the usual sense. Conversely, a self-complementary graph G on [p] vertices can be looked upon as a p-partite self-complementa[ry] graph with each set in the p-partition a singleton. Thus th[e] concept of r-psc graphs is a generalisation of the concept [of] self-complementary graphs.

Let (G,P) be r-psc. An <u>r-partite complementing permutation</u> (abbreviated <u>r-pcp</u>) of (G,P) is an isomorphism between G and $\overline{G}(P)$, i.e., a bijection $\sigma : V(G) \longrightarrow V(G)$ such that $\sigma(u)\,\sigma(v) \in E(\overline{G}(P))$ iff $uv \in E(G)$. We denote by $\mathcal{C}((G,P))$ the class of all r-partite complementing permutations of the r-psc graph (G,P). Note that if σ is an r-pcp of (G,P) then σ^{-1} may not be an r-pcp of (G,P). To put it is another way, an r-pcp of (G,P) may not be an r-pcp of $(\overline{G}(P), P)$ even though $(\overline{G}(P), P)$ is r-psc. A cycle of an r-pcp is said to be <u>pure</u> if it permutes only vertices belonging to a single set of P and is said to be <u>mixed</u> otherwise. For a cycle τ, the symbol $\langle \tau \rangle$ denotes the set of all vertices permuted by τ and the symbol $|\tau|$ stands for the cardinality or $\langle \tau \rangle$. Further we use I_τ to denote the set

$$\{ i \mid 1 \leq i \leq r,\ A_i \cap \langle \tau \rangle \neq \emptyset \}.$$

We now define two important subclasses of $\mathcal{C}((G,P))$ as follows:

$$\mathcal{C}_p((G,P)) = \{ \sigma \in \mathcal{C}((G,P)) \mid \text{all cycles of } \sigma \text{ are pure} \},$$

$$\mathcal{C}_m((G,P)) = \{ \sigma \in \mathcal{C}((G,P)) \mid \text{all cycles of } \sigma \text{ are mixed} \}.$$

Finally, if (G,P) is an r-partitioned graph which is not r-psc, then we define $\mathcal{C}((G,P))$ to be the empty set.

CHAPTER 1

COMPLEMENTING PERMUTATIONS AND THEIR PROPERTIES

In this chapter we study the properties of complementing permutations of an r-psc graph and establish analogues of some well-known theorems on self-complementary graphs.

We start with some basic observations on r-psc graphs and their complementing permutations. These observations will be frequently used in the course of the thesis.

OBSERVATION 1.1. If (G,P) is r-psc and $\sigma \in \mathcal{C}((G,P))$, then $uv \in E(\overline{G}(P))$ iff $\sigma^{-1}(u)\ \sigma^{-1}(v) \in E(G)$.

OBSERVATION 1.2. Let (G,P) be r-psc and $\sigma \in \mathcal{C}((G,P))$. If X is an invariant property of graphs and a subgraph H of G has property X, then the subgraph induced by $\sigma(V(H))$ in $\overline{G}(P)$ also has property X.

OBSERVATION 1.3. Let (G,P) be r-psc and $\sigma \in \mathcal{C}((G,P))$. If σ_j, $1 \leq j \leq n$ are cycles of σ and $\left| \bigcup_{j=1}^{n} I_{\sigma_j} \right| = k$, then the k-partitioned subgraph induced by $\bigcup_{j=1}^{n} \langle \sigma_j \rangle$ in G (where the k-partition under consideration is that induced by P) is k-psc with $\prod_{j=1}^{n} \sigma_j$ as a k-pcp.

<u>OBSERVATION 1.4</u>. If (G,P) is r-psc, then $|E(G)| = \frac{1}{2} \sum_{i<j} \sum n_i n_j$. Hence $\sum_{i<j} \sum n_i n_j$ is even. In particular, if $r = 2$ then, either n_1 or n_2 is even. Also, if G is self-complementary, then $|V(G)| \equiv 0$ or $1 \pmod 4$.

We now describe a method of constructing new r-psc graphs from a given r-psc graph. Let (G,P) be an r-psc graph, $\sigma \in \mathfrak{S}((G,P))$, τ a cycle of σ and let k be a positive integer. Then we define an r-partitioned graph (G^k_τ, P^k_τ) as follows :

$V(G^k_\tau) = S \bigsqcup T$, where $S = V(G) - \langle \tau \rangle$ and $T = \langle \tau \rangle \times \{1,2,\ldots,k\}$. If u,v are two distinct vertices then they are adjacent in G^k_τ iff

either (i) $u,v \in S$ and $uv \in E(G)$

or (ii) $u \in S, v \in T$ and $uy \in E(G)$ where $v = (y,j)$

or (iii) $u \in T, v \in S$ and $xv \in E(G)$ where $u = (x,i)$

or (iv) $u,v \in T$ and $xy \in E(G)$ where $u = (x,i)$ and $v = (y,j)$.

The partition P^k_τ of $V(G^k_\tau)$ consists of the sets $B_1, B_2, \ldots, B_r$ where $u \in B_j$ iff

either $u \in S$ and $u \in A_j$

or $u \in T$ and $x \in A_j$ where $u = (x,i)$.

We then have the following

THEOREM 1.5. (G^k_τ, P^k_τ) is r-psc.

PROOF : We define a bijection σ^k_τ between $V(G^k_\tau)$ and $V(\overline{G^k_\tau}(P^k_\tau))$ as follows :

$\sigma^k_\tau(u) = \sigma(u)$ if $u \in S$

$= (\tau(x), i)$ if $u \in T$, where $u = (x, i)$.

Then clearly σ^k_τ is an isomorphism between G^k_τ and $\overline{G^k_\tau}(P^k_\tau)$. Hence (G^k_τ, P^k_τ) is r-psc and the theorem is proved. □

Given a self-complementary graph G, and a complemen permutation σ of G, it is well-known (See Ringel [17], Sachs [18]) that except for a possible fixed point, all cy of σ have lengths $\equiv 0 \pmod 4$. An analogous result for graphs is given in the following

THEOREM 1.6. Let (G,P) be r-psc and $\sigma \in \mathcal{C}((G,$
Then the cycles of σ satisfy the following properties :

(i) There exists a set A_h of P such that $\langle \tau \rangle \subseteq$ for all pure cycles τ of σ having odd length.

(ii) Let t be a non-negative integer and τ a cycle of σ. If $\langle \tau \rangle$ intersects k sets of P in exactly $2t+1$ vertices each and is disjoint from the remaining sets of P then $k \equiv 0$ or $1 \pmod 4$. Further, if $k \geq 2$ and τ^2 is an automorphism of the subgraph induced by $\langle \tau \rangle$, then $k \equiv 0 \pmod 4$.

<u>PROOF</u> : (i). If possible, let τ, Ψ be two pure cycles of σ having odd length, $\langle \tau \rangle \subseteq A_i$, $\langle \Psi \rangle \subseteq A_j$ and $i \neq j$. Then, by Observation 1.3, the subgraph induced by $\langle \tau \rangle \cup \langle \Psi \rangle$ with the bipartition $\{ \langle \tau \rangle, \langle \Psi \rangle \}$ is bipsc. Hence by Observation 1.4, $|\tau| . |\Psi|$ is even, a contradiction. This proves (i).

(ii). Without loss of generality, let $I_\tau = \{1, 2, \ldots, k\}$. Let H be the subgraph induced by $\langle \tau \rangle$ in G and let $Q = \{ \langle \tau \rangle \cap A_i \mid 1 \leq i \leq k \}$. Then by Observation 1.3, the k-partitioned graph (H, Q), is k-psc. Hence by Observation 1.4, $(2t+1)^2 k(k-1)/4$ is an integer and so $k \equiv 0$ or $1 \pmod 4$.

Suppose now $k \geq 2$ and τ^2 is an automorphism of H. We now claim that $|\tau|$ is even. If possible, let $|\tau| = 2\alpha+1$ for some α. Since $k \geq 2$, $E(H) \neq \emptyset$. Let $uv \in E(H)$. Then since τ^2 is an automorphism of H, it follows that $\tau^{2\alpha+2}(u)\, \tau^{2\alpha+2}(v) \in E(H)$, i.e. $\tau(u)\, \tau(v) \in E(H)$, a contradiction since $\tau(u)$ and $\tau(v)$ are adjacent in $\bar{H}(Q)$. Hence

$|\tau|$ is even. But $|\tau| = (2t+1)k$. Hence $k \equiv 0 \pmod 4$.
This completes the proof of the theorem. □

COROLLARY 1.7. If (G,P) is bipsc, $\sigma \in \mathcal{C}((G,P))$ and a cycle τ of σ intersects each of A_1, A_2 in exactl t vertices, then t is even.

Let (G,P) be r-psc and $\sigma \in \mathcal{C}((G,P))$. we define σ to be P-invariant if σ maps each A_i into some A_j. We denote by $\mathcal{C}^*((G,P))$ the class of all P-invariant r-pc

We now show that if σ is P-invariant and $\sigma(A_i) \subseteq A$ then equality holds. For this define a digraph D (with lo allowed) on the vertex set $\{A_1,\ldots,A_r\}$ by joining A_i to A if $\sigma(A_i) \subseteq A_j$. Clearly then every vertex of D has outd exactly 1 and indegree at least 1, hence the indegree of e vertex is exactly 1. From this we immediately have

OBSERVATION 1.8. Let (G,P) be r-psc and $\sigma \in \mathcal{C}^*$ Then, $u,v \in A_i$ for some i iff $\sigma(u), \sigma(v) \in A_j$ for some

The P-invariant complementing permutations have many interesting properties. The rest of this chapter deals wit complementing permutations and their structural properties. first prove the following

THEOREM 1.9. If (G,P) is r-psc and $\sigma \in \mathcal{C}^*((G,P$ then $\sigma^2 \in \mathrm{Aut}(G)$.

PROOF : Let $u,v \in V(G)$. If u,v belong to some set of P then $\sigma^2(u)$, $\sigma^2(v)$ also belong to some set of P. If u,v belong to different sets of P, then $\sigma(u)$, $\sigma(v)$ as well as $\sigma^2(u)$, $\sigma^2(v)$ belong to different sets of P and $uv \in E(G)$ iff $\sigma(u)\,\sigma(v) \in E(\overline{G}(P))$ iff $\sigma(u)\,\sigma(v) \notin E(G)$ iff $\sigma^2(u)\,\sigma^2(v) \notin E(\overline{G}(P))$ iff $\sigma^2(u)\,\sigma^2(v) \in E(G)$. This proves the theorem. □

The corresponding result for self-complementary graphs can be deduced as a corollary.

COROLLARY 1.10. Let G be self-complementary. If σ is a complementing permutation of G, then $\sigma^2 \in \mathrm{Aut}\,(G)$.

This corollary follows from the fact that if P is the partition of $V(G)$ consisting of singleton sets and $|V(G)|=p$, then (G,P) is p-psc and $\sigma \in \mathcal{C}^*((G,P))$.

One can also prove the following theorem on the cycle lengths of a complementing permutation of a self-complementary graph.

THEOREM 1.11. (Ringel [17], Sachs [18]). Let G be self-complementary and σ be a complementing permutation of G. Then either $|V(G)| \equiv 0 \pmod 4$ and all cycles of σ have lengths

$\equiv$ 0 (mod 4), or, $|V(G)| \equiv 1$ (mod 4) and all but one cycle of σ have lengths $\equiv 0$ (mod 4), the remaining cycle having length one.

PROOF : Let $|V(G)| = p$. By Corollary 1.10, $\sigma^2 \in$ Aut We now consider G as a p-psc graph, where the p-partition V(G) consists of singleton sets. By Theorem 1.6 (i) it now follows that σ has at most one cycle of length 1. Further any cycle τ of σ with $|\tau| \geq 2$ satisfies the hypothe of Theorem 1.6 (ii) with $t = 0$ and $k = |\tau|$. Since $\sigma^2 \in$ Aut (G), it follows that if $|\tau| \geq 2$, then $|\tau| \equiv$ ((mod 4). This proves the theorem. □

In the case of connected bipsc graphs Theorem 1.9 re to the following

THEOREM 1.12. Let (G,P) be connected bipsc. Then $\mathcal{C}((G,P)) = \mathcal{C}^*((G,P))$ and $\sigma^2 \in$ Aut (G) for all $\sigma \in$

PROOF : Let $\sigma \in \mathcal{C}((G,P))$. Let $u,v \in A_i$ for som The since (G,P) is a connected bipartitioned graph, the between u and v in G is even and so by Observation 1 distance between $\sigma(u)$ and $\sigma(v)$ in $\overline{G}(P)$ is also even. follows that $\sigma(u), \sigma(v) \in A_j$ for some j. Thus $\sigma \in \mathcal{C}$ The rest of the theorem follows from Theorem 1.9. □

Let (G,P) be r-psc and $\sigma \in \mathcal{C}((G,P))$. A cycle τ of σ is said to be <u>k-periodic</u> if τ is of the form

$$(u_{11}\ u_{21} \ldots u_{k1}\ u_{12}\ u_{22} \ldots u_{k2} \ldots u_{1\alpha}\ u_{2\alpha} \ldots u_{k\alpha})$$

where $u_{tj} \in A_{i_t}$, $1 \leq j \leq \alpha$, $1 \leq t \leq k$ and $i_1, i_2, \ldots, i_k$ are distinct indices.

The cycles of a P-invariant complementing permutation have nice periodic structures. This is established in the following

<u>THEOREM 1.13</u>. Let (G,P) be r-psc and $\sigma \in \mathcal{C}^*((G,P))$. Let τ be a cycle of σ with $|I_\tau| = k$. Then

(i) τ is k-periodic.

(ii) If Ψ is any other cycle of σ with $I_\Psi \cap I_\tau \neq \emptyset$, then (a) $I_\Psi = I_\tau$ and (b) if τ takes vertices in A_i to A_j then so does Ψ.

<u>PROOF</u> : Let $i, i_2, \ldots, i_\ell$ be distinct indices in I_τ and u be a vertex in $\langle \tau \rangle \cap A_{i_1}$ such that $\tau^t(u) \in A_{i_{t+1}}$ when $1 \leq t \leq \ell-1$ and $\tau^\ell(u) \in A_{i_1}$. Since $\sigma \in \mathcal{C}^*((G,P))$, it follows that $\sigma(A_{i_t}) = A_{i_{t+1}}$ when $1 \leq t \leq \ell-1$ and $\sigma(A_{i_\ell}) = A_{i_1}$. Since τ is a cycle of σ and $|I_\tau| = k$, we get $\ell = k$. Thus

τ is k-periodic and the first part of the theorem is proved. The second part follows easily. □

The following Corollary is immediate from Theorem 1.13.

<u>COROLLARY 1.14</u>. Let (G,P) be r-psc, $\sigma \in \mathcal{C}^*((G,P))$ and τ be a cycle of σ. If $i,j \in I_\tau$, then $n_i = n_j$.

The consequences of Theorem 1.13 in the case of connect bipsc graphs can be summed up in the following

<u>COROLLARY 1.15</u>. Let (G,P) be connected bipsc. Then $\mathcal{C}((G,P)) = \mathcal{C}_p((G,P)) \bigcup \mathcal{C}_m((G,P))$. Further if $\sigma \in \mathcal{C}_m((G,P))$, and τ is a cycle of σ, then $|\tau| \equiv 0$ (mod and τ takes vertices alternatively from A_1 and A_2.

<u>PROOF</u> : Let $\sigma \in \mathcal{C}((G,P))$. By Theorem 1.12, $\sigma \in \mathcal{C}^*((G,P))$. Now if $\sigma \notin \mathcal{C}_p((G,P))$ then, for some cycl τ of σ, $|I_\tau| = 2$. It now follows by Theorem 1.13 that all cycles of σ are 2-periodic and thus $\sigma \in \mathcal{C}_m((G,P))$. This proves the first part of the Corollary. The second part foll easily from Theorem 1.13 and Corollary 1.7. □

CHAPTER 2

SOME CLASSES OF MULTIPARTITE SELF-COMPLEMENTARY GRAPHS

In this chapter we characterise when certain simple graphs like trees, forests, unicyclic graphs and cacti are r-psc. Throughout this chapter G will stand for a graph with p vertices and q edges.

We first characterise all r-psc graphs whose components are trees or unicyclic graphs. For $r = 2$, the characterisation is given in Theorem 2.1 and for $r \geq 3$, in Theorem 2.2.

Let G be a graph with k components, s of which are unicyclic and the remaining are trees. Then the following equation holds for G :

$$q = p - k + s \qquad \ldots(2.1)$$

Using this equation we prove the following

THEOREM 2.1. A bipartitioned graph (G,P) with k components, s of which are unicyclic and the remaining are trees, is bipsc iff exactly one of the following conditions holds :

(a) $n_1 = 1$, $n_2 = 2q$,

(b) $n_1 = 2$, $n_2 = d$,

(c) (G,P) is one of the bipartitioned graphs listed in Figure 2.1.

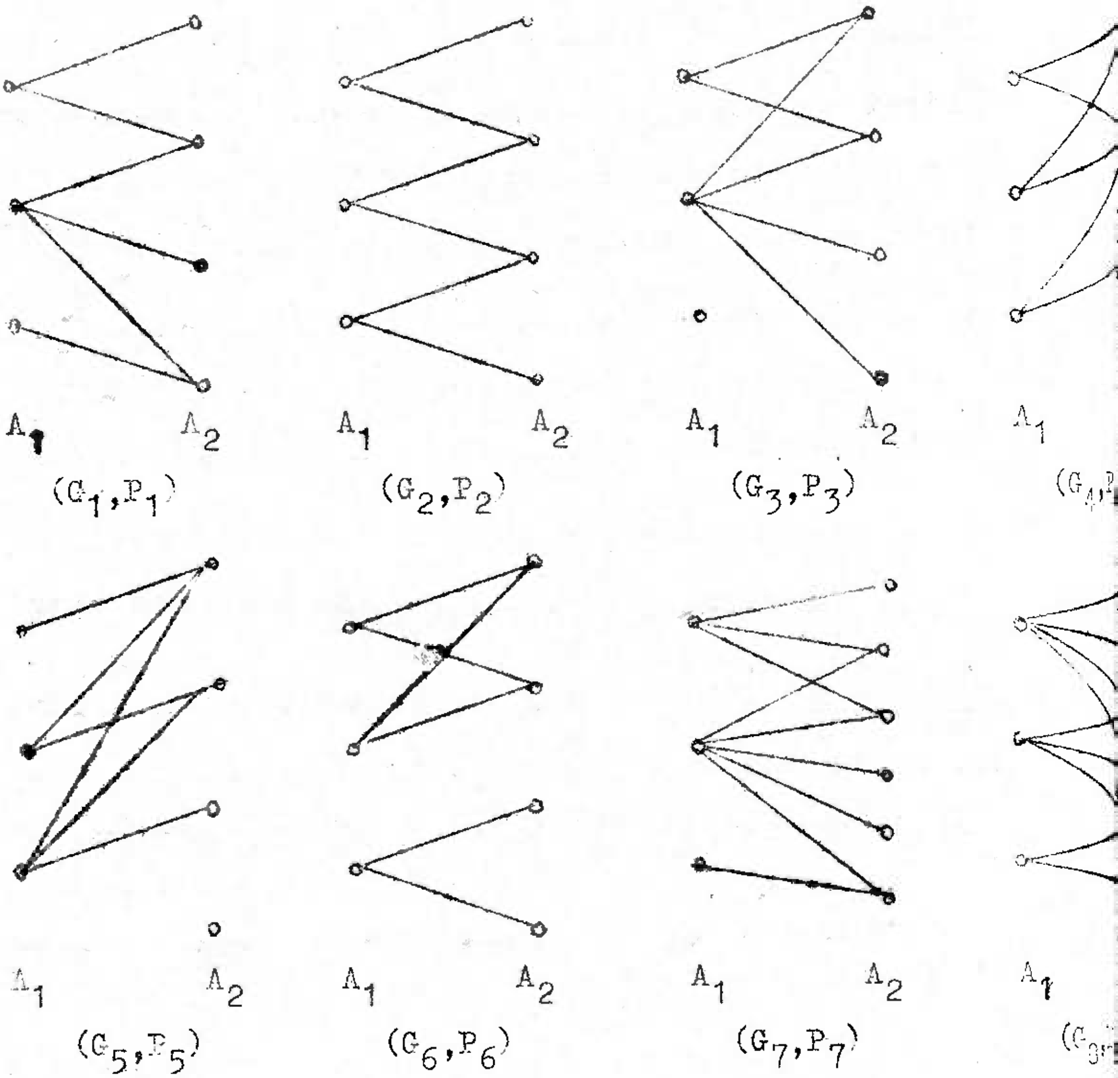


FIGURE 2.1

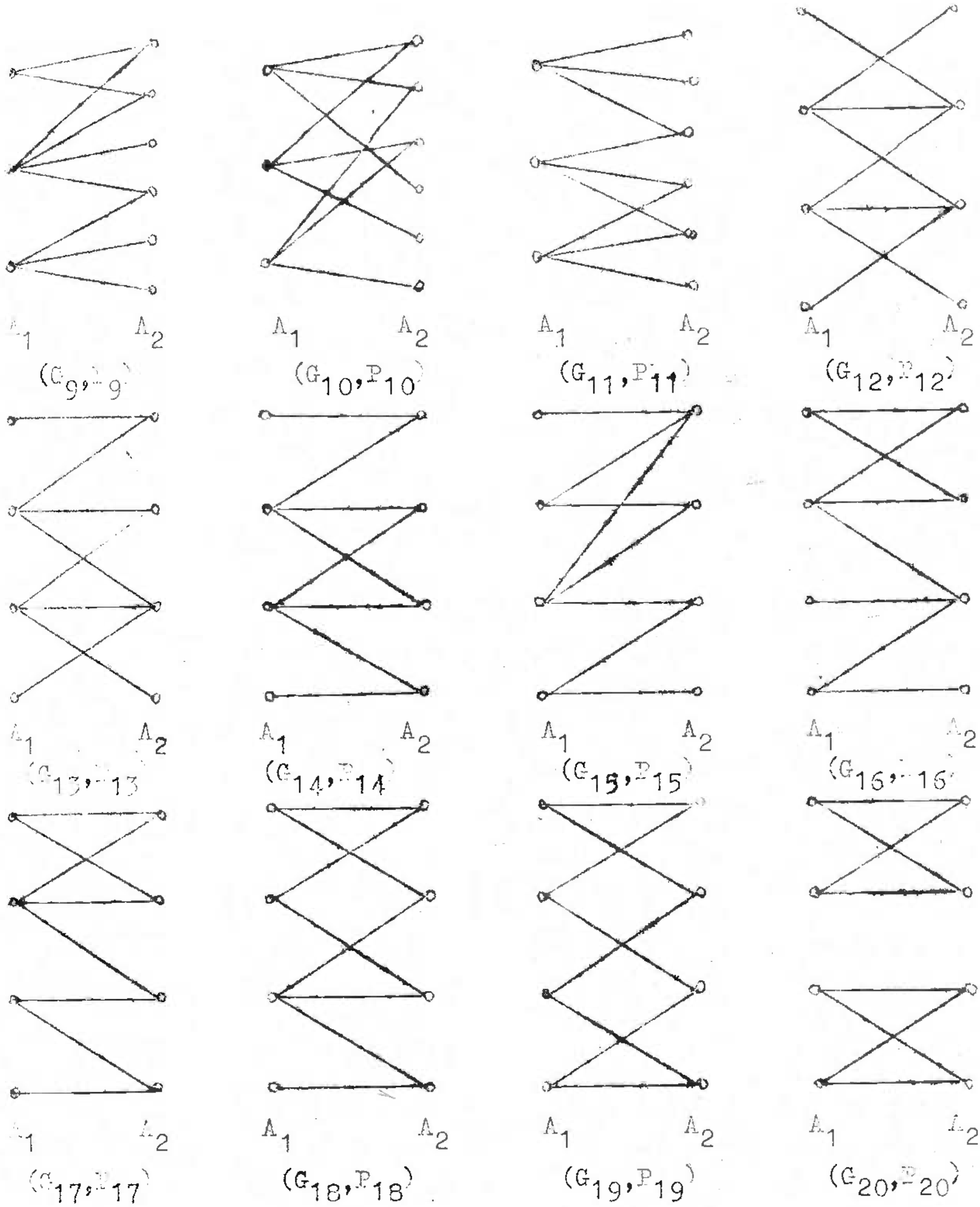


FIGURE 2.1 (Contd.)

PROOF : (Necessity). Let (G,P) be a bipsc graph [...] components, s of which are unicyclic and k-s of which are [...] Then $p = n_1+n_2$ and $q = \frac{n_1 n_2}{2}$. Now if $n_1 = 1$, then (a) ho[lds] and if $n_1 = 2$, then (b) holds. So let $n_1 \geq 3$. Substitu[ting] $p = n_1+n_2$ and $q = \frac{n_1 n_2}{2}$ in (2.1) and simplifying, we obt[ain]

$$(n_1 - 2)(n_2 - 2) = 2(s - k + 2) \quad \ldots($$

Since $n_2 \geq n_1 \geq 3$, it follows that $k = s$ or $s+1$. We acc[ordingly] consider two cases :

Case 1. $k = s+1$. Then $n_1 = 3$, $n_2 = 4$ and so [...] It can now be easily verified that if $s = 0$, then (G,P) [is] one of the graphs (G_1,P_1), (G_2,P_2), and if $s = 1$, then [(G,P)] is the graph (G_i,P_i) for some i, $3 \leq i \leq 6$, exhibite[d in] Figure 2.1. Thus (c) holds in this case.

Case 2. $k = s$. Then either (i) $n_1 = 3$, $n_2 = 6$, k [...] or (ii) $n_1 = n_2 = 4$, $1 \leq k = s \leq 2$. It can now be easil[y] verified that if (i) holds, then (G,P) is the graph (G[$_i$,P$_i$)] for some i, $7 \leq i \leq 11$, if (ii) holds, and $k = s = 1$, [then] (G,P) is the graph (G_i,P_i) for some i, $12 \leq i \leq 19$ [and if] (ii) holds and $k = s = 2$, then (G,P) is the graph (G[...] exhibited in Figure 2.1. Thus (c) holds in this case al[so].

This completes the proof of necessity.

(Sufficiency). Let (G,P) be any bipartitioned graph. We will show that if any of (a), (b), (c) holds, then (G,P) is bipsc. For this let $A_1 = \{u_1,\ldots,u_{n_1}\}$ and $A_2 = \{v_1,\ldots,v_{n_2}\}$.

First if (a) holds, then $n_1 = 1$, $n_2 = 2q$ and without loss of generality, we may assume that in G, u_1 is adjacent to $v_1, v_2, \ldots, v_q$. Clearly now $\sigma = (u_1) \prod_{j=1}^{q} (v_j \; v_{2q+1-j}) \varepsilon \; \mathcal{C}_p((G,P))$ and (G,P) is bipsc.

Next, if (b) holds, then $n_1 = 2$, $n_2 = q$. Let $k = |N_G(u_1) \cap N_G(u_2)|$ and $d = d_G(u_1)$. Then without loss of generality we may assume that in G, u_1 is adjacent to $v_1,\ldots,v_d$ and u_2 is adjacent to $v_1,\ldots,v_k, v_{d+1},\ldots,v_{q-k}$. Clearly now $\sigma = (u_1 u_2) \prod_{j=1}^{k} (v_j v_{q+1-j}) \prod_{j=k+1}^{q-k} (v_j) \; \varepsilon \; \mathcal{C}_p((G,P))$ and (G,P) is bipsc.

Finally if (c) holds, then (G,P) is the graph (G_i,P_i) for some i, $1 \leq i \leq 20$ given in Figure 2.1, and it can be easily verified that (G,P) is bipsc.

This completes the proof of sufficiency and Theorem 2.1 is proved. □

We next prove the following

<u>THEOREM 2.2.</u> Let $r \geq 3$ and (G,P) be an r-partit graph with k components, s of which are unicyclic, the ing $k-s$ being trees. Then (G,P) is r-psc iff exactly the following conditions holds :

(a) $r = 3$ and (G,P) is one of the tripartitioned grap listed in Figure 2.2,

(b) $r = 4$ and (G,P) is the 4-partitioned graph given Figure 2.3,

(c) $r = 5$ and (G,P) is one of the two 5-partitioned graphs given in Figure 2.4.

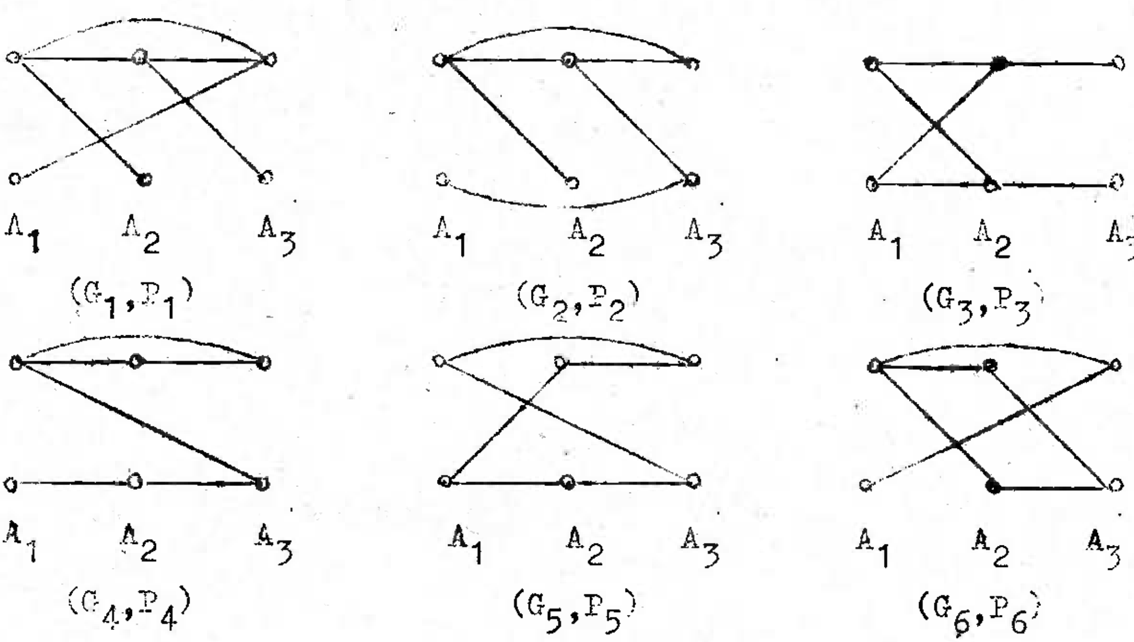


FIGURE 2.2

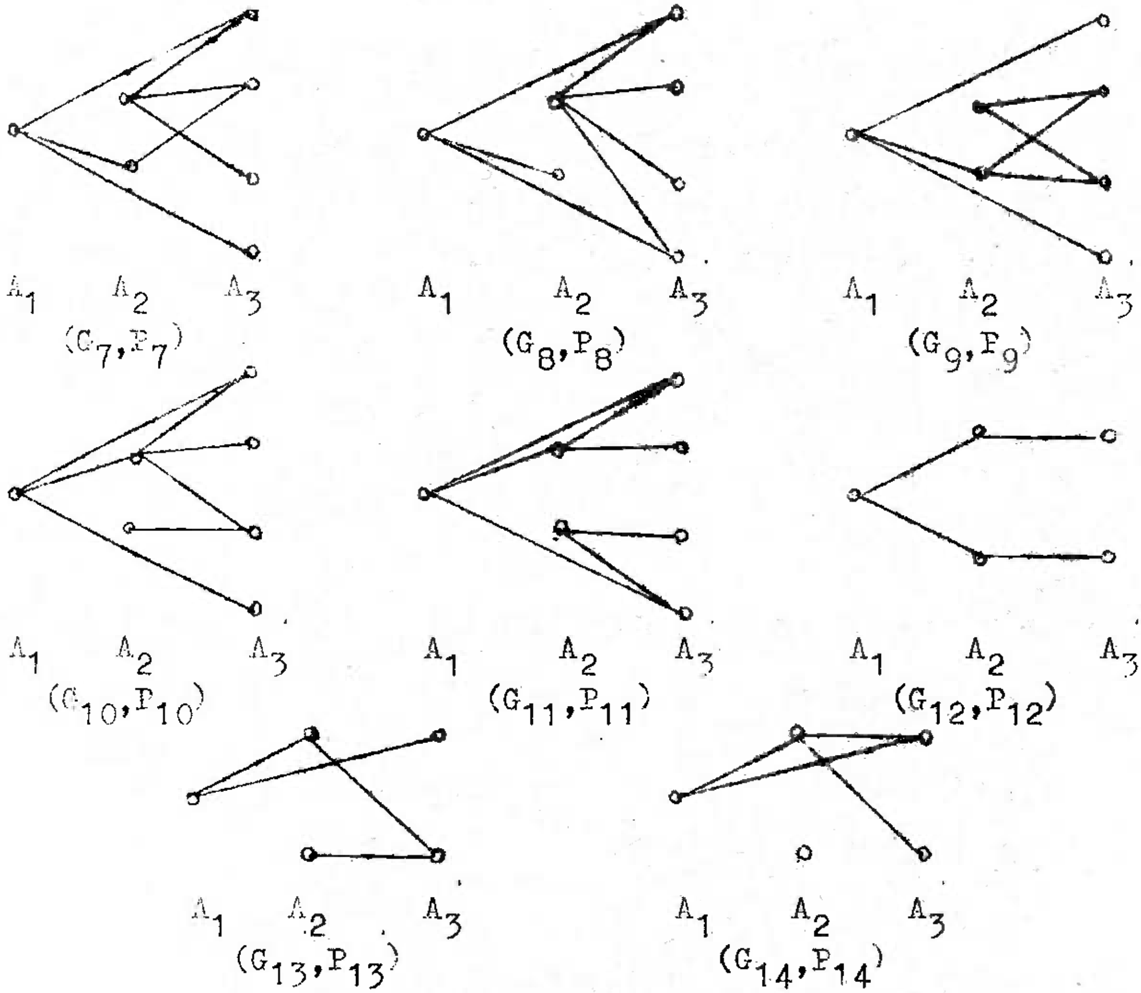


FIGURE 2.2 (Contd.)

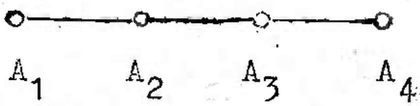


FIGURE 2.3

FIGURE 2.4

PROOF : Sufficiency is a matter of simple verification. To prove the necessity, let (G,P) be an r-psc graph with k components, s of which are unicyclic and $k-s$ of which are trees. By (2.1) we have

$$q - p = s - k \leq 0. \qquad \ldots(2.3)$$

Also, since $p = \sum_{i=1}^{r} n_i$ and $q = \frac{1}{2} \sum_{i<j}\sum n_i n_j$, it follows that

$$2(q-p) = (n_1-2) \sum_{i=2}^{r} n_i + (n_2n_3-2n_1) + n_2 \sum_{i=4}^{r} n_i + \sum_{i=3}^{r} \sum_{j=i+1}^{r} n_i n_j . \qquad \ldots(2.4)$$

First let $n_1 \geq 2$. Then $n_2n_3 \geq 2n_1$ and so by (2.3) it follows that $r = 3$, $n_1 = n_2 = n_3 = 2$ and $k = s$. It can now be easily verified that (G,P) is the graph (G_i,P_i) for some i, $1 \leq i \leq 6$, given in Figure 2.2, and (a) holds.

Next let $n_1 = 1$. Then from (2.4) we have

$$2(q-p) = (n_2-2) \sum_{i=3}^{r} n_i + \left(\sum_{i=3}^{r} n_i - n_2 - 2\right) + \sum_{i=3}^{r} \sum_{j=i+1}^{r} n_i n_j . \qquad \ldots(2.5)$$

If now $n_2 \geq 2$, then by (2.3) we have that $r = 3$ and so $(n_2-1)(n_3-1) \leq 3$. Hence it follows that $n_2 = 2$, $n_3 \leq 4$. Also since q is an integer it follows that n_3 is even. It can now be easily verified that (G,P) is the graph (G_i,P_i) for some i, $7 \le i \le 14$, given in Figure 2.2, and (a) holds. So let $n_2 = 1$. Then since $q = \frac{1}{2} \sum\sum_{i<j} n_i n_j$, it follows that $r \ge 4$. Also from (2.5) we have

$$2(q-p) = \sum_{i=3}^{r} \sum_{j=i+1}^{r} n_i n_j - 3 . \qquad \ldots(2.6)$$

By (2.3), it now follows that $r \le 5$. Thus $r = 4$ or 5. Now if $r = 4$, then by (2.3) and (2.6) it follows that $n_3 = 1$, $n_4 \le 3$ and since $q = \frac{1}{2} \sum\sum_{i<j} n_i n_j$, it also follows that n_4 is odd. But there is no 4-psc graph with $(n_1, n_2, n_3, n_4) = (1,1,1,3)$. It can now be easily verified that (G,P) is the graph shown in Figure 2.3 and (b) holds. Finally, if $r = 5$, then by (2.3) and (2.6) it follows that $n_3 = n_4 = n_5 = 1$. It can now be easily verified that (G,P) is one of the two graphs given in Figure 2.4 and (c) holds.

This completes the proof of necessity and Theorem 2.2 is proved. □

Next for $r \ge 2$, we characterise when an r-partitioned cactus (G,P) is also r-psc. If G is a cactus, then G satisfies (See Rao [11]),

$$2q \le 3(p-1) \qquad \ldots(2.7)$$

as well as

$$q = p - 1 + s \qquad \ldots (2.C$$

where s is the number of cycles in G. Also by induction on the number of blocks one can easily prove the following

LEMMA 2.3. If a cactus has an independent set of siz α then it has at most $p-\alpha-1$ cycles.

Using these results we prove the following

THEOREM 2.4. A bipartitioned graph (G,P) is a bipse cactus iff (G,P) is one of the graphs given in Figure 2.5.

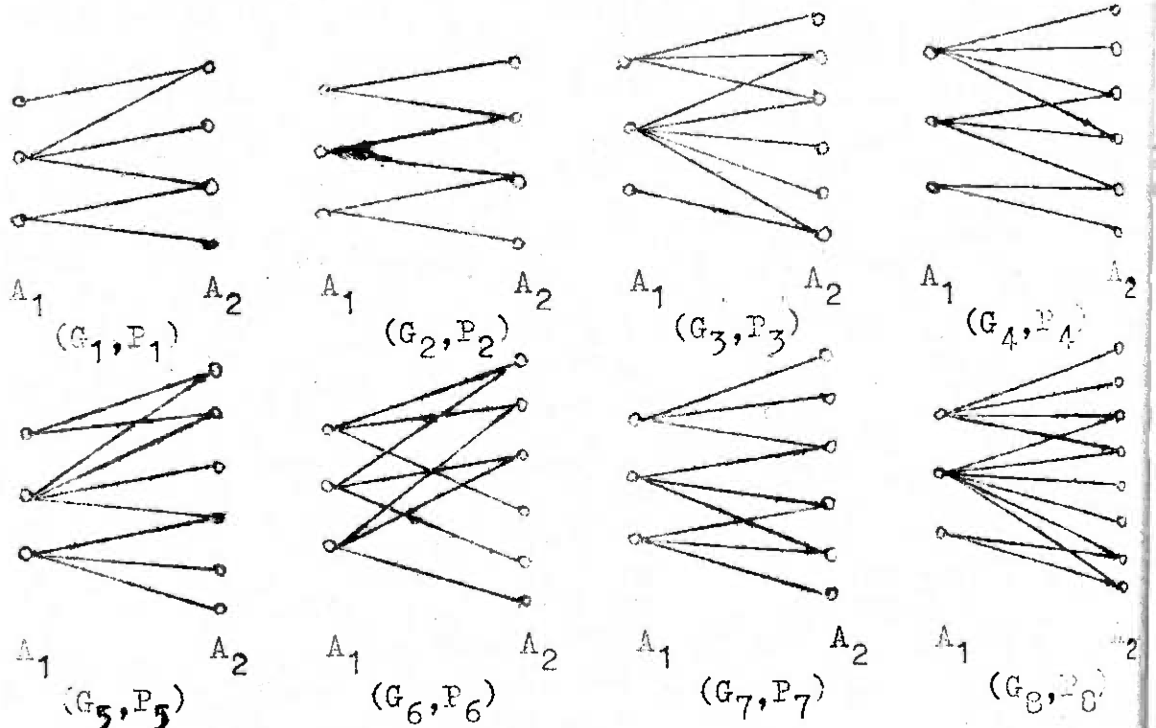


FIGURE 2.5

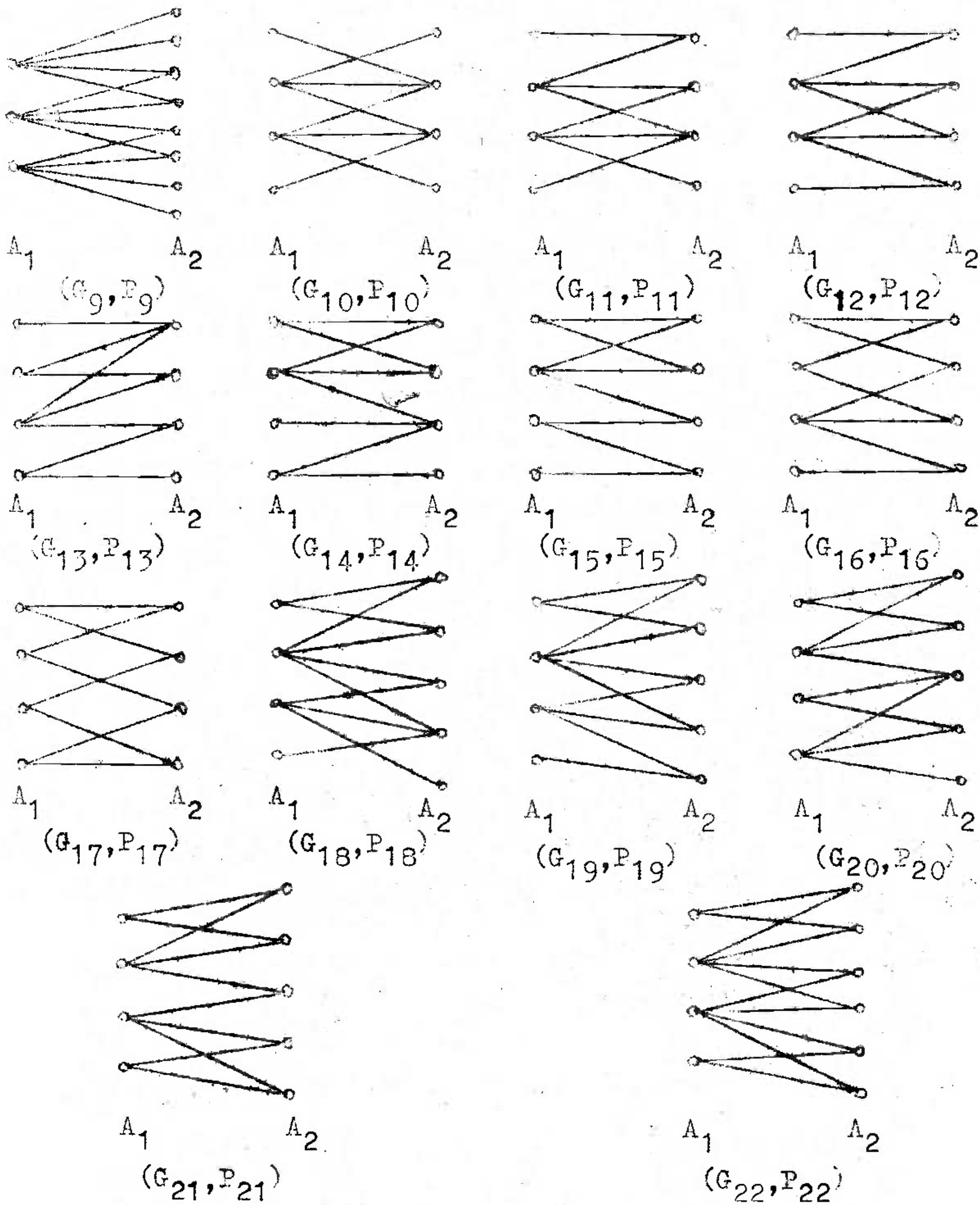


FIGURE 2.5 (Contd.)

PROOF : It can be easily verified all the graphs in Figure 2.5 are bipsc cacti.

To prove the converse, let (G,P) be a bipsc cactus. Then $p = n_1 + n_2$, $q = \frac{n_1 n_2}{2}$. Since G is connected it follo that $n_1 \geq 3$. Also by (2.7), we have

$$(n_1 - 3)(n_2 - 3) \leq 6$$

and so $n_1 \leq 5$. Let s be the number of cycles in G. The by Lemma 2.3, $s \leq n_1 - 1$. Hence by (2.8) we have

$$n_1 n_2 = 2q = 2(p+s-1) \leq 2(2n_1 + n_2 - 2)$$

and so

$$n_2 \leq \frac{4n_1 - 4}{n_1 - 2}.$$

If now $n_1 = 3$, then $n_2 \leq 8$ and by Observation 1.4, n_2 is even. It can now be easily verified that (G,P) is the gra (G_i, P_i) for some i, $1 \leq i \leq 9$, given in Figure 2.5. Next i $n_1 = 4$, then $n_2 \leq 6$. It can now be easily verified that (G, is the graph (G_i, P_i) for some i, $10 \leq i \leq 22$. Finally, i $n_1 = 5$, then $n_2 \leq 5$, a contradiction. This proves Theorem

Next for $r \geq 3$, we characterise when a given r-parti-tioned cactus is r-psc in the following

<u>THEOREM 2.5</u>. Let $r \geq 3$ and (G,P) be an r-parti-tioned cactus. Then (G,P) is r-psc iff exactly one of the

following conditions holds :

(a) $r = 3$ and (G,P) is one of the tripartitioned graphs listed in Figure 2.6. ,

(b) $r = 4$ and (G,P) is the 4-partitioned graph given in Figure 2.3. ,

(c) $r = 5$ and (G,P) is one of the two 5-partitioned graphs given in Figure 2.4.

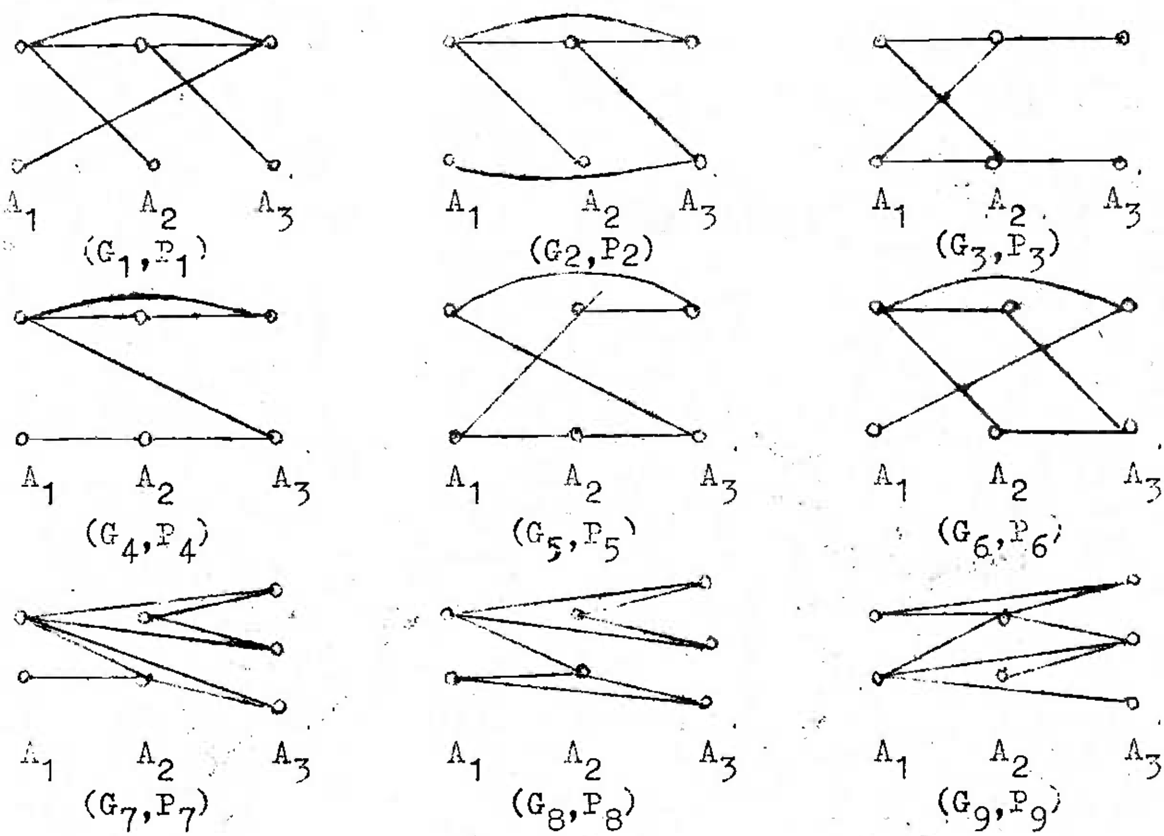


FIGURE 2.6

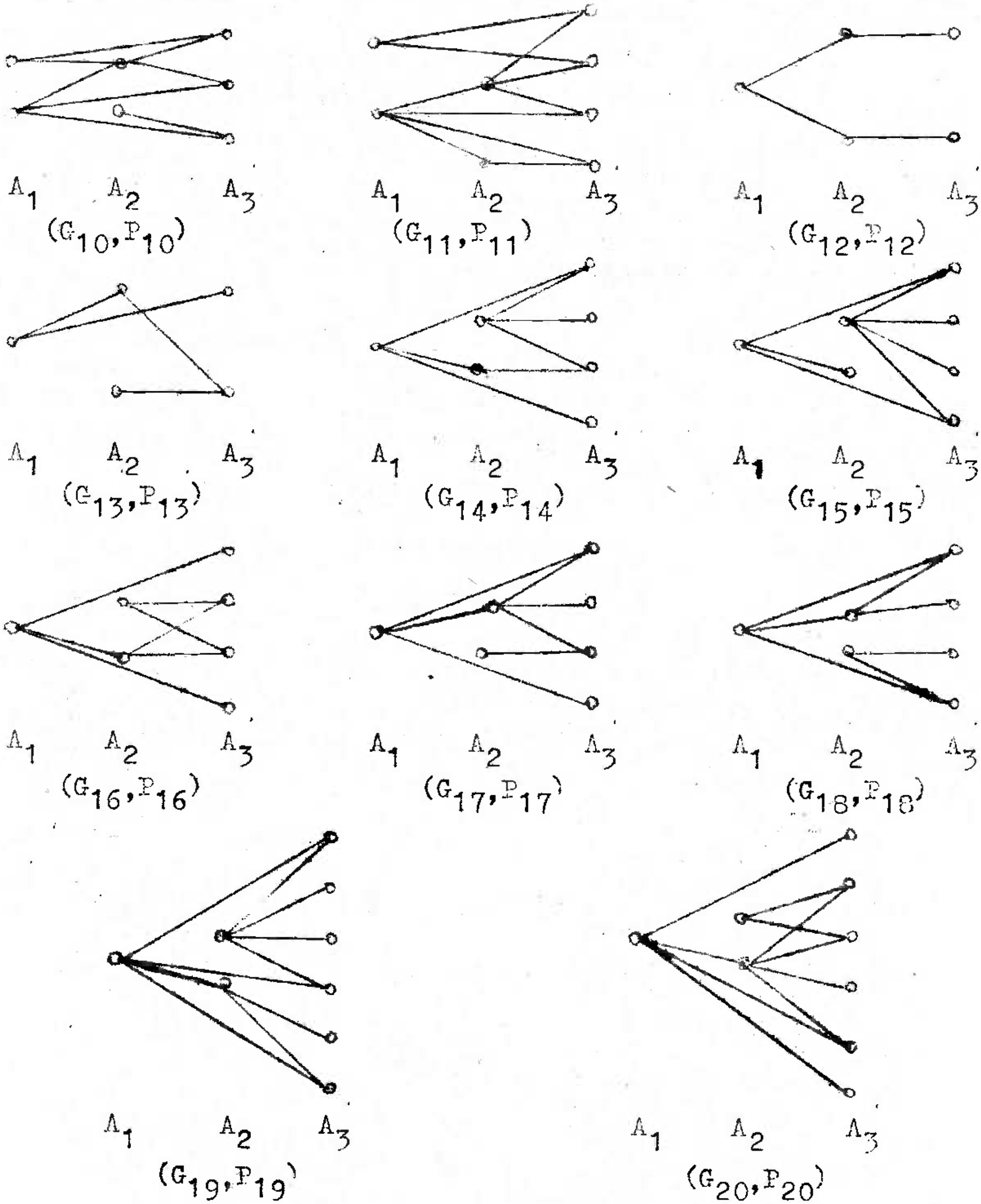


FIGURE 2.6 (Contd.)

PROOF : Sufficiency can be easily verified. To prove the necessity, let (G,P) be an r-psc cactus. Then $p = \sum_{i=1}^{r} n_i$ and $q = \frac{1}{2} \sum_{i<j} \sum n_i n_j$. Now from (2.7) we have

$$(n_1-3) \sum_{i=2}^{r} n_i + (n_2 n_3 - 3n_1) + n_2 \sum_{i=4}^{r} n_i + \sum_{i=3}^{r} \sum_{j=i+1}^{r} n_i n_j + 3 \leq 0 \qquad \ldots(2.9)$$

From this it follows that $n_1 \leq 2$. We now consider two cases as follows :

Case 1. $n_1 = 2$. Thus from (2.9) we have

$$(n_2-2) \sum_{i=3}^{r} n_i + \left(\sum_{i=3}^{r} n_i - n_2 - 3 \right) + \sum_{i=3}^{r} \sum_{j=i+1}^{r} n_i n_j \leq 0 \qquad \ldots(2.10)$$

From this it follows that $r = 3$. Also then (2.10) reduces to

$$(n_2 - 1)(n_3 - 1) \leq 4 \qquad \ldots(2.11)$$

If now $n_2 = 3$, then by (2.11) $n_3 = 3$ and q is not an integer, a contradiction. Thus $n_2 = 2$, and G has an independent set of size p-4. By Lemma 2.3, it follows that G has at most 3 cycles. Hence by (2.8)

$$4(n_3 + 1) = 2q \leq 2(n_3 + 6)$$

and so $n_3 \leq 4$. It can now be easily verified that the only 3-psc cacti with $(n_1, n_2, n_3) \in \{(2,2,2), (2,2,3), (2,2,4)\}$

are the graphs (G_i, P_i), $1 \leq i \leq 11$, given in Figure 2.6. Thus (a) holds in this case.

Case 2. $n_1 = 1$. We now consider three subcases as follows :

Case 2(a). $r = 3$. Then by (2.9) we obtain

$$(n_2 - 2)(n_3 - 2) \leq 4 \qquad \ldots(2.12)$$

and so $n_2 \leq 4$. Since q is an integer both n_2 and n_3 are even. If now $n_2 = 2$, then by Lemma 2.3, G has at most 2 cycles and so by (2.8) it follows that $n_3 \leq 6$. Again if $n_2 = 4$, then by (2.12), $n_3 = 4$. It can now be easily verified that the only 3-psc cacti with $(n_1, n_2, n_3) \in \{(1,2,2), (1,2,4), (1,2,6), (1,4,4)\}$ are the graphs (G_i, P_i), $12 \leq i \leq 20$, given in Figure 2.6. Thus (a) holds in this subcase.

Case 2(b). $r = 4$. Then by (2.9) we obtain

$$(n_2 - 2)(n_3 + n_4) + (n_3 n_4 - 2n_2) \leq 0 \qquad \ldots(2.13)$$

and so $n_2 \leq 2$.

First let $n_2 = 1$. Then from (2.13) we get

$$(n_3 - 1)(n_4 - 1) \leq 3$$

and so $n_3 \leq 2$. But since q is an integer and $n_1 = n_2 = 1$ it follows that both n_3 and n_4 are odd. Hence $n_3 = 1$.

then by Lemma 2.3, G has at most 2 cycles and so by (2.8), $n_4 \leq 5$. It can now be easily verified that the 4-partitioned graph given in Figure 2.3 is the only 4-psc cactus with $n_1 = n_2 = n_3 = 1$.

Next let $n_2 = 2$. Then from (2.13) we get $n_3 = n_4 = 2$. It can now be easily verified that there is no 4-psc cactus with $n_1 = 1$ and $n_2 = n_3 = n_4 = 2$.

Thus (b) holds in this subcase.

<u>Case 2(c)</u>. $r \geq 5$. Then from (2.9) we get

$$(n_2-2) \sum_{i=3}^{r} n_i + (n_3 \sum_{i=4}^{r} n_i - 2n_2) + \sum_{i=4}^{r} \sum_{j=i+1}^{r} n_i n_j \leq 0 \qquad \ldots(2.14)$$

and so $n_2 = 1$. Substituting this in (2.14) we get

$$(n_3 - 1) \sum_{i=4}^{r} n_i + (\sum_{i=4}^{r} \sum_{j=i+1}^{r} n_i n_j - n_3 - 2) \leq 0 .$$

From this it follows that $n_3 = n_4 = 1$. It also follows that either (i) $r = 5$ and $n_5 \leq 3$, or (ii) $r = 6$ and $n_5 = n_6 = 1$. But in the latter case q is not an integer. Thus $r = 5$, $n_1 = n_2 = n_3 = n_4 = 1$, and $n_5 \leq 3$. It can now be easily verified that the only 5-psc cacti satisfying these conditions are the graphs given in Figure 2.4. Thus (c) holds in this subcase.

This completes the proof of necessity and Theorem 2.5 is proved. □

CHAPTER 3

MULTIPARTITE COMPLEMENTARY GRAPHS AND THEIR DIAMETERS

In this chapter, we study the diameters of an r-partitioned graph (G,P) and its r-partite complement $\overline{G}(P)$. It is well known that the diameter of a self-complementary graph is either 2 or 3. The problem of determining the range of diameters for bipsc graphs is solved in Theorem 3.2 and the corresponding problem for r-psc graphs with $r \geq 3$ is solved in Theorem 3.5.

Given a connected r-partitioned graph (G,P) with diameter λ, we choose and fix u_o, $v_o \in V(G)$ such that $d_G(u_o, v_o) = \lambda$. Further we define

$$B_\mu = \{u \in V(G) \mid d_G(u_o, u) = \mu\} \text{ if } \mu \in \{0,1,\ldots,\lambda\}$$
$$= \emptyset \text{ otherwise.}$$

Then clearly $B_\mu \neq \emptyset$ for $\mu \in \{0,1,\ldots,\lambda\}$ and $\{B_o, B_1, \ldots, B_\lambda\}$ is a partition of $V(G)$.

As a preliminary to the determination of the range of diameters for bipsc graphs we now prove the following

THEOREM 3.1. If (G,P) is a connected bipartitioned graph with diameter at least seven, then $\overline{G}(P)$ has diameter at most four.

PROOF : Let λ be the diameter of G, $7 \leq \lambda < \infty$. Without loss of generality we assume that $u_0 \in A_1$. Then $B_\mu \subseteq A_1$ for all even μ and $B_\mu \subseteq A_2$ for all odd μ. We first observe the following :

Observation 1. If $0 \leq \mu \leq \lambda$ and $0 \leq t \leq 8$ then either $0 \leq \mu - t \leq \lambda$ and so $B_{\mu-t} \neq \emptyset$ or $0 \leq \mu + 8 - t \leq \lambda$ and so $B_{\mu+8-t} \neq \emptyset$.

Observation 2. If $u \in B_\mu$ and $v \in B_{\mu+2t+1}$ with $t \geq 1$, then u and v are adjacent in $\overline{G}(P)$.

Now, let $u,v \in V(G)$. We shall show that the distance between u and v in $\overline{G}(P)$ is at most 4. Without loss of generality let $u \in B_\mu$, $v \in B_\eta$ with $\mu \leq \eta$. We consider the following two cases :

Case 1. $\mu = \eta$. By Observation 1, there exists $w \in B_{\mu-3} \cup B_{\mu+5}$. By Observation 2, uw, vw are edges of $\overline{G}(P)$ and so $d_{\overline{G}(P)}(u,v) \leq 2$.

Case 2. $\mu < \eta$. We now consider the following two subcases :

Case 2.1. $B_\mu, B_\eta \subseteq A_i$ for some $i \in \{1,2\}$. Then $\eta = \mu+2t$ for some $t \geq 1$. Now if there exists $w \in B_{\mu-3} \cup B$ then by Observation 2, uw and vw are edges of $\overline{G}(P)$ and we a done. Otherwise $\mu \leq 2$ and $\eta + 2 \geq \lambda$. Since $\lambda \geq 7$, we have $t \geq 2$. If $t = 2$, then $7 \leq \lambda \leq \mu + 6$ and so $(\mu,\eta) = (1,5)$ o (2,6). Let $x \in B_{\mu+1}$, $y \in B_{\eta-1}$ and $z \in B_7$. If $(\mu,\eta) = (1,5$ then u y z x v is a 4-path in $\overline{G}(P)$ and if $(\mu,\eta) = (2,6)$, then $u y u_o x v$ is a 4-path in $\overline{G}(P)$. Finally if $t \geq 3$, then let $w \in B_{\mu+3}$. Then uw, vw are edges of $\overline{G}(P)$ and we are d

Case 2.2. $B_\mu \subseteq A_i$, $B_\eta \subseteq A_{3-i}$ for some $i \in \{1,2\}$. Then $\eta = \mu + 2t + 1$ for some $t \geq 0$. If $t \geq 1$, then by Observation 2, uv is an edge of $\overline{G}(P)$. If $t = 0$, then $\eta =$ and by Observation 1, there exist $w \in B_{\mu-2} \cup B_{\mu+6}$ and $x \in B_{\mu-5} \cup B_{\mu+3}$. By Observation 2, u x w v is a 3-path in $\overline{G}($

Thus for all $u,v \in V(G)$, $d_{\overline{G}(P)}(u,v) \leq 4$. This proves theorem. □

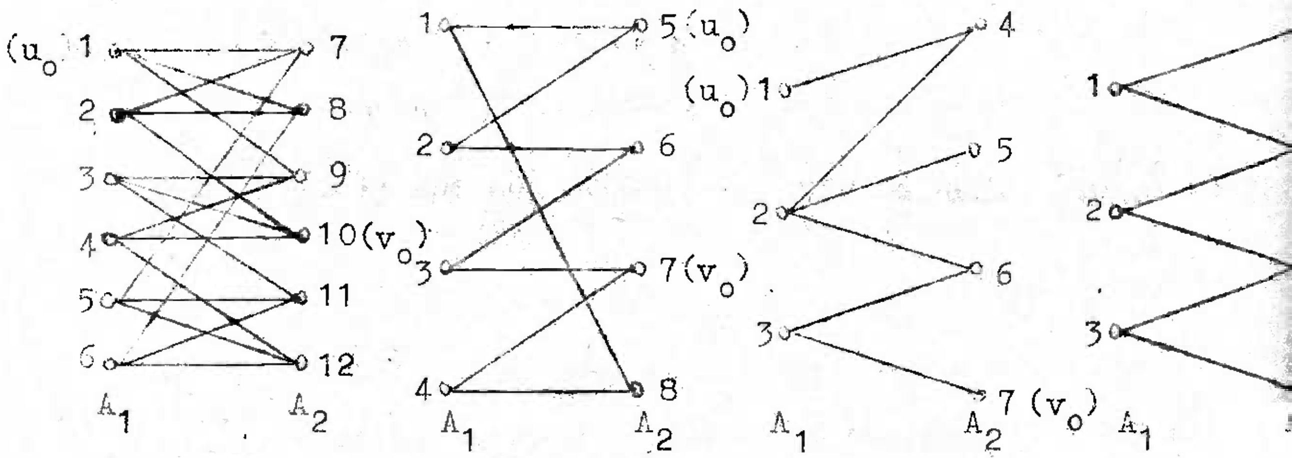
We now give the range of diameters in bipsc graphs in following

THEOREM 3.2. If (G,P) is a connected bipsc graph wi diameter λ, then $3 \leq \lambda \leq 6$. Further if $\lambda \in \{3,4,5,6\}$, th there is a bipsc graph with diameter λ on p vertices iff $p \geq p_\lambda$ where $(p_3, p_4, p_5, p_6) = (12, 8, 7, 7)$.

<u>PROOF</u> : Let (G,P) be a connected bipsc graph with diameter λ. By Theorem 3.1 it follows that $\lambda \leq 6$. But if $\lambda \leq 2$ then G is a complete bipartite graph and so (G,P) is not bipsc. Thus we have $3 \leq \lambda \leq 6$ and the first part of the theorem is proved.

Next, given $\lambda \in \{3,4,5,6\}$, we construct in Figure 3.1 a bipsc graph (G,P) with diameter λ on p_λ vertices and also specify an element σ in $\mathcal{G}_p((G,P))$ which has a cycle τ of length one.

Now given $k \geq 1$, we consider the bipartitioned graph $(G_\tau^{k+1}, P_\tau^{k+1})$ as constructed on page 13. By Theorem 1.5, $(G_\tau^{k+1}, P_\tau^{k+1})$ is bipsc on $(p_\lambda + k)$ vertices. Clearly the diameter of G_τ^{k+1} is λ. Thus there is a bipsc graph with diameter λ on p vertices if $p \geq p_\lambda$. This proves the 'if part' of the second statement in the theorem. The 'only if' part will be proved in Theorem 3.8. This completes the proof of Theorem 3.2. □



$\lambda = 3$ $\quad$ $\lambda = 4$ $\quad$ $\lambda = 5$ $\quad$ $\lambda = 6$

$\sigma = (1)(2)(3\ 5)(4\ 6)(7\ 12)(8\ 11)(9\ 10) \in \mathcal{G}_p((G,P))$

$\sigma = (1)(3)(2\ 4)(5\ 6)(7\ 8) \in \mathcal{G}_p((G,P))$

$\sigma = (1\ 2)(3)(4\ 7)(5\ 6) \in \mathcal{G}_p((G,P))$

$\sigma = (1\ 3)(2)(4\ 5)(6\ 7$ … $\in \mathcal{G}_p(($ …

FIGURE 3.1

Next, to determine the range of diameters for r-psc graphs with $r \geq 3$, we first prove the following preliminary

<u>THEOREM 3.3</u>. Let $r \geq 3$ and (G,P) be a connected r-partitioned graph. If the diameter of G is at least six then the diameter of $\overline{G}(P)$ is at most four.

<u>PROOF</u> : Let the diameter of G be λ and assume that $\lambda \geq 6$. For any integer μ, define

$$S_{\mu i} = B_{\mu} \cap A_i \ , \ 1 \leq i \leq r .$$

Clearly, $\{S_{\mu i} \mid 0 \leq \mu \leq \lambda, 1 \leq i \leq r\}$ is a partition of $V(G)$. We now make some observations which will be used repeatedly.

Observation 1. If $\mu \geq 1$ and $S_{\mu i} \neq \emptyset$, then $S_{\mu-1,j} \neq \emptyset$ for some $j \neq i$. Also, if $\mu < \lambda$ and $B_\mu \subseteq A_i$, then $S_{\mu+1,i} = \emptyset$.

Observation 2. If $0 \leq \mu \leq \lambda-1$ and $B_\mu \cup B_{\mu+1} \subseteq A_i \cup A_j$, then $S_{\mu i} \cup S_{\mu+1,i} \neq \emptyset$ and $S_{\mu j} \cup S_{\mu+1,j} = \emptyset$.

Observation 3. If $0 \leq \mu \leq \lambda$ and $0 \leq t \leq 7$ then either $0 \leq \mu-t \leq \lambda$ and so $B_{\mu-t} \neq \emptyset$, or, $0 \leq \mu+7-t \leq \lambda$ and so $B_{\mu+7-t} \neq \emptyset$.

Now, let $u, v \in V(G)$. We shall show that the distance between u and v in $\overline{G}(P)$ is at most 4. Without loss of generality let $u \in S_{\mu i}$, $v \in S_{\eta j}$ with $\mu \leq \eta$ and $i \leq j$. We consider the following four cases :

Case 1. $\mu = \eta$, $i = j$. By Observation 3, either $B_{\mu-3} \neq \emptyset$, or $B_{\mu+4} \neq \emptyset$. It now follows that for some $k \neq i$, $S^* = S_{\mu-3,k} \cup S_{\mu-2,k} \cup S_{\mu+3,k} \cup S_{\mu+4,k} \neq \emptyset$. Let $w \in S^*$. Then uw, vw are edges of $\overline{G}(P)$ and we are done.

Case 2. $\mu = \eta$, $i < j$. If for some $k \neq i,j$ and some $\alpha \notin \{\mu-1, \mu, \mu+1\}$, we have $S_{\alpha k} \neq \emptyset$, then for any $w \in S_{\alpha k}$,

we get a 2-path u w v in $\overline{G}(P)$. Otherwise $B_\alpha \subseteq A_i \cup A_j$ for all $\alpha \notin \{\mu-1, \mu, \mu+1\}$, and by Observation 2 it follows that $S_{\alpha i} \cup S_{\alpha+1,i} \neq \emptyset$ and $S_{\alpha j} \cup S_{\alpha+1,j} \neq \emptyset$ for all $\alpha \in \{0,1,\ldots,\mu-3\} \underset{\mu+1}{\cup} \{\mu+2, \mu+3,\ldots,\lambda-1\}$. Also since $r \geq 3$, it follows that $\bigcup_{\alpha=\mu-1}^{\mu+1} S_{\alpha k} \neq \emptyset$ for some $k \neq i,j$. Further by Observation 3, $B_{\mu-4} \cup B_{\mu+3} \neq \emptyset$ and $B_{\mu-3} \cup B_{\mu+4} \neq \emptyset$. Now if $S_{\mu-1,k} \cup S_{\mu k} \neq \emptyset$ for some $k \neq i,j$, then we take $x \in S_{\mu-1,k} \cup S_{\mu k}$, $y \in S_{\mu-4,i} \cup S_{\mu-3,i} \cup S_{\mu+2,i} \cup S_{\mu+3,i}$ and $z \in S_{\mu-4,j} \cup S_{\mu-3,j} \cup S_{\mu+2,j} \cup S_{\mu+3,j}$; Otherwise $S_{\mu+1,k} \neq \emptyset$ for some $k \neq i,j$ and we take
$x \in S_{\mu+1,k}$, $y \in S_{\mu-3,i} \cup S_{\mu-2,i} \cup S_{\mu+3,i} \cup S_{\mu+4,i}$ and $z \in S_{\mu-3,j} \cup S_{\mu-2,j} \cup S_{\mu+3,j} \cup S_{\mu+4,j}$. In either case u z x y v is a 4-path in $\overline{G}(P)$ and we are done.

<u>Case 3</u>. $\mu < \eta$, $i = j$. If $S_{\alpha k} \neq \emptyset$ for some $k \neq i$ and some $\alpha \notin \{\mu-1, \mu, \mu+1, \eta-1, \eta, \eta+1\}$, then for any $w \in S_{\alpha k}$, u w v is a 2-path in $\overline{G}(P)$ and we are done. Otherwise $B_\alpha \subseteq A_i$ for all $\alpha \notin \{\mu-1, \mu, \mu+1, \eta-1, \eta, \eta+1\}$. By Observation 1, it now follows that $\mu \leq 2$, $\lambda \leq \eta+2$ and $\eta \leq \mu+4$. Also since $\lambda \geq 6$, it follows that $\eta \geq \mu+2$. We now consider the following two subcases :

Case 3.1. $\mu = 0$. Then $\eta = 4$. By Observation 1 we have $S_{1k} \neq \emptyset$ for some $k \neq i$. Now, if there exist ℓ, m, $\ell \neq m \neq i$, such that $\bigcup_{\alpha \leq 2} S_{\alpha\ell} \neq \emptyset$, $\bigcup_{\alpha \geq 4} S_{\alpha m} \neq \emptyset$, then for $w \in \bigcup_{\alpha \leq 2} S_{\alpha\ell}$, $x \in \bigcup_{\alpha \geq 4} S_{\alpha m}$, $u\,x\,w\,v$ is a 3-path in $\overline{G}(P)$. Otherwise, $\bigcup_{\alpha \neq 3} B_\alpha \subseteq A_i \cup A_k$ and so, by Observation 2, $S_{5k} \cup S_{6k} \neq \emptyset$. Since $r \geq 3$, it also follows that $S_{3m} \neq \emptyset$ for some $m \neq i,k$. Now let $y_1 \in S_{5k} \cup S_{6k}$, $y_2 \in S_{3m}$ and $y_3 \in S_{1k}$. Then $u\,y_1 y_2 y_3\,v$ is a 4-path in $\overline{G}(P)$ and we are done.

Case 3.2. $\mu \geq 1$. Then by Observation 1, $S_{\mu-1,k} \neq \emptyset$ for some $k \neq i$. Now if there exist $\ell, m, \ell \neq m \neq i$, such that $\bigcup_{\alpha \leq \mu} S_{\alpha\ell} \neq \emptyset$, $\bigcup_{\alpha \geq \mu+2} S_{\alpha m} \neq \emptyset$, then, since $\eta \geq \mu+2$, it follows that for $w \in \bigcup_{\alpha \leq \mu} S_{\alpha\ell}$, $x \in \bigcup_{\alpha \geq \mu+2} S_{\alpha m}$, $u\,x\,w\,v$ is a 3-path in $\overline{G}(P)$. Otherwise, since $\lambda \geq \mu+4$ we have $\bigcup_{\alpha \neq \mu+1} B_\alpha \subseteq A_i \cup A_k$ and by Observation 2 it follows that $S_{\mu+3,k} \cup S_{\mu+4,k} \neq \emptyset$. Since $r \geq 3$, we also have $S_{\mu+1,m} \neq \emptyset$ for some $m \neq i,k$. Now let $y_1 \in S_{\mu+3,k} \cup S_{\mu+4,k}$, $y_2 \in S_{\mu+1,m}$ and $y_3 \in S_{\mu-1,k}$. Then $u\,y_1 y_2 y_3\,v$ is a 4-path in $\overline{G}(P)$ and we are done.

Case 4. $\mu < \eta$, $i < j$. If $\eta \geq \mu+2$, then uv is an edge of $\overline{G}(P)$ and we are done. So let $\eta = \mu+1$. Now, if $S_{\alpha k} \neq \emptyset$ for some $k \neq i,j$ and some $\alpha \notin \{\mu-1,\mu,\mu+1,\mu+2\}$, then for $w \in S_{\alpha k}$, $u\ w\ v$ is a 2-path in $\overline{G}(P)$. Otherwise $B_\alpha \subseteq A_i \cup A_j$, for all $\alpha \notin \{\mu-1,\mu,\mu+1,\mu+2\}$. But since $r \geq 3$, $S_{\beta k} \neq \emptyset$ for some $k \neq i,j$ and some $\beta \in \{\mu-1,\mu,\mu+1,\mu+2\}$. We now consider the following four subcases.

Case 4.1. $\mu \leq 1$. Then $\bigcup_{\beta=0}^{3} S_{\beta k} \neq \emptyset$. Also since $B_5 \cup B_6 \subseteq A_i \cup A_j$, by Observation 2 we have, $S_{5i} \cup S_{6i} \neq \emptyset$ and $S_{5j} \cup S_{6j} \neq \emptyset$. Let $x \in S_{5j} \cup S_{6j}$, $y \in \bigcup_{\beta=0}^{3} S_{\beta k}$, $z \in S_{5i} \cup S_{6i}$. Then $u\ x\ y\ z\ v$ is a 4-path in $\overline{G}(P)$ and we are done.

Case 4.2. $\mu = 2$. As before $S_{5i} \cup S_{6i} \neq \emptyset$, $S_{5j} \cup S_{6j} \neq \emptyset$. Let $x \in B_o$, $y \in S_{5i} \cup S_{6i}$ and $z \in S_{5j} \cup S_{6j}$. If $x \notin A_i$, then $u\ x\ y\ v$ is a 3-path in $\overline{G}(P)$ and if $x \in A_i$, then $u\ z\ x\ v$ is a 3-path in $\overline{G}(P)$. In either case, we are done.

Case 4.3. $\mu = 3$. Then $B_o \cup B_1 \subseteq A_i \cup A_j$ and so by Observation 2, $S_{oi} \cup S_{1i} \neq \emptyset$, $S_{oj} \cup S_{1j} \neq \emptyset$. Let $x \in B_6$, $y \in S_{oi} \cup S_{1i}$ and $z \in S_{oj} \cup S_{1j}$. If $x \notin A_i$, then $u\ x\ y\ v$

is a 3-path in $\overline{G}(P)$ and if $x \in A_i$, then $u\ z\ x\ v$ is a 3-path in $\overline{G}(P)$. In either case, we are done.

<u>Case 4.4</u>. $\mu \geq 4$. Then $\bigcup_{\beta=3}^{\mu+2} S_{\beta k} \neq \emptyset$ for some $k \neq i,j$. Also, as in Case 4.3, $S_{oi} \cup S_{1i} \neq \emptyset$, $S_{oj} \cup S_{1j} \neq \emptyset$. Let $x \in S_{oj} \cup S_{1j}$, $y \in \bigcup_{\beta=3}^{\mu+2} S_{\beta k}$, $z \in S_{oi} \cup S_{1i}$. Then $u\ x\ y\ z\ v$ is a 4-path in $\overline{G}(P)$ and we are done.

Thus we have shown that for any $u, v \in V(G)$, the distance between u and v in $\overline{G}(P)$ is at most four. This proves Theorem 3.3. □

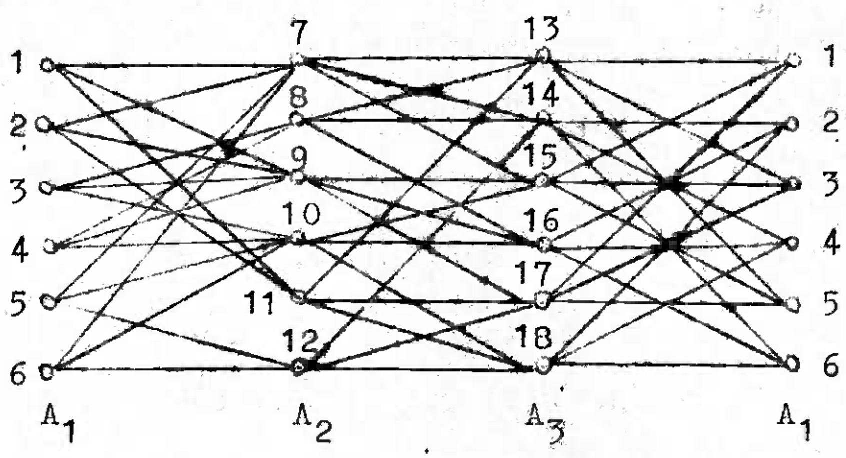
<u>COROLLARY 3.4</u>. If $r \geq 3$ and (G,P) is a connected r-partitioned graph with diameter at least six, then $\overline{G}(P)$ is connected.

We now determine the range of values taken by the diameter of an r-psc graph (G,P) with $r \geq 3$, in the following

<u>THEOREM 3.5</u>. Let $r \geq 3$. If (G,P) is connected r-psc with diameter λ, then $2 \leq \lambda \leq 5$. Further, there is an infinite class of r-psc graphs with diameter λ for each $\lambda \in \{2,3,4,5\}$.

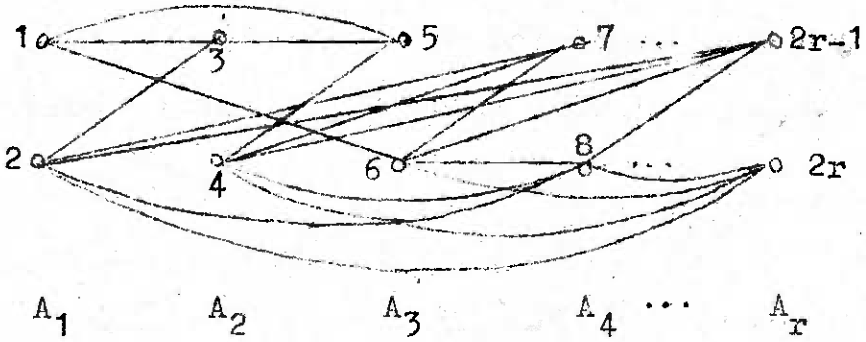
<u>PROOF</u> : Let $r \geq 3$ and (G,P) be a connected r-psc graph with diameter λ. Then $\overline{G}(P)$ also has diameter λ. It now follows by Theorem 3.3 that $\lambda \leq 5$. But if $\lambda = 1$ then G

is complete, $r = |V(G)|$ and so (G,P) is not r-psc, a contradiction. Hence $2 \leq \lambda \leq 5$ and the first part of the theorem is proved.



$r = 3,\ \lambda = 2$

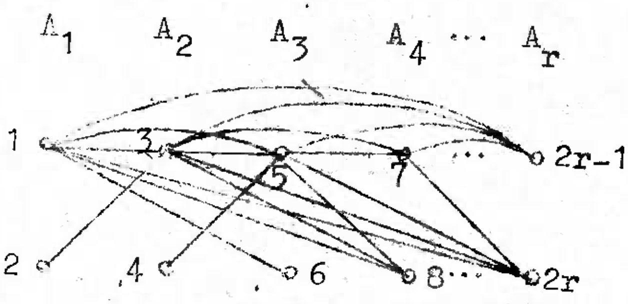
$\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 12)(10\ 11)(13\ 17)$
$(14\ 18)(15)(16)\ \ \varepsilon\ \mathcal{C}_p((G,P))$



$r \geq 4,\ \lambda = 2$

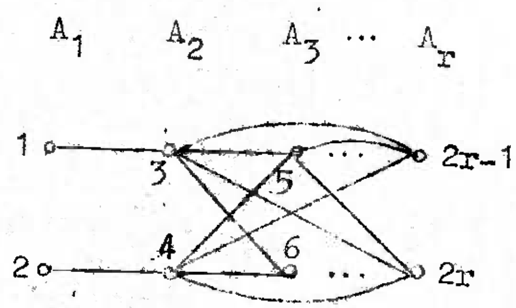
$$\sigma = \prod_{i=1}^{r} (2i-1\ \ 2i)\ \varepsilon\ \mathcal{C}_p((G,P))$$

FIGURE 3.2



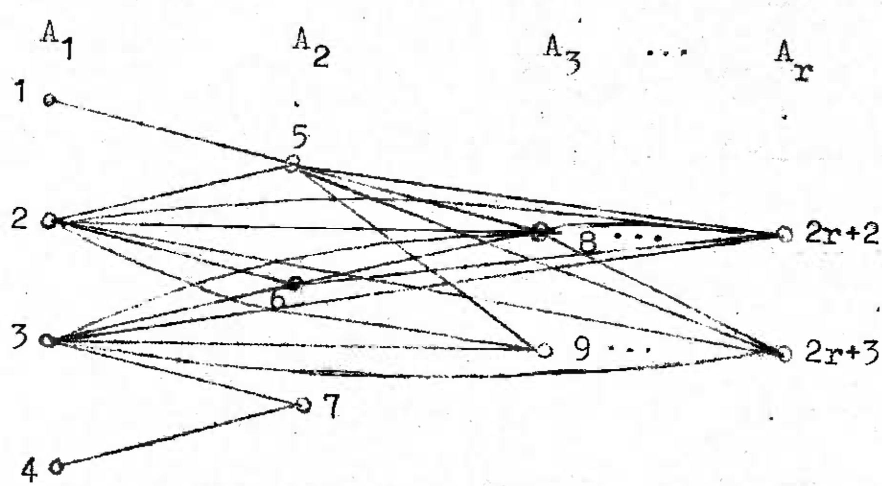
$r \geq 3,\ \lambda = 3,\ u_o = 2,\ v_o = 4$

$\sigma = \prod_{i=1}^{r} (2i-1\ \ 2i)\ \varepsilon\ \mathcal{C}((G,P))$



$r \geq 3,\ \lambda = 4,\ u_o = 1,\ v_o = 2$

$\sigma = (1\ 3\ 2\ 4) \prod_{i=3}^{r} (2i-1\ \ 2i)\ \varepsilon\ \mathcal{C}((G,P))$



$r \geq 3,\ \lambda = 5,\ u_o = 1,\ v_o = 4$

$\sigma = (1\ \ 2)\ (3\ \ 4)\ (5\ \ 7)\ (6) \prod_{i=3}^{r} (2i+2\ \ 2i+3)$

$\varepsilon\ \mathcal{C}((G,P))$

FIGURE 3.2 (Contd.)

To prove the second part of the theorem, we exhibit an r-psc graph (G,P) with diameter λ for each $r \geq 3$ and each $\lambda \in \{2,3,4,5\}$ in Figure 3.2. Now if $\sigma \in \mathcal{C}((G,P))$ and τ is a cycle of σ then for any positive integer k, the graph (G^k_τ, P^k_τ) as constructed on page 13, is r-psc with diameter λ. This gives us an infinite class of r-psc graphs for each $r \geq 3$ and each $\lambda \in \{2,3,4,5\}$ and Theorem 3.5 is proved. □

Ringel [17] and Sachs [18] proved that every self-complementary graph has diameter 2 or 3. We prove a generalisation of this in the following

<u>THEOREM 3.6</u>. Let $r \geq 3$ and (G,P) be r-psc. If there exists $\sigma \in \mathcal{C}^*((G,P))$ such that any cycle of σ having length > 1 intersects at least three sets of P, then the diameter of G is either 2 or 3.

<u>PROOF</u> : Let $\sigma \in \mathcal{C}^*((G,P))$ be such that any cycle of σ having length > 1, intersects at least three sets of P. By Theorem 1.9, $\sigma^2 \in \mathrm{Aut}\,(G)$. Let $u,v \in V(G)$. We first prove the following claims.

<u>Claim 1</u>. If $\sigma(u) \neq u$, then $d_G(u, \sigma(u)) \leq 2$.

Suppose $\sigma(u) \neq u$. Then by hypothesis and Theorem 1.13 the cycle of σ containing u is k-periodic for some $k \geq 3$.

Thus, u, $\sigma(u)$, $\sigma^2(u)$ all belong to different sets of P. Now if $u\,\sigma(u) \in E(G)$ we are done. Otherwise $u\,\sigma(u) \in E(\overline{G}(P))$ and so $\sigma^{-1}(u)\,u \in E(G)$. Since $\sigma^2 \in \mathrm{Aut}\,(G)$, it follows that $\sigma(u)\,\sigma^2(u) \in E(G)$. Now, either $\sigma^{-1}(u)\,\sigma(u) \in E(G)$ or $u\,\sigma^2(u) \notin E(\overline{G}(P))$, and hence $u\,\sigma^2(u) \in E(G)$. Thus either $u\,\sigma^{-1}(u)\,\sigma(u)$ or $u\,\sigma^2(u)\,\sigma(u)$ is a 2-path in G. This proves the claim.

Claim 2. If $\sigma(u) \neq u$ and $\sigma(v) \neq v$, then either $\sigma(u)$, v belong to different sets of P or u, $\sigma(v)$ belong to different sets of P.

If the claim is false, there exist A_i and A_j such that $\sigma(u), v \in A_i$ and $\sigma(v), u \in A_j$. Since $\sigma \in \mathfrak{G}^*((G,P))$, it follows that $\sigma(A_i) = A_j$ and $\sigma(A_j) = A_i$. Also by hypothesis and since $\sigma(u) \neq u$, we have $i \neq j$. But then σ has a 2-periodic cycle, contradicting the hypothesis. This prove the claim.

We shall now prove that for any $u,v \in V(G)$, $d_G(u,v) \leq 3$. We consider the following three cases :

Case 1. $\sigma(u) \neq u$, $\sigma(v) \neq v$. By Claim 2, we may assume without loss of generality that $\sigma(u)$, v belong to different sets of P. Now if $\sigma(u)\,v \in E(G)$ then by Claim 1, $d_G(u,v) \leq 3$.

Otherwise $\sigma(u)\ v \in E(\overline{G}(P))$ and so $u\,\sigma^{-1}(v) \in E(G)$. By Claim 1, $d_G(\sigma^{-1}(v), v) \leq 2$ and so $d_G(u,v) \leq 3$.

<u>Case 2</u>. σ sends exactly one of u,v to itself. Without loss of generality assume that $\sigma(u) \neq u$, $\sigma(v) = v$. If $uv \in E(G)$ we are done. Otherwise $uv \notin E(G)$, hence $\sigma(u)\ \sigma(v) \notin E(\overline{G}(P))$, i.e. $\sigma(u)\ v \notin E(\overline{G}(P))$. Now if $\sigma(u), v \in A_i$ for some i, then since $\sigma(v) = v$, it follows that $\sigma(A_i) = A_i$. Since $\sigma(u) \in A_i$, it also follows that $u \in A_i$. But $u \neq \sigma(u)$ and so if τ is the cycle of σ containing u then τ has length > 1 and $\langle \tau \rangle \subseteq A_i$, contradicting the hypothesis. Hence $\sigma(u)$, v belong to different sets of P. Since $\sigma(u)\ v \notin E(\overline{G}(P))$, it follows that $\sigma(u)\ v \in E(G)$. Now by Claim 1 we have $d_G(u,v) \leq 3$.

<u>Case 3</u>. $\sigma(u) = u$, $\sigma(v) = v$. By Theorem 1.6 (i), $u,v \in A_i$ for some i. Choose and fix an element w in some A_j, $j \neq i$. By Theorem 1.6 (i), $\sigma(w) \neq w$. Now by hypothesis and Theorem 1.13, the cycle containing w is k-periodic for some $k \geq 3$. Thus, w, $\sigma(w)$, $\sigma^2(w)$ belong to different sets of P. Also since $\sigma(A_i) = A_i$ we have $w, \sigma(w), \sigma^2(w) \notin A_i$. Now if uw, vw are edges of G we are done. Otherwise without loss of generality we assume that $uw \notin E(G)$. Then $u\,\sigma(w) \notin E(\overline{G}(P))$ and so $u\,\sigma(w) \in E(G)$. Now if $v\,\sigma(w) \in E(G)$ then we are done.

Otherwise, $v\,\sigma(w) \notin E(G)$ and so $v\,\sigma(w) \in E(\overline{G}(P))$. Since $\sigma^{-1}(v) = v$, it follows that $vw \in E(G)$. Also, since $\sigma^2 \in \mathrm{Aut}\,(G)$, we have $v\,\sigma^2(w) \in E(G)$. Now if $w\,\sigma(w) \in E(G)$ then $u\,\sigma(w)\,w\,v$ is a 3-path in G ; otherwise $\sigma(w)\,\sigma^2(w) \in E(G)$, and so $u\,\sigma(w)\,\sigma^2(w)\,v$ is a 3-path in G. In either case $d_G(u,v) \leq 3$.

This completes the proof of Theorem 3.6. □

COROLLARY 3.7. (Ringel [17], Sachs [18]). Every self-complementary graph G with more than one vertex has diameter 2 or 3.

PROOF : Let P be the partition of $V(G)$ consisting of singleton sets. Then (G,P) is p-psc where $p = |V(G)|$. Further every complementing permutation of the self-complementary graph G is also an element of $\mathfrak{C}^*((G,P))$. By Theorem 1.11 we also have that if $p \geq 2$ then every cycle of a complementing permutation of G having length > 1, intersects at least four sets of P. The corollary now follows from the theorem. □

We will now deal with an essentially Nordhaus-Gaddum type of problem, for a bipartitioned graph and its bipartite complement.

Let f be a graph theoretic parameter and p a positive integer. The Nordhaus-Gaddum problem for f (cf [10]) is to determine upper and lower bounds (preferably sharp) for $f(G) + f(\overline{G})$ and $f(G).f(\overline{G})$, where G is a graph on p vertices and $\overline{G}$ its ordinary complement. One can also consider the problem of determining all triplets (a,b,p) for which there exists a graph G such that $|V(G)| = p$, $f(G) = a$, $f(\overline{G}) = b$. In the class of bipartitioned graphs the corresponding problems are (i) to determine upper and lower bounds for $f(G) + f(\overline{G}(P))$ and $f(G).f(\overline{G}(P))$, where (G,P) is a bipartitioned graph on p vertices and $\overline{G}(P)$ is its bipartite complement, and (ii) to enumerate all triplets (a,b,p) for which there exists a bipartitioned graph (G,P) such that $|V(G)| = p$, $f(G) = a$ and $f(\overline{G}(P)) = b$. A solution of the second problem necessarily provides a solution for the first problem. Below we solve problem (ii) when f stands for the diameter of a graph. In this context we define a triplet (a,b,p) to be realisable if there exists a bipartitioned graph (G,P) on p vertices such that the diameter of G is a and the diameter of $\overline{G}(P)$ is b. Such a (G,P) is called a realisation of (a,b,p). If $\min(a,b) = 1$ then clearly the only realisable triplets are $(\infty,1,2)$ and $(1,\infty,2)$. We now enumerate all realisable triplets (a,b,p) with $\min(a,b) \geq 2$ in the following

<u>THEOREM 3.8</u>. Let min (a,b) $\geq$ 2. If (a,b,p) is realisable, then so is (a,b,p+1). The smallest value of p, if it exists for which (a,b,p) is realisable is given in the table below :

<u>TABLE 1</u>

a \ b	2	3	4	5	6	7	8	$9 \leq b < \infty$	∞
2	–	–	–	–	–	–	–	–	3
3	–	12	12	10	9	10	10	b + 1	4
4	–	12	8	8	8	8	9	–	5
5	–	10	8	7	8	–	–	–	6
6	–	9	8	8	7	–	–	–	–
7	–	10	8	–	–	–	–	–	–
8	–	10	9	–	–	–	–	–	–
$9 \leq a < \infty$	–	a +1	–	–	–	–	–	–	–
∞	3	4	5	6	–	–	–	–	3

<u>PROOF</u> : To prove the first part, let (a,b,p) be realisable and let (G,P) be a realisation of (a,b,p). Fix a vertex u in G. We construct a graph H from G by adding a

new vertex u' to $V(G)$ and joining it to all the vertices to which u is joined. We obtain a bipartition Q of $V(H)$ by including u' in the set of P containing u. Then (H,Q) is a bipartitioned graph on $p+1$ vertices. It can be easily verified that H has diameter a and $\overline{H}(P)$ has diameter b. Thus (H,Q) is a realisation of (a,b,p). This proves the first part of the theorem.

We next prove the following

Claim A. If in Table 1, a blank ('__') corresponds to a pair (a,b), then (a,b,p) is not realisable for any p.

Note that if (G,P) is a bipartitioned graph then the bipartite complement of $(\overline{G}(P),P)$ is isomorphic to G. Thus (a,b,p) is realisable iff (b,a,p) is so. Hence it suffices to prove Claim A for pairs (a,b) with $a \geq b$. We break up the proof into several steps.

1°. If $a < \infty$, then $(a,2,p)$ is not realisable for any p.

This follows since if a bipartite graph has diameter 2 then it is complete bipartite and so its bipartite complement (with respect to the unique bipartition) is disconnected.

2°. If $6 \leq b < \infty$, then (∞,b,p) is not realisable for any p.

For this, let $b < \infty$ and (G,P) be a realisation of (∞,b,p). If G has an isolated vertex or G has at least three components, each containing at least an edge then, since $\overline{G}(P)$ is connected, it follows that the diameter of $\overline{G}(P)$ is at most four. If not, then G has exactly two components, each containing at least an edge and since $\overline{G}(P)$ is connected it follows that the diameter of $\overline{G}(P)$ is at most five. Thus if (∞,b,p) is realisable for some p, then either $b \leq 5$ or $b = \infty$.

3°. If $9 \leq a < \infty$ and $4 \leq b \leq a$ then (a,b,p) is not realisable for any p.

By using techniques similar to those used in the proof of Theorem 3.1, one can prove that if (G,P) is a connected bipartitioned graph with diameter at least nine then the diameter of $\overline{G}(P)$ is at most three. From this 3° follows easily.

4°. If $a = 7$ or 8 and $5 \leq b \leq a$, then (a,b,p) is not realisable for any p.

This follows easily from Theorem 3.1.

This proves Claim A completely. We next prove the following

Claim B. If a positive integer p^* corresponds to a pair (a,b) in Table 1, then (a,b,p^*) is realisable.

Since Table 1 represents a symmetric matrix, it suffices to prove Claim B (as it sufficed to prove Claim A) for pairs (a,b) with $a \geq b$. Our method of proof is as follows:

For each positive integer p^* corresponding to a pair (a,b) in Table 1, we exhibit in Figure 3.3 a realisation of (a,b,p^*). Below each graph in Figure 3.3, we also give the triplet which is realised by the graph.

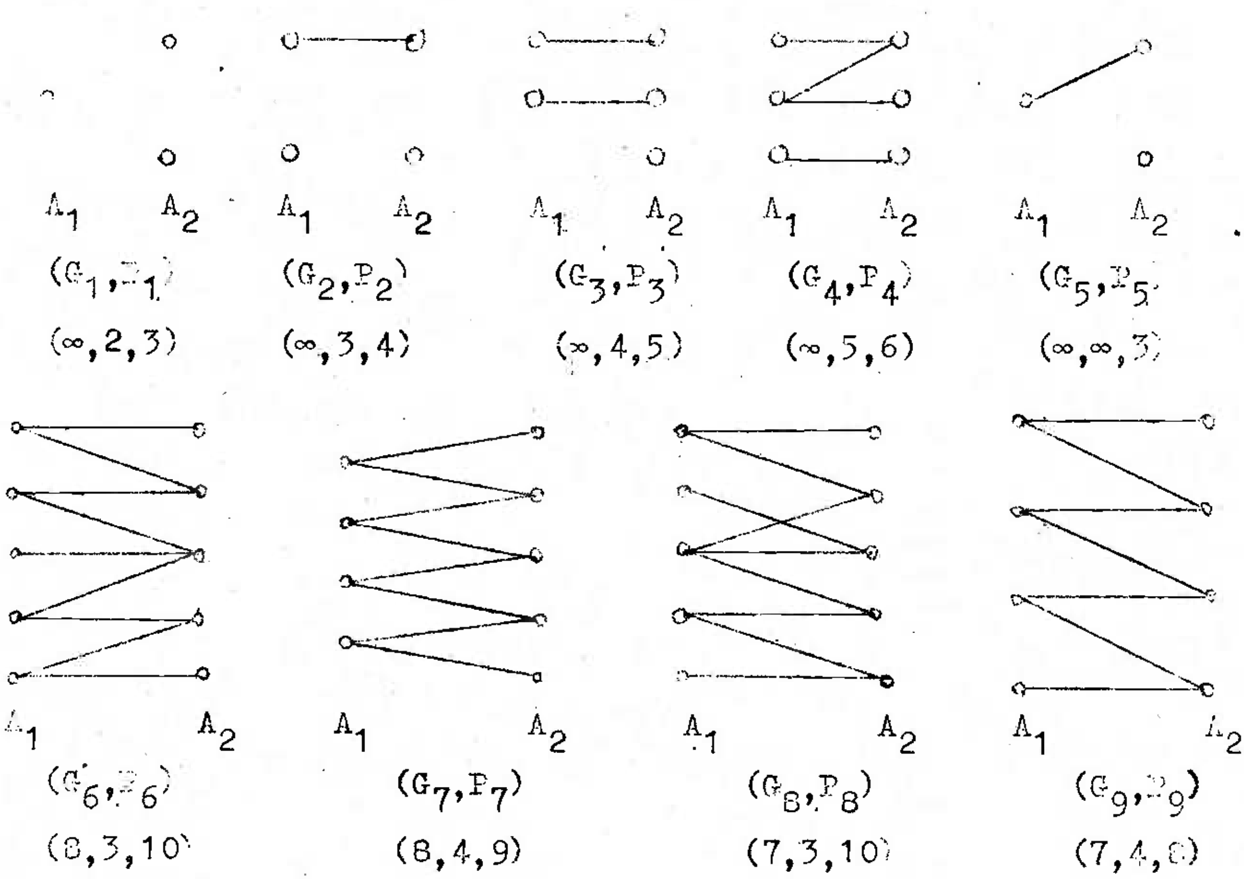
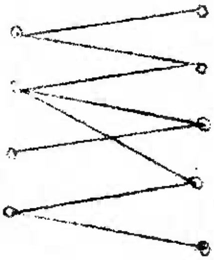
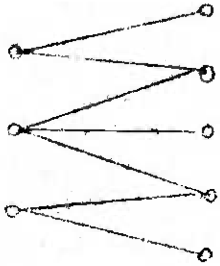


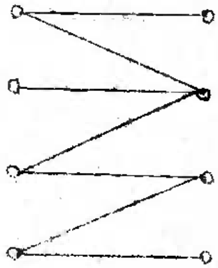
FIGURE 3.3



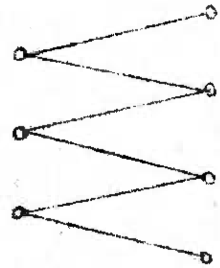
(G_{10}, P_{10})
(6,3,9)



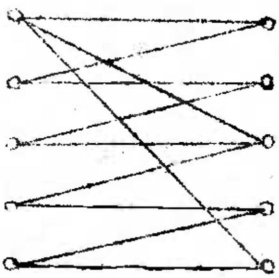
(G_{11}, P_{11})
(6,4,8)



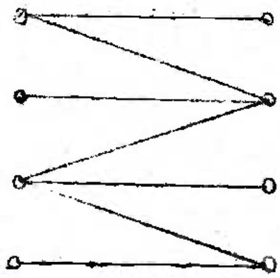
(G_{12}, P_{12})
(6,5,8)



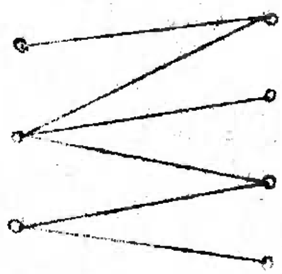
(G_{13}, P_{13})
(6,6,7)



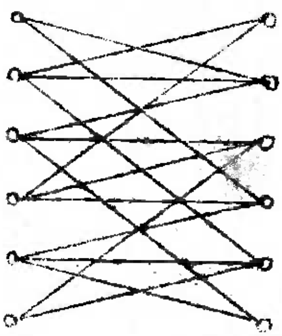
(G_{14}, P_{14})
(5,3,10)



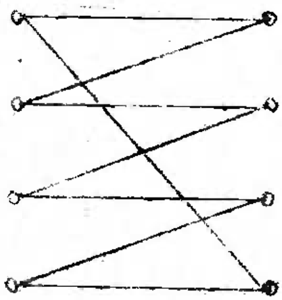
(G_{15}, P_{15})
(5,4,8)



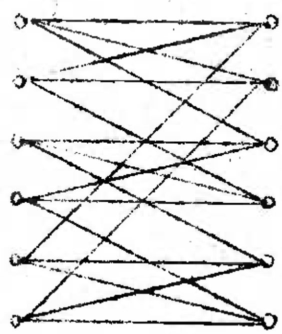
(G_{16}, P_{16})
(5,5,7)



(G_{17}, P_{17})
(4,3,12)



(G_{18}, P_{18})
(4,4,8)



(G_{19}, P_{19})
(3,3,12)

FIGURE 3.3 (Contd.)

To complete the proof of Claim B, we have show

that if $9 \leq a < \infty$, then $(a,3,a+1)$ is rea

follows, since the path of length a with i

is a realisation of $(a,3,a+1)$ for all

proves Claim B.

We will next prove the

<u>Claim C.</u> If a pos

pair (a,b) in Table

$p \geq p^*$.

As bef (a,b)

with a teps. <u>In</u>

<u>wha</u> <u>some</u> p, <u>then</u>

ealisable then $p \geq a+1$.

a realisation of (a,b,p).

diameter a, there is a path of length

+1.

a+1, for all a, $9 \leq a < \infty$. Further,

p^* (, $p^*(7,4) = 8$, $p^*(6,6) = 7$.

This follows from Claim B and 1^{0} above.

3° If $b < \infty$ and (∞,b,p) is realisable then $p \geq b+1$. Further $p^*(\infty,\infty) = 3$.

The first statement follows since if $b < \infty$ and (G,P) is a realisation of (∞,b,p) then $\overline{G}(P)$ has a path of length b. The second statement follows easily from Claim B.

4° If $a,b < \infty$ and (a,b,p) is realisable, then $p \geq 7$.

This follows since for any connected bipartitioned graph on six or less vertices, $\overline{G}(P)$ is disconnected.

5° $p^*(5,5) = 7$.

This follows from Claim B and 4° above.

6° If (a,b,p) is realisable and either (i) $a \notin \{5,6\}$, or (ii) $a,b < \infty$, $a \neq b$, then $p \geq 8$.

Indeed, by 4°, $p \geq 7$. Further, on seven vertices the only connected bipartitioned graph (G,P) for which $\overline{G}(P)$ is also connected are the graphs G_{13} and G_{16}, shown in Figure 3.3. Note that G_{13} is a realisation of $(6,6,7)$ and G_{16} of $(5,5,7)$. This proves 6°.

7° $p^*(6,4) = p^*(6,5) = p^*(5,4) = p^*(4,4) = 8$.

This follows from Claim B and 6° above.

8°. If (G,P) is a realisation of $(a,3,p)$, then given two vertices u_1, u_2 in a set of P, there is a vertex v in the other set such that u_1v, $u_2v \in E(G)$.

Suppose not. Then $d_{\bar{G}(P)}(u_1,u_2) \geq 4$, a contradiction. This proves 8°.

9°. If $(8,3,p)$ is realisable, then $p \geq 10$.

For this, let (G,P) be a realisation of $(8,3,p)$. Take a diametrical path, say, $u_1v_1u_2v_2u_3v_3u_4v_4u_5$ in G. Then by 8°, there is a vertex v which is at odd distance from u_2 in G such that u_2v, $u_4v \in E(G)$. Thus $p \geq 10$.

10°. If $(7,3,p)$ is realisable, then $p \geq 10$.

For this, let (G,P) be a realisation of $(7,3,p)$. Take a diametrical path $u_1v_1u_2v_2u_3v_3u_4v_4$ in G. Then by 8°, there are vertices u,v such that in G, u is at odd distance from v_1, v is at odd distance from u_2 and uv_1, uv_3, u_2v, $u_4v \notin E(G)$. Thus $p \geq 10$.

11°. If $(6,3,p)$ is realisable, then $p \geq 9$.

Proof of this is similar to that of 10°.

12°. If $(5,3,p)$ is realisable, then $p \geq 10$.

13°. If $(4,3,p)$ is realisable, then $p \geq 12$.

We omit the proofs of 12° and 13° as these are rather lengthy. However these proof techniques are similar in principle to those used above.

14°. If $(3,3,p)$ is realisable, then $p \geq 12$.

For this let (G,P) be a realisation of $(3,3,p)$. If $n_1 \leq 4$ then, since G has diameter 3, there are vertices $v_1, v_2 \in A_2$ such that $N_G(v_1) \bigcup N_G(v_2) = A_1$. Thus $d_{\overline{G}(P)}(v_1,v_2) \geq 4$, a contradiction. Hence $n_1 \geq 5$. But if $n_1 = 5$, then the degree of any vertex in A_2 is at most two in G. Now since G has diameter 3, it follows that for any pair of vertices u_1,u_2 in A_1, there corresponds a vertex v in A_2 such that $N_G(v) = \{u_1,u_2\}$. Thus $n_2 \geq \binom{5}{2} = 10$ and $p \geq 15$. Finally if $n_1 \geq 6$, then since $n_2 \geq n_1$, we have $p \geq 12$. This proves 14°.

This proves Claim C completely and Theorem 3.8 is proved. □

All the results in this chapter, except Theorem 3.8 will appear in [6].

CHAPTER 4

PATH LENGTHS IN MULTIPARTITE SELF-COMPLEMENTARY GRAPHS

In this chapter we consider the problem of determining the maximum length of a path in r-psc graphs. The problem is completely solved for connected bipsc graphs (G,P) with $\mathcal{C}_m((G,P)) \neq \emptyset$. Further sufficient conditions are obtained for the existence of a hamiltonian path in r-psc graphs : for $r = 2$ in Theorem 4.3 and for $r \geq 4$ in Theorem 4.5.

The following lemma will be used to establish certain structural properties of a connected bipsc graph (G,P) with $\mathcal{C}_m((G,P)) \neq \emptyset$, even though the lemma is stated here in a slightly more general form.

<u>LEMMA 4.1</u>. Let (G,P) be bipsc and $\sigma \in \mathcal{C}((G,P))$. Let $\ell \geq 1$ and $\tau = (u_1\ u_2 \ldots u_{4\ell-1} u_{4\ell})$ be a cycle of σ wit $u_i \in A_1$ if i is odd and $u_i \in A_2$ if i is even. Let H b the subgraph of G induced by $\langle \tau \rangle$. Then one of (a), (b), holds :

(a) $\ell = 1$; $u_1 u_2$ and $u_3 u_4 \in E(G)$ or, $u_1 u_4$ and $u_3 u_2 \in E(G)$.

(b) H has a hamiltonian cycle C such that for any u_i with i even there exist $j \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ such that $u_j u_i u_k$ is a part of C.

(c) H has two vertex-disjoint cycles C_1 and C_2, each of length 2ℓ such that C_1 contains all u_i with $i \equiv 1 \pmod 4$ and, either all u_i with $i \equiv 0 \pmod 4$ or all u_i with $i \equiv 2 \pmod 4$.

PROOF : Consider the bipartition Q of $V(H)$ with sets B_1 and B_2, where $B_1 = \{u_1, u_3, \ldots, u_{4\ell-1}\}$ and $B_2 = \{u_2, u_4, \ldots, u_{4\ell}\}$. By Observation 1.3, (H,Q) is bipsc and $\tau \in \mathcal{C}^*((H,Q))$. Hence by Theorem 1.9, $\tau^2 \in \mathrm{Aut}(H)$. Now, either $u_1u_2 \in E(G)$ or $u_1u_{4t} \in E(G)$. Since $\tau^2 \in \mathrm{Aut}(H)$, it follows that either $u_iu_{i+1} \in E(G)$ for all odd i or, $u_iu_{i-1} \in E(G)$ for all odd i, where the suffixes are reduced modulo 4ℓ. If $\ell = 1$, (a) follows.

If $\ell > 1$, then we consider four cases and in each case show that either (b) or (c) holds.

Case 1. $u_1u_2 \in E(G)$ and $u_1u_4 \in E(G)$. Then clearly, $u_i\, u_{i+1} \in E(G)$ and $u_i\, u_{i+3} \in E(G)$ for all odd i. To show that (b) holds in this case we consider the following hamiltonian cycle C

$$C : u_1\, u_4\, u_3\, u_6\, u_5\, u_8 \ldots u_{4\ell-3}\, u_{4\ell}\, u_{4\ell-1}\, u_2\, u_1 .$$

Case 2. $u_1u_2 \in E(G)$ and $u_1u_4 \notin E(G)$. Then $u_i u_{i+1} \in E(G)$ for all odd i, $u_2u_5 \in E(G)$ and hence $u_iu_{i+3} \in E(G)$ for all even i. To show that (c) holds in this case we consider the following cycles C_1 and C_2

$$C_1 : u_1 u_2 u_5 u_6 u_9 u_{10} \cdots u_{4\ell-3} u_{4\ell-2} u_1$$
$$C_2 : u_3 u_4 u_7 u_8 u_{11} u_{12} \cdots u_{4\ell-1} u_{4\ell} u_3.$$

Case 3. $u_1u_2 \notin E(G)$ and $u_1u_4 \in E(G)$. Then $u_2u_3 \in E(G)$ and hence $u_i u_{i+1} \in E(G)$ for all even i. Further $u_i u_{i+3} \in E(G)$ for all odd i. In this case (c) holds as is shown by the following cycles C_1 and C_2

$$C_1 : u_1 u_4 u_5 u_8 u_9 u_{12} \cdots u_{4\ell-3} u_{4\ell} u_1$$
$$C_2 : u_2 u_3 u_6 u_7 u_{10} u_{11} \cdots u_{4\ell-2} u_{4\ell-1} u_2.$$

Case 4. $u_1u_2 \notin E(G)$ and $u_1u_4 \notin E(G)$. Then $u_2u_3 \in E(G)$ and $u_2u_5 \in E(G)$. Hence $u_i u_{i+1} \in E(G)$ and $u_i u_{i+3} \in E(G)$ for all even i. In this case (b) holds as is shown by the following hamiltonian cycle C

$$C : u_3 u_2 u_5 u_4 u_7 u_6 \cdots u_{4\ell-1} u_{4\ell-2} u_1 u_{4\ell} u_3.$$

This completes the proof of Lemma 4.1. □

The maximum length of a path in a connected bipsc graph (G,P) with $\mathfrak{C}_m((G,P)) \neq \emptyset$ is now determined in the following

<u>THEOREM 4.2.</u> Every connected bipsc graph (G,P) with $\mathcal{C}_m((G,P)) \neq \emptyset$ has a $(p-3)$-path, where $p = |V(G)|$. Further for each $p \equiv 0 \pmod 4$, $p \geq 8$, there exists a connected bipsc graph (G,P) on p vertices such that $\mathcal{C}_m((G,P)) \neq \emptyset$ and G has no $(p-2)$-path. Also for each $p \equiv 0 \pmod 4$, $p \geq 12$, there exists a connected bipsc graph (H,Q) on p vertices such that $\mathcal{C}_m((H,Q)) = \emptyset$ and the maximum length of a path in H is $\frac{p}{2}+2$.

<u>PROOF</u> : Let $\sigma \in \mathcal{C}_m((G,P))$ and let $\sigma = \sigma_1\sigma_2\ldots\sigma_\lambda$ be the disjoint cycle representation of σ. By Theorem 1.12, $\sigma^2 \in \mathrm{Aut}\,(G)$ and by Corollary 1.15, each σ_i takes vertices alternately from A_1 and A_2. Further $|\sigma_i| \equiv 0 \pmod 4$ for all i. We now consider two cases :

<u>Case 1</u>. $\lambda=1$. Without loss of generality, we assume that

$$\sigma = (u_1\ u_2\ \ldots\ u_{4\ell-1}\ u_{4\ell})$$

where $u_s \in A_1$ (resp. A_2) if s is odd (resp. even), and $p = 4\ell$. Since G is connected, $\ell \geq 2$. It now follows by Lemma 4.1 that either G has a hamiltonian cycle or G has two vertex-disjoint cycles, each of length 2ℓ. In the latter case, since G is connected, it follows easily that G contains a hamiltonian path. This completes Case 1.

<u>Case 2</u>. $\lambda > 1$. Without loss of generality, we assume that for all i, $1 \le i \le \lambda$

$$\sigma_i = (u_{i1}\ u_{i2} \cdots u_{i,4\ell_i - 1}\ u_{i,4\ell_i})$$

where $u_{is} \in A_1$ (resp. A_2) if s is odd (resp. even).

Let G_i be the subgraph of G induced by $\langle \sigma_i \rangle$. We shall call the vertex u_{is} of G_i <u>even</u> or <u>odd</u> according as s is even or odd. By Lemma 4.1 we have

<u>Observation 1</u>. One of the following holds :

(1) G_i is hamiltonian and given u_{is} with s even there is a $t \equiv 1 \pmod 4$ (resp. $t \equiv 3 \pmod 4$) such that u_{is}, u_{it} appear consecutively in a hamiltonian cycle of G,

(2) G_i does not satisfy (1) and $V(G_i)$ can be partitioned into two sets V_{i1}, V_{i2} such that if u_{is}, $u_{it} \in V_{i1}$ (resp. V_{i2}) and $s - t \equiv 0 \pmod 2$, then $s - t \equiv 0 \pmod 4$. Further $G[V_{i1}]$, $G[V_{i2}]$ are either both hamiltonian or both K_2's.

We say that σ_i <u>is of type</u> 1 or 2 according as G_i satisfies condition (1) or (2). Without loss of generality, we assume that σ_i is of type 1 if $1 \le i \le \theta$ and σ_i is of type 2 if $\theta + 1 \le i \le \lambda$. We choose and fix V_{i1} and V_{i2} as in Observation 1 for $i = \theta+1, \theta+2, \ldots, \lambda$.

Given i, j, $1 \leq i \neq j \leq \lambda$, we define $G_i < G_j$ if some even vertex of G_i is adjacent to some odd vertex of G_j. If $G_i \nless G_j$, then in particular $u_{i2} u_{j1} \notin E(G)$ and so $u_{i3} u_{j2} \in E(G)$ implying that $G_j < G_i$. Thus, either $G_i < G_j$ or $G_j < G_i$ (or both).

After a suitable relabelling of $\sigma_1, \ldots, \sigma_\theta$, we now assume by Rédei's Theorem [16], that

$$G_1 < G_2 < \ldots < G_\theta .$$

Similarly we also assume

$$G_{\theta+1} < G_{\theta+2} < \ldots < G_\lambda .$$

We now define two subgraphs G_{i1} and G_{i2} of G_i for $i = \theta+1, \theta+2, \ldots, \lambda$. Let $G_{\theta+1,k} = G_{\theta+1}[V_{\theta+1,k}]$, $k = 1,2$. After defining G_{i1} and G_{i2}, define

$$G_{i+1,k} = G_{i+1}[V_{i+1,k}] , k = 1,2$$

if some even vertex of G_{i1} is joined to some odd vertex of $V_{i+1,1}$. Otherwise we define

$$G_{i+1,k} = G_{i+1}[V_{i+1,3-k}] , k = 1,2 .$$

We now make a few observations.

<u>Observation 2</u>. Let $1 \leq i \neq j \leq \lambda$. If $G_i < G_j$ then every even vertex of G_i is joined to some odd vertex of G_j

and every odd vertex of G_j is joined to some even vertex of G_i.

<u>Observation 3</u>. Let $\theta+1 \leq i \leq \lambda-1$. Then every even vertex of G_{ik} is joined to some odd vertex of $G_{i+1,k}$ and every odd vertex of $G_{i+1,k}$ is joined to some even vertex of G_{ik}, $k = 1,2$.

<u>Observation 4</u>. If for σ_i of type 2 and σ_j of type 1, $G_i < G_j$ then either (i) for each $s \equiv 1 \pmod 4$, u_{js} is adjacent to some even vertex of G_{i1} and $u_{j,s+2}$ is adjacent to some even vertex of G_{i2} or (ii) for each $s \equiv 3 \pmod 4$, u_{js} is adjacent to some even vertex of G_{i1} and $u_{j,s+2}$ is adjacent to some even vertex of G_{i2}.

<u>Observation 5</u>. If for σ_i of type 2 and σ_j of type 1, $\sigma_i < \sigma_j$ then for any $k \in \{1,2\}$ and any even vertex u in G_j, there exist an odd vertex v in G_j and an even vertex w in G_{ik} such that uv is in a hamiltonian cycle in G_j and $vw \in E(G)$.

<u>Observation 6</u>. Let $1 \leq i \neq j \leq \lambda$. If $G_i \nless G_j$, u is an odd vertex of G_i and v an even vertex of G_j, then $uv \in E(G)$.

We now give hamiltonian paths μ in $\bigcup_{i=1}^{\theta} G_i$, η_k in $\bigcup_{i=\theta+1}^{\lambda} G_{ik}$, $k = 1,2$, which will be used in constructing a $(p-3)$-path in G. We exhibit μ in Figure 4.1 by a broken line. It is constructed as follows : start at an arbitrary

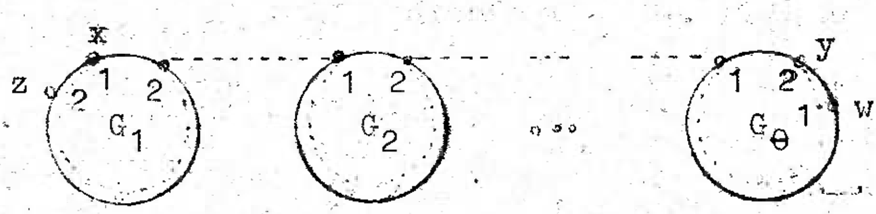


FIGURE 4.1

(odd and even vertices are denoted by 1 and 2 respectively.)

odd vertex x of G_1, trace a hamiltonian path of G_1, then go to some odd vertex of G_2 (this is possible by Observation 2, since $G_1 < G_2$), then trace a hamiltonian path of G_2. Proceed like this until $G_{\theta-1}$ is covered and an odd vertex of G_θ is reached. Then trace a hamiltonian path of G_θ ending in an even vertex y. We note that μ can also be constructed by starting from an arbitrary even vertex y in G_θ and going backwards and in this process by Observation 1, the final odd vertex x can be chosen to be some u_{1s} with $s \equiv 1 \pmod 4$ or $3 \pmod 4$ at will. In either case we will denote the vertices adjacent to x and y in μ by z and w respectively.

The path η_k is obtained exactly like μ with $G_{\theta+1,k}, G_{\theta+2,k}, \ldots, G_{\lambda k}$ replacing $G_1, G_2, \ldots, G_\theta$, and x_k, y_k, z_k, w_k replacing x,y,z,w respectively, except that if y_k is arbitrary then x_k cannot be chosen at will. Further, by Observation 1, η_k can be chosen such that instead of the initial vertex x_k, the next vertex z_k is arbitrary (see Figure 4.1).

We are now ready to show the existence of a $(p-3)$-path in G. We deal with the cases $\theta > 0$ and $\theta = 0$ separately.

<u>Case 2.1.</u> $\theta > 0$. Here we consider four subcases ;

<u>Case 2.1.1.</u> $\theta = \lambda$ or ($G_{\theta+1} < G_1$ and $G_\theta < G_\lambda$). By Observation 4, we may assume without loss of generality that for all $s \equiv 1 \pmod 4$, u_{1s} is adjacent to some even vertex of $G_{\theta+1,2}$. Then the $(p-3)$-path is obtained by tracing η_1 from x_1 to w_1, then going to some even vertex y of G_θ, then tracing μ backwards choosing x to be some u_{1s} with

$s \equiv 1 \pmod 4$, then going to some even vertex z_2 of $G_{\theta+1,2}$ and finally tracing η_2 from z_2 to y_2.

We note that if $\theta = \lambda$, the above path is actually a hamiltonian path.

Case 2.1.2 $G_{\theta+1} \prec G_1$ and $G_\theta \nprec G_\lambda$. Then the (p-3)-path is obtained by tracing η_2 backwards from y_2 to z_2, then going to some odd vertex x in G_1, then tracing μ from x to w, then going to some even vertex y_1 of $G_{\lambda 1}$ (this is possible by Observation 6), then tracing η_1 backwards from y_1 to x_1.

Case 2.1.3. $G_{\theta+1} \nprec G_1$ and $G_\theta \prec G_\lambda$. Then the (p-3)-path is obtained by tracing η_1 from x_1 to w_1, then going to some even vertex y of G_θ, then tracing μ backwards from y to z, then going to some odd vertex x_2 of $G_{\theta+1,2}$ (this is possible by Observation 6), then tracing η_2 from x_2 to y_2.

Case 2.1.4. $G_{\theta+1} \nprec G_1$ and $G_\theta \nprec G_\lambda$. Then the (p-3)-path is obtained by tracing η_1 from x_1 to y_1, then going to the odd vertex w in μ (this is possible by observation 6 since w is an odd vertex of G_θ and $G_\theta \nprec G_\lambda$), then tracing μ backwards from w to z, then going to some odd vertex x_2 of $G_{\theta+1,2}$ (this is possible by Observation 6), then tracing η_2 from x_2 to y_2.

Case 2.2. $\theta = 0$. Here we consider three subcases :

Case 2.2.1. $G_\lambda \nless G_1$. Then the (p-3)-path is obtained by tracing η_1 from x_1 to w_1, then going to some even vertex z_2 in η_2 (this is possible by Observation 6), then tracing η_2 from z_2 to y_2.

Case 2.2.2. $G_\lambda < G_1$ and there exist an even vertex y_1 of $G_{\lambda 1}$ and an odd vertex x_2 of G_{12} such that $x_2y_1 \in E(G)$. In this case a hamiltonian path can be obtained which contains the edge x_2y_1, traces η_1 backwards from y_1 to x_1 and traces η_2 from x_2 to y_2.

Case 2.2.3. $G_\lambda < G_1$ and no even vertex of $G_{\lambda 1}$ is adjacent to any odd vertex of G_{12}. In this case by Observation 2, every even vertex of $G_{\lambda 1}$ is adjacent to some odd vertex of G_{11} and since $\sigma^2 \in \text{Aut}(G)$, every even vertex of $G_{\lambda 2}$ is adjacent to some odd vertex of G_{12}.

Now, since G is connected, for some i,j, there exist $u_{is} \in V(G_{i1})$ and $u_{jt} \in V(G_{j2})$ such that $u_{is}u_{jt} \in E(G)$. Without loss of generality we assume that s is odd and t is even (Otherwise we interchange the roles of G_{h1} and G_{h2} for all h, $1 \leq h \leq \lambda$).

Let $W_k = \bigcup_{h=1}^{\lambda} V(G_{hk})$, $k = 1,2$. Then in Figure 4.2 we construct a hamiltonian path P1 in $G[W_1]$ which has u_{is} as an end vertex and a path P2 in $G[W_2]$ which covers all but one vertex of W_2 and has u_{jt} as an end vertex. Then P1, P2 and the edge $u_{is}u_{jt}$ gives us a (p-2)-path in G, as is indicated by a broken line in the figure.

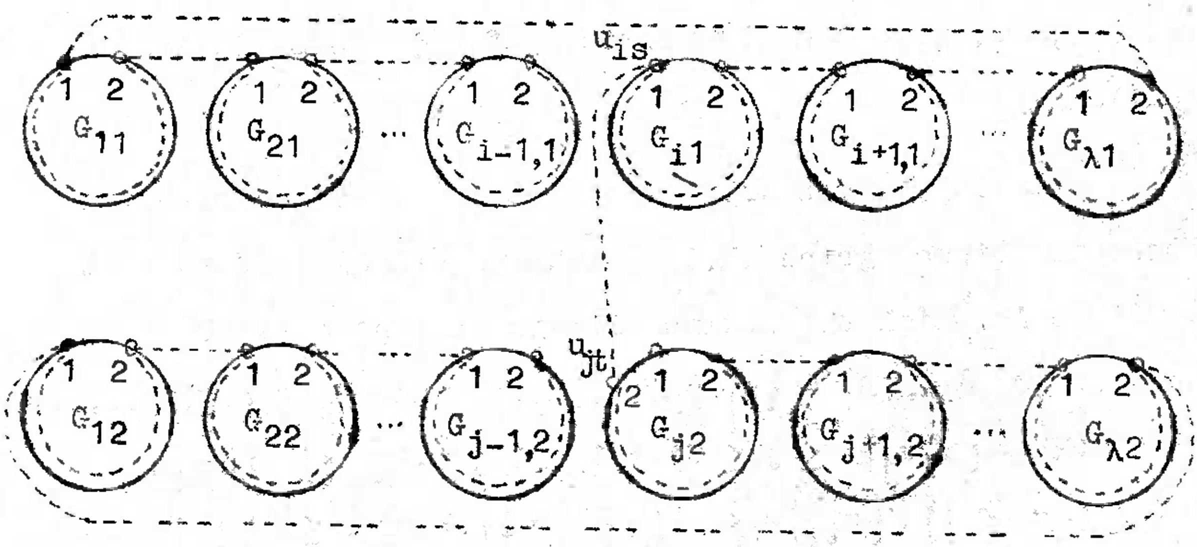


FIGURE 4.2

(odd and even vertices are denoted by 1 and 2 respectively)

Thus we have shown that every connected bipsc graph (G,P) with $\mathcal{C}_m((G,P)) \neq \emptyset$ has a (p-3)-path, where $p = |V(G)|$.

Next, given an integer $t \geq 2$, we construct a connected bipsc graph (G,P), with $\mathcal{C}_n((G,P)) \neq \emptyset$, on $p = 4t$ vertices which has no $p-2$ path. The sets of P are $A_1 = \{u_1, u_2, \ldots, u_{2t}\}$ and $A_2 = \{v_1, v_2, \ldots, v_{2t}\}$. The vertices u_i and v_j are joined in G iff $i \leq j \leq 2t-i$, $1 \leq i \leq t$, or, $2t + 2 - i \leq j \leq i$, $t + 1 \leq i \leq 2t$. Clearly, (G,P) is connected bipsc and $\sigma = \prod_{i=1}^{t} (u_i v_i u_{2t+1-i} v_{2t+1-i}) \in \mathcal{C}_m((G,P))$. Further $u_t, u_{t+1}, v_1, v_{2t}$ are end-vertices of G. Thus G has no $(p-2)$-path.

Finally, given any integer $t \geq 3$, we construct a connected bipsc graph (H,Q) with $\mathcal{C}_n((H,Q)) = \emptyset$ on $4t$ vertices in which the maximum length of a path is $2t + 2$. The sets of Q are $B_1 = \{u_1, u_2, \ldots, u_{2t}\}$ and $B_2 = \{v_1, v_2, \ldots, \theta_{2t}\}$ and $E(H) = \{u_i v_j \mid 1 \leq j \leq 2t-1, 1 \leq i \leq t-1\} \cup \{u_t v_j \mid 2 \leq j \leq 2t\} \cup \{u_{t+1} v_1\} \cup \{u_i v_{2t} \mid t+2 \leq i \leq 2t\}$. Then clearly (H,Q) is connected bipsc, $\mathcal{C}_n((H,Q)) = \emptyset$ and $\sigma = (\prod_{i=1}^{t} (u_i u_{2t+1-i})) (\prod_{i=1}^{2t} (v_i)) \in \mathcal{C}_p((H,Q))$. Now since u_i, $t+1 \leq i \leq 2t$ are end-vertices of H, it follows that a path of H can include at most $t + 2$ vertices of B_1, and since H is bipartitioned, it follows that if a path includes $t + 2$ vertices of B_1 it is a maximal path. Now one such path in H is $u_{t+1}\ v_1\ u_1\ v_2\ u_2 \dots v_{t-1}\ u_{t-1}\ v_t\ u_t\ v_{2t}\ u_{t+2}$. Thus in H the maximum length of a path is $2t+2$.

This completes the proof of Theorem 4.2. □

We next prove the following theorem, which gives a sufficient condition for the existence of a hamiltonian path in a bipsc graph.

<u>THEOREM 4.3</u>. Let (G,P) be bipsc and $\sigma \in \mathfrak{S}_m((G,P))$ have the disjoint cycle representation

$$\sigma = \sigma_1\ \sigma_2 \dots \sigma_\lambda .$$

Let G_i be the subgraph of G induced by $\langle \sigma_i \rangle$. If G_i is connected for all i, then G has a hamiltonian path.

<u>PROOF</u> : Fix i, $1 \leq i \leq \lambda$. By hypothesis (G_i,P_i) is connected bipsc and $\sigma_i \in \mathfrak{S}_m((G_i,P_i))$, where P_i is the restriction of P to G_i. By Theorem 1.12 and Corollary 1.15, $\sigma_i^2 \in \mathrm{Aut}\,(G_i)$, σ_i takes vertices alternately from A_1 and A_2 and $|\sigma_i| \equiv 0 \pmod 4$.

Let $\sigma_i = (u_{i1}\ u_{i2} \dots u_{i,4\ell_i-1}\ u_{i,4\ell_i})$, where $u_{is} \in A_1$ (resp. A_2) if s is odd (resp. even). As before we call u_{is} <u>odd</u> or <u>even</u> according as s is odd or even. Since G_i is connected, it follows by Lemma 4.1 that one of the following holds :

(1) G_i is hamiltonian,

(2) $V(G_i)$ can be partitioned into two subsets V_{i1} and V_{i2} of size $2\ell_i$ each such that V_{i1} contains all u_{is} with $s \equiv 1 \pmod 4$ and either all u_{is} with $s \equiv 0 \pmod 4$ or all u_{is} with $s \equiv 2 \pmod 4$. Further $G[V_{ik}]$ has a hamiltonian cycle C_{ik}, $k = 1,2$.

We now prove the following

<u>Claim</u> : Given $u_{is} \in V(G_i)$, there exists a hamiltonian path in G_i with u_{is} as an end-vertex.

Indeed, if G_i satisfies (1), our claim follows trivially. Suppose now G_i satisfies (2). Then since G_i is connected, some vertex u_{is_o} of V_{i1} is joined to some vertex u_{it_o} of V_{i2}. Since $\sigma_i^2 \in$ Aut (G_i), it follows by (2) that any vertex of V_{ik} is joined to some vertex of $V_{i,3-k}$, $k = 1,2$. Now, given $u_{is} \in V_{ik}$, let u_{it} be a vertex adjacent to u_{is} on C_{ik}. To get the required hamiltonian path, trace a hamiltonian path of C_{ik} from u_{is} to u_{it} then go to some vertex of $V_{i,3-k}$ and trace a hamiltonian path of $C_{i,3-k}$. This proves the claim. We note that since G_i has $4\ell_i$ vertices, the end vertices of any hamiltonian path of G_i have different parities.

Given i, j, $1 \leq i \neq j \leq \lambda$, we define $G_i < G_j$ if an even vertex of G_i is adjacent to some odd vertex of G_j. Then, as in Theorem 4.2, we have either $G_i < G_j$ or $G_j < G_i$. Also, if $G_i < G_j$, then every even vertex of G_i is adjacent to some odd vertex of G_j and every odd vertex of G_j is adjacent to some even vertex of G_i. By Rédei's Theorem [16], it now follows that the cycles of σ may be suitably relabelled so that

$$G_1 < G_2 < \ldots < G_\lambda .$$

We are now ready to give a hamiltonian path in G. Trace a hamiltonian path in G_1, starting from an odd vertex of G_1, then go to an odd vertex of G_2 and trace a hamiltonian path in G_2. Proceed like this until an odd vertex of G_λ is reached, then trace a hamiltonian path in G_λ. This gives us a hamiltonian path in G.

This completes the proof of Theorem 4.3. □

Next, in Theorem 4.5, we give sufficient conditions for the existence of a hamiltonian path in an r-psc graph, $r \geq 4$. We first prove the following preliminary lemma.

<u>LEMMA 4.4</u>. Let (G,P) be r-psc with $r \geq 4$ and $\sigma \in \mathcal{G}^*((G,P))$. Let τ be a cycle of σ with $|I_\tau| \geq 4$.

Let H be the subgraph of G induced by $\langle \tau \rangle$ and u an arbitrary vertex of H. Let $\ell = |\tau|$. Then one of the following holds :

(a) for any integer s, $0 \leq s \leq \frac{\ell}{2} - 1$, there is a hamiltonian path μ in H in which the vertices $\tau^{2s}(u)$, $\tau^{2s+2}(u)$ appear consecutively and which has $\tau^{2t+1}(u)$, $\tau^{2t+3}(u)$ as end vertices, for some t, $0 \leq t \leq \frac{\ell}{2} - 1$.

(b) for any integer s, $0 \leq s \leq \frac{\ell}{2} - 1$, there is a hamiltonian path μ in H in which the vertices $\tau^{2s+1}(u)$, $\tau^{2s+3}(u)$ appear consecutively and which has $\tau^{2t}(u)$, $\tau^{2t+2}(u)$ as end vertices, for some t, $0 \leq t \leq \frac{\ell}{2} - 1$.

REMARK : Note that by Theorem 1.13 and Theorem 1.6 (ii), ℓ is even.

PROOF : By Theorem 1.9, $\sigma^2 \varepsilon$ Aut (G). Let $m = |I_\tau|$. Since $m \geq 4$, by Theorem 1.13 (i) we have that τ is m-periodic. Hence if $v \varepsilon \langle \tau \rangle$, then v, $\tau(v)$, $\tau^2(v)$, $\tau^3(v)$ all belong to different sets of P. Without loss of generality, we now assume that $u\tau(u) \varepsilon E(G)$. For, if not, then $\tau^{-1}(u)u \varepsilon E(G)$ and (a) or (b) holds for u according as

(b) or (a) holds for $\tau^{-1}(u)$. Since $\sigma^2 \in \mathrm{Aut}(G)$, it follows that for any s, $0 \leq s \leq \frac{\ell}{2} - 1$, we have $\tau^{2s}(u)\, \tau^{2s+1}(u) \in E(G)$.

We now consider the following two cases :

<u>Case 1</u>. $u\, \tau^3(u) \in E(G)$. Since $\sigma^2 \in \mathrm{Aut}(G)$, for any s, $0 \leq s \leq \frac{\ell}{2} - 1$, we have $\tau^{2s}(u)\, \tau^{2s+3}(u) \in E(G)$. Let μ_1 be the $(\ell-3)$-path $\tau(u)\, u\, \tau^3(u)\, \tau^2(u), \tau^5(u)\, \tau^4(u) \ldots \tau^{\ell-3}(u)\, \tau^{\ell-4}(u)$. Since $\sigma^2 \in \mathrm{Aut}(G)$, either $\tau^{\ell-4}(u)\, \tau^{\ell-2}(u) \in E(G)$ or $\tau^{\ell-1}(u)\, \tau(u) \in E(G)$.

We obtain a hamiltonian path μ in H by combining μ_1 with the 2-path $\tau^{\ell-4}(u)\, \tau^{\ell-2}(u)\, \tau^{\ell-1}(u)$ or with the 2-path $\tau^{\ell-2}(u)\, \tau^{\ell-1}(u)\, \tau(u)$ according as $\tau^{\ell-4}(u)\, \tau^{\ell-2}(u) \in E(G)$ or $\tau^{\ell-1}(u)\, \tau(u) \in E(G)$.

Note that either $\tau^{\ell-4}(u), \tau^{\ell-2}(u)$ appear consecutively in μ and μ has $\tau^{\ell-1}(u)$ and $\tau^{\ell+1}(u)$ as its end vertices, or, $\tau^{\ell-1}(u), \tau^{\ell+1}(u)$ appear consecutively in μ and μ has $\tau^{\ell-4}(u)$ and $\tau^{\ell-2}(u)$ as its end vertices. Now since ℓ is even and $\sigma^2 \in \mathrm{Aut}(G)$, it follows that either (a) or (b) holds.

<u>Case 2</u>. $u\, \tau^3(u) \notin E(G)$. Then we have $\tau(u)\, \tau^4(u) \in E(G)$. Since $\sigma^2 \in \mathrm{Aut}(G)$, it follows that $\tau^{2s+1}(u)\, \tau^{2s+4}(u) \in E(G)$, for all s, $0 \leq s \leq \frac{\ell}{2} - 1$. Further,

either $u\tau^2(u) \in E(G)$, or $\tau^{\ell-3}(u)\tau^{\ell-1}(u) \in E(G)$. We now construct a hamiltonian path μ as follows :

Let μ_1 be the path $u\,\tau(u)\,\tau^4(u)\,\tau^5(u)\,\tau^8(u)\,\ldots$, the last term being $\tau^{\ell-3}(u)$ or $\tau^{\ell-1}(u)$ according as $\ell \equiv 0 \pmod 4$ or $\ell \equiv 2 \pmod 4$. Let μ_2 be the path $\tau^2(u)\,\tau^3(u)\,\tau^6(u)\,\tau^7(u)\,\tau^{10}(u)\,\ldots$, the last term being $\tau^{\ell-1}(u)$ or $\tau^{\ell-3}(u)$ according as $\ell \equiv 0 \pmod 4$ or $\ell \equiv 2 \pmod 4$. Then μ is obtained by combining μ_1 and μ_2, using the edge $u\tau^2(u)$ or $\tau^{\ell-3}(u)\tau^{\ell-1}(u)$ whichever exists.

Note that either u, $\tau^2(u)$ appear consecutively in μ and μ has $\tau^{\ell-3}(u)$ and $\tau^{\ell-1}(u)$ as its end vertices, or, $\tau^{\ell-3}(u)$, $\tau^{\ell-1}(u)$ appear consecutively in μ and μ has u and $\tau^2(u)$ as its end vertices. Now since ℓ is even and $\sigma^2 \in \mathrm{Aut}\,(G)$, it follows that either (a) or (b) holds.

This completes the proof of Lemma 4.4. □

<u>THEOREM 4.5</u>. Let $r \geq 4$ and (G,P) be r-psc. If there exists $\sigma \in \mathcal{C}^*((G,P))$ such that σ has at most one cycle of length 1 and every other cycle τ of σ satisfies $|I_\tau| \geq 4$, then G has a hamiltonian path.

<u>PROOF</u> : By Theorem 1.9, $\sigma^2 \in \mathrm{Aut}\,(G)$. We now consider two cases.

Case 1. σ has no cycle of length 1. Let

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_\lambda$$

be the disjoint cycle representation of σ. Let G_i be the subgraph induced by $\langle \sigma_i \rangle$ and $\ell_i = |\sigma_i|$. By Theorem 1.13 and Theorem 1.6 (ii), ℓ_i is even.

For each i, $1 \leq i \leq \lambda$, fix a $u_i \varepsilon \langle \sigma_i \rangle$. Then define a vertex $v \varepsilon \langle \sigma_i \rangle$ to be odd (resp. even) if for some integer s, $0 \leq s \leq \frac{\ell_i}{2} - 1$, we have $v = \sigma_i^{2s+1}(u_i)$ (resp. $v = \sigma_i^{2s}(u_i)$). Two odd (resp. even) vertices $v_1, v_2 \varepsilon \langle\sigma_i\rangle$ are said to be consecutive if either $v_1 = \sigma_i^2(v_2)$, or $v_2 = \sigma_i^2(v_1)$. Now, by Lemma 4.4, we have

Observation 1. For each i, $1 \leq i \leq \lambda$, one of the following holds :

(a) for any two consecutive even vertices $v_i, w_i \varepsilon \langle \sigma_i \rangle$, there is a hamiltonian path η_i in G_i in which v_i, w_i appear consecutively and which has two consecutive odd vertices $x_i, y_i \varepsilon \langle \sigma_i \rangle$ as end vertices.

(b) G_i does not satisfy (a) and for any two consecutive odd vertices $v_i, w_i \varepsilon \langle \sigma_i \rangle$, there is a hamiltonian path η_i in G_i in which v_i, w_i appear consecutively and which has two consecutive even vertices $x_i, y_i \varepsilon \langle \sigma_i \rangle$ as end vertices.

Given i,j with $1 \leq i \neq j \leq \lambda$, we define $G_i < G_j$ if some vertex v of G_i is adjacent to some vertex w of G_j with v odd or even according as G_i satisfies (a) or (b) and w even or odd according as G_j satisfies (a) or (b).

Suppose now $G_i \nless G_j$. Since $|I_{\sigma_i}|$, $|I_{\sigma_j}| \geq 4$, there exist vertices v of G_i and w of G_j such that v,w belong to different sets of P, with v odd or even according as G_i satisfies (a) or (b) and w even or odd according as G_j satisfies (a) or (b). Now $vw \notin E(G)$, hence $\sigma(v)\,\sigma(w) \in E(G)$ and it follows that $G_j < G_i$.

Thus, for any two cycles σ_i and σ_j of σ, either $G_i < G_j$ or $G_j < G_i$. Hence, by Rédei's Theorem [16], σ_i's can be suitably relabelled so that

$$G_1 < G_2 < \ldots < G_\lambda .$$

Using the fact that $\sigma^2 \in \mathrm{Aut}\,(G)$, we have the following

<u>Observation 2</u>. If $G_i < G_j$ then (i) given any vertex v of G_i with v odd or even according as G_i satisfies (a) or (b) there exists some vertex w of G_j with w even or odd according as G_j satisfies (a) or (b), such that $vw \in E(G)$, and (ii) given any vertex w of G_j with w even or odd according as G_j satisfies (a) or (b) there exists some

vertex v of G_i with v odd or even according as G_i satisfies (a) or (b), such that $vw \in E(G)$.

To complete Case 1, we will prove the following claim by induction on i.

Claim : for $1 \leq i \leq \lambda$, there exists a hamiltonian path μ_i of $G_1 \cup G_2 \cup \ldots \cup G_i$ which has as end vertices two consecutive odd or two consecutive even vertices of G_i according as G_i satisfies (a) or (b).

Let v_1, w_1 be two consecutive even or consecutive odd vertices of G_1 according as G_1 satisfies (a) or (b). Take η_1 as given in Observation 1 to be μ_1. This proves the claim for $i = 1$. Given μ_{i-1}, we construct μ_i as follows. Since $G_{i-1} < G_i$, the end vertices x_{i-1} and y_{i-1} of μ_{i-1} are adjacent to some vertices v_i and w_i respectively where v_i and w_i are two consecutive even or consecutive odd vertices of G_i according as G_i satisfies (a) or (b). Now μ_i is obtained by combining μ_{i-1} with $\eta_i - v_i w_i$, using the edges $x_{i-1} v_i$ and $y_{i-1} w_i$, where η_i is as given in Observation 1. Thus the claim is proved and μ_λ is a hamiltonian path of G. This completes case 1.

Case 2. σ has a cycle τ of length one. Let

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_\lambda \tau$$

be the disjoint cycle representation of σ. Let $\langle \tau \rangle = \{u\}$. Without loss of generality assume that $u \in A_r$. Then by Theorem 1.13 and the hypothesis it follows that $A_r = \{u\}$.

Let $H = G - u$ and Q the restriction of P to $V(H)$. Then (H,Q) is $(r-1)$ - psc and $\sigma_1 \sigma_2 \ldots \sigma_\lambda \in \mathcal{C}^*((H,Q))$ satisfies Case 1. Let μ_λ be the hamiltonian path of H obtained as in Case 1. We now make a few observations. In what follows G_i, v_i, w_i, x_i, y_i have the same meaning as in Case 1.

<u>Observation 3</u>. Either u is adjacent to all even vertices of G_i or u is adjacent to all odd vertices of G_i.

<u>Observation 4</u>. Either (i) v_i and w_i are both even and x_i and y_i are both odd or (ii) v_i and w_i are both odd and x_i and y_i are both even.

Now if u is adjacent to x_λ then we obtain a hamiltonian path of G by combining μ_λ with the edge $x_\lambda u$. If u is not adjacent to x_λ then by Observations 3 and 4, u is adjacent to v_λ. Let i be the smallest integer such that u is adjacent to v_i. If $i = 1$, we get a hamiltonian path of G by replacing the edge $v_1 w_1$ of μ_λ by the 2-path $v_1 u w_1$. If $i > 1$, then since u is not adjacent to v_{i-1}, it follows that u is adjacent to x_{i-1} and a hamiltonian path of G is obtained by replacing the edge $x_{i-1} v_i$ of μ_λ by the 2-path $x_{i-1} u v_i$.

This completes the proof of the theorem. $\square$

From Theorem 4.5 and Theorem 1.11 we immediately have the following

COROLLARY 4.6 (Clapham [1]). Every self-complementary graph has a hamiltonian path.

We now show by examples that the conditions on the cycle lengths of σ in Theorem 4.5 cannot be omitted.

Let t be an integer ≥ 1. We construct a 4-psc graph (G,P) with $\sigma \in \mathcal{C}^*((G,P))$ on $p = 6t+1$ vertices such that σ has exactly two cycles σ_1, σ_2 with $|\sigma_1| = 1$, $|I_{\sigma_2}| = 3$ and G has no hamiltonian path. The sets of P are $A_1 = \{u\}$, $A_2 = \{v_1, \ldots, v_{2t}\}$, $A_3 = \{w_1, \ldots, w_{2t}\}$ and $A_4 = \{x_1, \ldots, x_{2t}\}$ and $E(G) = \{u\,v_i,\ u\,w_{t+i},\ u\,x_i \mid 1 \leq i \leq t\} \cup \{v_i w_{t+j},\ w_{t+i}\,x_j,\ x_i v_j \mid 1 \leq i,j \leq t\} \cup \{v_i w_j,\ w_{t+i}\,x_{t+j},\ x_i v_{t+j} \mid 1 \leq i,j \leq t\}$. Clearly (G,P) is bipsc and $\sigma = (u)\,(v_1 w_1 x_1 v_{t+1}\, w_{t+1}\, x_{t+1}\, v_2 w_2 x_2\, v_{t+2}\, w_{t+2}\, x_{t+2} \ldots v_t w_t x_t v_{2t}\, w_{2t}\, x_{2t}) \in \mathcal{C}^*((G,P))$. We will now show that G has no hamiltonian path. We note that given any path μ in G, there exists i_o, j_o, k_o, $t+1 \leq i_o, k_o \leq 2t$, $1 \leq j_o \leq t$, such that v_{i_o}, w_{j_o}, x_{k_o} are not cut vertices of μ. Thus at least one of v_{i_o}, w_{j_o}, x_{k_o} does not belong to μ. Hence μ is not hamiltonian. Thus G has no hamiltonian path.

Next, let t be an integer ≥ 1. We construct a 5-psc graph (G,P) with $\sigma \in \mathcal{C}^*((G,P))$ on $p = 4t+2$ vertices such that σ has two cycles of length 1, every other cycle τ of σ has $|I_\tau| = 4$ and G has no hamiltonian path. The sets of P are $A_1 = \{u_1, u_2\}$, $A_2 = \{v_1, \ldots, v_t\}$, $A_3 = \{w_1, \ldots, w_t\}$, $A_4 = \{x_1, \ldots, x_t\}$, $A_5 = \{y_1, \ldots, y_t\}$ and $E(G) = \{u_i w_j, u_i x_j \mid i = 1,2,\ 1 \leq j \leq t\} \cup \{v_i w_j, w_i x_j, x_i y_j \mid 1 \leq i,j \leq t\}$. Clearly (G,P) is bipsc and $\sigma = (u_1)(u_2)\left(\prod_{i=1}^{t}(v_i w_i y_i x_i)\right) \in \mathcal{C}^*((G,P))$. Now since $|A_2 \cup A_3| = 2t$ and $G - A_2 - A_3$ is disconnected with $2t+2$ components, it follows that G has no hamiltonian path.

These examples show that the conditions given in Theorem 4.5 are best possible.

All the results in this chapter will appear in [7].

CHAPTER 5

DISCONNECTED MULTIPARTITE SELF-COMPLEMENTARY GRAPHS

In this chapter we deal exclusively with disconnected r-psc graphs. Disconnected r-psc graphs without isolated vertices are completely characterised. It is also established that for every disconnected bipsc graph (G,P), $\mathcal{T}_p((G,P))$ is non-empty.

We first characterise disconnected r-psc graphs which do not have any isolated vertex in the following

THEOREM 5.1. If (G,P) is a disconnected r-partitioned graph without isolated vertices, then (G,P) is r-psc iff $r = 2$ and for $i = 1,2$, A_i can be partitioned into two sets A_{i1} and A_{i2} such that $|A_{j1}| = |A_{j2}|$ for some j and

$$E(G) = \bigcup_{t=1}^{2} \{uv \mid u \in A_{1t}, v \in A_{2t}\}.$$

PROOF : Let (G,P) be disconnected r-psc and have no isolated vertex. Let G_t, $1 \leq t \leq k$ be the connected components of G.

By hypothesis, there exists an edge uv in G_1 and an edge xy in G_2. Now in $\bar{G}(P)$, every vertex is adjacent to at least one of u,v,x,y. Also either $ux, vy \in E(\bar{G}(P))$ or $uy, vx \in E(\bar{G}(P))$. Hence $\bar{G}(P)$ has at most two components. Hence $k = 2$. If now $r \geq 3$, then we can choose the above edges uv and xy such that at least three of u,v,x,y belong to different A_i's. It is then easy to see that $\bar{G}(P)$ is connected, a contradiction. Thus $r = 2$.

Define now $A_{it} = A_i \cap V(G_t)$, $i, t \in \{1,2\}$. Since G has no isolated vertices, $A_{it} \neq \emptyset$.

We now show that $uv \in E(G)$ if $u \in A_{1t}$ and $v \in A_{2t}$. Indeed if uv is not an edge in G, then in $\bar{G}(P)$, uv is an edge. Further, in $\bar{G}(P)$, every vertex of A_{1t} is adjacent to every vertex of $A_{2,3-t}$ and every vertex of A_{2t} is adjacent to every vertex of $A_{1,3-t}$. Hence $\bar{G}(P)$ is connected, a contradiction. Thus, for $t = 1,2$

$$E(G_t) = \{uv \mid u \in A_{1t}, v \in A_{2t}\}.$$

Now let $|A_{it}| = n_{it}$. Then $G = K_{n_{11},n_{21}} \cup K_{n_{12},n_{22}}$ and $\bar{G}(P) = K_{n_{11},n_{22}} \cup K_{n_{12},n_{21}}$. Since G and $\bar{G}(P)$ are isomorphic, it follows that $n_{11} = n_{12}$ or $n_{21} = n_{22}$. This proves the 'only if part' of the theorem. The 'if part' is trivial and the theorem is proved. $\square$

In the next three theorems, we study disconnected bipsc graphs (G,P) and show that $\mathcal{C}_p((G,P))$ is always non-empty. We will find it convenient to use A,B for the sets of P (instead of A_1, A_2). Also, given a bipartitioned graph (G,P), if $A_1 \subseteq A$ and $B_1 \subseteq B$ then $G[A_1|B_1]$ denotes the bipartitioned graph (H,Q) where $H = G[A_1 \cup B_1]$ and the sets of Q are A_1 and B_1. Further, $\bar{G}[A_1|B_1]$ will be used to denote the bipartite complement of $G[A_1|B_1]$.

THEOREM 5.2. Let (G,P) be a disconnected bipartitioned graph having no isolated vertex. Then (G,P) is bipsc iff there exist a partition $\{A_1,A_2\}$ of A and a partition $\{B_1,B_2\}$ of B such that

$$E(G) = \bigcup_{i=1}^{2} \{uv | u \in A_i , v \in B_i\}$$

and either $|A_1| = |A_2|$ or $|B_1| = |B_2|$. Further, if $\mathcal{C}_m((G,P)) \neq \emptyset$, then $|A_1| = |A_2| = |B_1| = |B_2|$.

PROOF : The first part of the theorem follows directly from Theorem 5.1. Suppose now $\mathcal{C}_m((G,P))$ has an element σ. The $\sigma(A_1) \in \{A_2,B_1,B_2\}$. But if $\sigma(A_1) = A_2$, then $\sigma(B_1) = B_1$, a contradiction. Hence either $\sigma(A_1) = B_1$ or $\sigma(A_1) = B_2$. We consider two cases accordingly.

Case 1. $\sigma(A_1) = B_1$. Then $\sigma(B_1) = A_2$ and $\sigma(B_2) = A_1$. Thus, $|B_2| = |A_1| = |B_1| = |A_2|$.

Case 2. $\sigma(A_1) = B_2$. Then $\sigma(B_1) = A_1$ and $\sigma(A_2) = B_1$. Thus, $|A_2| = |B_1| = |A_1| = |B_2|$.

This completes the proof of Theorem 5.2. □

THEOREM 5.3. Let (G,P) be disconnected bipsc and let $u \in A$ be an isolated vertex of G. Then either $\mathcal{G}((G,P)) = \mathcal{G}_p((G,P))$, or, for some positive integer m, A can be partitioned into $(m+1)$ sets $A_o, A_1, \ldots, A_m$ and B into m sets $B_1, B_2, \ldots, B_m$ such that $|A_i| = |B_j|$ for all i,j, $0 \leq i \leq m$, $1 \leq j \leq m$ and

$$E(G) = \bigcup_{i=1}^{m} \bigcup_{j=i}^{m} \{uv | u \in A_i \text{ , } v \in B_j\}.$$

PROOF : First of all, we observe that all the isolated vertices of G belong to A, because otherwise $\overline{G}(P)$ is connected.

Suppose now $\mathcal{G}((G,P)) \neq \mathcal{G}_p((G,P))$. Fix a σ in $\mathcal{G}((G,P)) - \mathcal{G}_p((G,P))$. Define

$$A_o = \{u \in A | u \text{ is an isolated vertex of } G\}$$

and

$$A_1 = \sigma(A_o).$$

We then prove the following :

1°. $A_1 \subseteq A - A_o$ and $G[A_1|B] = K$.

To prove this, let $u \in A_o$. Then $d_{\overline{G}(P)}(\sigma(u)) = 0$. If $\sigma(u) \in B$, then $d_G(\sigma(u)) = |A|$, a contradiction since $d_G(u) = 0$. Thus $\sigma(u) \in A$. If $\sigma(u) \in A_o$, then $d_{\overline{G}(P)}(\sigma(u)) = |B|$, a contradiction. Hence $\sigma(u) \in A - A_o$. Also since $d_{\overline{G}(P)}(\sigma(u)) = 0$, we have $G[\sigma(u)|B] = K$. Now 1° follows easily.

2°. $\sigma(A-A_o) = B$ and $\sigma(B) = A-A_1$.

To prove this, note that $G[A-A_o|B] \simeq \overline{G}[A-A_1|B]$ and σ (restricted to $(A-A_o) \cup B)$ is an isomorphism between them. But $G[A-A_o|B]$ is connected, since $G[A_1|B] = K$ and no vertex of $A-A_o$ is isolated in G. Hence either $\sigma(A-A_o) = A-A_1$, or $\sigma(A-A_o) = B$. If $\sigma(A-A_o) = A-A_1$, then $\sigma(B) = B$ and so $\sigma \in \mathcal{C}_p((G,P))$, a contradiction. Thus $\sigma(A-A_o) = B$ and so $\sigma(B) = A-A_1$. This proves 2°.

We now define B_i, A_{i+1}, $i = 1,2,\ldots$ inductively as follows :

$$B_i = \sigma(A_i) \ , \ A_{i+1} = \sigma(B_i).$$

Clearly $B_1 \subseteq B$. We now prove the following :

3°. Let t be any positive integer. Suppose that $A_i \subseteq A-A_o$, $1 \leq i \leq t$. Then,

(i) $B_j \cap B_t = \emptyset$, $1 \leq j \leq t-1$,

(ii) $A_j \cap A_{t+1} = \emptyset$, $1 \leq j \leq t$,

(iii) $G[A - \bigcup_{j=1}^{t} A_j | B_t] = \overline{K}$

(iv) $G[A_{t+1} | B - \bigcup_{j=1}^{t} B_j] = K$.

We prove 3° by induction on t. Since $A_i \subseteq A-A_o$, it follows that $B_i \subseteq B$, $1 \leq i \leq t$.

First let $t = 1$. Then (i) is vacuously tree. Also $A_1 = \sigma(A_o)$ and $A_2 = \sigma(B_1)$. Since $A_o \cap B_1 = \emptyset$ and σ is one-one, (ii) follows. Also

$$\overline{G}[A-A_1 | B_1] = \overline{G}[\sigma(B) | \sigma(A_1)] \simeq G[A_1 | B] = K .$$

This proves (iii). Finally,

$$\overline{G}[A_2 | B-B_1] = \overline{G}[\sigma(B_1) | \sigma(A-A_o-A_1)]$$

$$\simeq G[A-A_o-A_1 | B_1]$$

$$= \overline{K} \text{ (by (iii))}$$

Now since $A_2 \subseteq A$, (iv) follows.

Next, assuming the result for $t = s-1$, we will prove it for $t = s$, where $s \geq 2$. To prove (i), first note that $B_j = \sigma(A_j)$, $1 \leq j \leq s$. By induction hypothesis, $A_j \cap A_s = \emptyset$, $1 \leq j \leq s-1$. Since σ is one-one, (i) follows. To prove (ii), note that $A_1 = \sigma(A_o)$ and $A_{j+1} = \sigma(B_j)$, $1 \leq j \leq s$. Now $A_o \cap B_s = \emptyset$ and by (i), $B_j \cap B_s = \emptyset$, $1 \leq j \leq s-1$. Since σ is one-one, (ii) follows. Next note that

$$\overline{G}\left[A - \bigcup_{j=1}^{s} A_j \middle| B_s\right] = \overline{G}\left[\sigma(B) - \bigcup_{j=1}^{s-1} \sigma(B_j) \middle| \sigma(A_s)\right]$$

$$\simeq G\left[A_s \middle| B - \bigcup_{j=1}^{s-1} B_j\right]$$

$$= K \text{ (by induction hypothesis).}$$

This proves (iii). Finally,

$$\overline{G}\left[A_{s+1} \middle| B - \bigcup_{j=1}^{s} B_j\right] = \overline{G}\left[\sigma(B_s) \middle| \sigma(A-A_o) - \bigcup_{j=1}^{s} \sigma(A_j)\right]$$

$$\simeq G\left[A - \bigcup_{j=0}^{s} A_j \middle| B_s\right]$$

$$= \overline{K} \text{ (by (iii)).}$$

Now since $B_s \subseteq B$, we have $A_{s+1} \subseteq A$ and (iv) follows.

This completes the induction and proves 3^o.

Now, let m be the smallest integer such that $A_{m+1} \cap A_o = \emptyset$. Then $A_i \subseteq A-A_o$, $1 \leq i \leq m$. Hence by 3^o, (i) - (iv) hold for any positive integer $t \leq m$. Let $u \in A_{m+1} \cap A_o$. Then

$$G[A-A_o|\sigma^{-1}(u)] = G[\sigma^{-1}(B)|\sigma^{-1}(u)] \simeq \overline{G}[u|B] = K$$

(the last step follows since $u \in A_o$). Also since $u \in A_{m+1}$, $\sigma^{-1}(u) \in B_m$. So from 3^o (iii) we have $G[A - \bigcup_{j=1}^{m} A_j|\sigma^{-1}(u)] = \overline{K}$.

Hence $(A-A_o) \cap (A - \bigcup_{j=1}^{m} A_j) = \emptyset$, i.e. $A - \bigcup_{j=0}^{m} A_j = \emptyset$ and $A = \bigcup_{j=0}^{m} A_j$. Clearly now $B = \sigma(A-A_o) = \sigma(\bigcup_{i=1}^{m} A_i) = \bigcup_{i=1}^{m} B_i$.

Further $|A_o| = |A_1|$, $|A_i| = |B_i| = |A_{i+1}|$ for $i = 1,\ldots,m-1$ and $|A_m| = |B_m|$. Hence $|A_i| = |B_j|$ for all i,j. Also by 3^o (iii) and (iv) we have

$$G[A - \bigcup_{j=1}^{t} A_j|B_t] = \overline{K}, \quad 1 \leq t \leq m$$

$$G[A_{t+1}|B - \bigcup_{j=1}^{t} B_j] = K, \quad 1 \leq t \leq m.$$

Now since $G[A_1|B] = K$ and $G[A_o|B] = \overline{K}$, it follows that

$$E(G) = \bigcup_{i=1}^{m} \bigcup_{j=i}^{m} \{uv|u \in A_i, v \in B_j\}.$$

This proves Theorem 5.3. $\square$

THEOREM 5.4. If (G,P) is a disconnected bipsc graph, then $\mathcal{C}_p((G,P)) \neq \emptyset$.

PROOF : We consider the following two cases :

Case 1. G does not have any isolated vertex. Then by Theorem 5.2, there exist a partition $\{A_1,A_2\}$ of A and a partition $\{B_1,B_2\}$ of B such that

$$E(G) = \bigcup_{i=1}^{2} \{uv \mid u \in A_i \, , \, v \in B_i\}$$

and either $|A_1| = |A_2|$ or $|B_1| = |B_2|$. Let $A_1 = \{u_1,\ldots,u_a\}$, $A_2 = \{u_{a+1},\ldots,u_{a+b}\}$, $B_1 = \{v_1,\ldots,v_c\}$, $B_2 = \{v_{c+1},\ldots,v_{c+d}\}$ and without loss of generality assume that $a = b$. Then clearly

$$\sigma = \prod_{i=1}^{a} (u_i \; u_{2a+1-i}) \prod_{j=1}^{c+d} (v_j)$$

belongs to $\mathcal{C}_p((G,P))$.

Case 2. G has at least one isolated vertex. Without loss of generality let $u \in A$ be an isolated vertex of G. Then by Theorem 5.3, either $\mathcal{C}((G,P)) = \mathcal{C}_p((G,P))$, or for some positive integer m, A can be partitioned into $(m+1)$ sets $A_o, A_1,\ldots,A_m$ and B into m sets $B_1, B_2,\ldots,B_m$ such that $|A_i| = |B_j| = \delta$, $0 \leq i \leq m$, $1 \leq j \leq m$ and

$$E(G) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} \{uv \mid u \in A_i, v \in B_j\}.$$

In the latter case label the vertices of A_i with $u_{(i-1)\delta+1}, \ldots, u_{i\delta}$ for $i = 1,2,\ldots,m$, label the vertices of A_0 with $u_{m\delta+1}, \ldots, u_{(m+1)\delta}$ and label the vertices of B_i with $v_{(i-1)\delta+1}, \ldots, v_{i\delta}$ for $i = 1,2,\ldots,m$. Then it can be easily seen that

$$\sigma = \prod_{i=1}^{\left[\frac{(m+1)\delta+1}{2}\right]} (u_i \; u_{(m+1)\delta+1-i}) \prod_{j=1}^{\left[\frac{m\delta+1}{2}\right]} (v_j \; v_{m\delta+1-j})$$

belongs to $\mathscr{C}_p((G,P))$.

This completes the proof of Theorem 5.4. □

PART II

DEGREE SEQUENCES OF BIPSC GRAPHS

In Part II we deal with bipsc graphs and their degree sequences.

Throughout Part II, we will use (G,P) to denote a bipartitioned graph and A,B will denote the sets of P.

Given a bipartitioned graph (G,P), if $A_1 \subseteq A$ and $B_1 \subseteq B$ then $G[A_1|B_1]$ denotes the bipartitioned graph (H,Q) where $H = G[A_1 \cup B_1]$ and the sets of Q are A_1 and B_1. Further if $A_1 = \{u_1,\dots,u_s\}$ and $B_1=\{v_1\dots,v_t\}$, then we will write $G[u_1\dots,u_s|v_1,\dots,v_t]$ to mean $G[A_1|B_1]$. Finally $\overline{G}[A_1|B_1]$ will be used to denote the bipartite complement of $G[A_1|B_1]$.

If (G,P) is a bipartitioned graph, let $A = \{u_1,\dots,u_m\}$ and $B = \{v_1,\dots,v_n\}$ where $d_G(u_1) \geq \dots \geq d_G(u_m)$ and $d_G(v_1) \geq \dots \geq d_G(v_n)$. Let $d_i = d_G(u_i)$ and $e_j = d_G(v_j)$, then the bipartitioned sequence $\pi((G,P)) = (d_1,\dots,d_m|e_1,\dots,e_n)$ is called the degree sequence of (G,P).

If $A = \{u_1,\dots,u_m\}$ and $B = \{v_1,\dots,v_n\}$, then we say that $S = (u_1,\dots,u_m|v_1,\dots,v_n)$ is an ordering of (G,P). The bipartitioned graph (G,P) with the ordering $(u_1,\dots,u_m|v_1,\dots,v_n)$ is said to be a realisation of the bipartitioned sequence $\pi = (d_1,\dots,d_m|e_1,\dots,e_n)$ if

$d_G(u_i) = d_i$ and $d_G(v_j) = e_j$ for all i and j. We also say that (G,P) *is a realisation of* π if (G,P) with some ordering S, is a realisation of π. A bipartitioned sequence π is said to be *graphic* if there is a realisation of π. Further, π is said to be *unigraphic* if given any two realisations (G,P) and (H,P) of π, there is an isomorphism σ from G onto H such that $\sigma(B) = B$.

Finally, if $w_1, w_2, \ldots, w_{2k}$ are distinct vertices of a graph G and if $w_i\, w_{i+1}$ (with $w_{2k+1} = w_1$) is an edge of G or not according as i is odd or even, then by an *interchange along* $(w_1, \ldots, w_{2k}, w_1)$ we mean removing the edges $w_i\, w_{i+1}$ for odd i and adding the edges $w_i\, w_{i+1}$ for even i.

CHAPTER 6

POTENTIALLY BIPSC BIPARTITIONED SEQUENCES

Throughout this chapter π will denote the bipartitioned sequence $(d_1,\ldots,d_m|e_1,\ldots,e_n)$ where $n \geq d_1 \geq \ldots \geq d_m \geq 0$ and $m \geq e_1 \geq \ldots \geq e_n \geq 0$.

A graphic bipartitioned sequence π is said to be potentially bipsc if there exists at least one bipsc realisation (G,P) of π.

In this chapter we characterise when a bipartitioned sequence π is potentially bipsc. This characterisation is in terms of the following three conditions C1, C2 and C3 on π.

$$C1: \begin{cases} d_i + d_{m+1-i} = n, & 1 \leq i \leq m \\ e_j + e_{n+1-j} = m, & 1 \leq j \leq n \end{cases}$$

$$C2: \begin{cases} m = n \text{ is even}, \\ d_i + e_{m+1-i} = m, & 1 \leq i \leq m \\ d_{2i-1} = d_{2i}, & 1 \leq i \leq \frac{m}{2}. \end{cases}$$

C3 : m, n and $\sum_{j=1}^{n/2} e_j - \sum_{i=1}^{m/2} d_i - \frac{mn}{4}$ are all even integers.

In what follows, we denote $\frac{m}{2}$ by r if m is even and $\frac{n}{2}$ by t if n is even. Moreover, if $m = n$ is even, then we denote $\frac{m}{2} = \frac{n}{2}$ by s.

In our characterisation we will use the following result by Clapham and Kleitman [2].

RESULT A. If $(f_1, f_2, \ldots, f_{4k})$ is a graphic sequence satisfying

(i) $f_1 \geq f_2 \geq \ldots \geq f_{4k} \geq 0$,

(ii) $f_i + f_{4k+1-i} = 4k-1$, $1 \leq i \leq 2k$,

(iii) $f_{2i-1} = f_{2i}$, $1 \leq i \leq k$,

then $(f_1, f_2, \ldots, f_{4k})$ is the degree sequence of a self-complementary graph G with a complementing permutation σ given by

$$\sigma = \prod_{i=1}^{k} (w_{2i-1}\ w_{4k+1-2i}\ w_{2i}\ w_{4k+2-2i})$$

where w_i is the vertex having degree f_i in G.

We can now state the main result of this chapter as

THEOREM 6.1. A graphic bipartitioned sequence $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ is potentially bipsc iff it satisfies at least one of the following conditions :

(1) C1 holds and exactly one of m and n is odd.

(2) C1 holds, both m and n are even, and either $d_s = d_{s+1} = t$, or $e_t = e_{t+1} = s$.

(3) C1 and C3 hold.

(4) C2 holds.

The necessity part of the theorem is proved in two cases. We prove that if (G,P) is a bipsc realisation of π, then π satisfies at least one of conditions (1), (2) and (3) in case $\mathcal{C}_p((G,P)) \neq \emptyset$ and π satisfies condition (4) in case $\mathcal{C}_p((G,P)) = \emptyset$. The sufficiency part is also split up into two cases. If π satisfies at least one of conditions (1), (2) and (3) then we use the principle of induction to prove that π is potentially bipsc. If π satisfies condition (4), we use Result A to prove the sufficiency.

<u>Proof of Necessity in Theorem 6.1</u> : To prove the necessity let (G,P) with the ordering $(u_1,\ldots,u_m|v_1,\ldots,v_n)$ be a bipsc realisation of π. We consider two cases now.

Case 1. $\mathcal{C}_p((G,P)) \neq \emptyset$. Let $\sigma \varepsilon \mathcal{C}_p((G,P))$. Then we will prove that π satisfies at least one of conditions (1), (2), (3).

We first prove that π satisfies C1. Since $\sigma(A) = A$, the sequence $(n-d_1, n-d_2, \ldots, n-d_m)$ is a rearrangement of $(d_1, d_2, \ldots, d_m)$. Similarly, $(m-e_1, m-e_2, \ldots, m-e_n)$ is a rearrangement of $(e_1, e_2, \ldots, e_n)$. Since $d_1 \geq d_2 \geq \ldots \geq d_m$ and $e_1 \geq e_2 \geq \ldots \geq e_n$, it easily follows that π satisfies C1.

Now since G and $\overline{G}(P)$ have the same number of edges, it follows that G has $\frac{mn}{2}$ edges and so at least one of m and n is even. If exactly one of m,n is even then π satisfies condition (1). So let both m and n be even.

If now σ contains an odd cycle we will show that π satisfies condition (2). Let τ be an odd cycle of σ. Without loss of generality, assume that $\langle \tau \rangle \subseteq A$. Let $|\tau| = 2\ell+1$ and $w \varepsilon \langle \tau \rangle$. Now by Theorem 1.9, $\sigma^2 \varepsilon \text{Aut}(G)$. So $d_G(\sigma^i(w)) = d_G(\sigma^{i+2}(w))$, $i = 0, 2, \ldots, 2\ell$. Thus $d_G(w) = d_G(\sigma^{2\ell+2}(w)) = d_G(\sigma(w))$. Since $d_G(\sigma(w)) = n - d_{\overline{G}(P)}(\sigma(w)) = n - d_G(w)$, it follows that $d_G(w) = t$. Now since m is even, σ contains another odd cycle Ψ with $\langle \Psi \rangle \subseteq A$. By the above argument, $\langle \Psi \rangle$ contains a vertex x with $d_G(x) = t$. Since $d_1 \geq \ldots \geq d_m$, π satisfies C1 and two d_i's are equal to t, it follows that π satisfies condition (2).

Finally, let all cycles of σ have even lengths. Then we may assume that the vertices in A and the vertices in B are so labelled that $d_1 \geq \ldots \geq d_m$, $e_1 \geq \ldots \geq e_n$, $\sigma(A_1) = A-A_1$ and $\sigma(B_1) = B-B_1$ where $A_1 = \{u_1,\ldots,u_s\}$ and $B_1 = \{v_1,\ldots,v_t\}$. Let q_1 be the number of edges in $G[A_1|B_1]$ and q_2 the number of edges in $G[A_1|B-B_1]$. Since $\sigma(A_1) = A-A_1$ and $\sigma(B-B_1) = B_1$, it follows that the number of edges in $\overline{G}[A-A_1|B_1]$ is q_2, so the number of edges in $G[A-A_1|B_1]$ is $st-q_2$. Hence

$$\sum_{j=1}^{t} e_j = q_1 + st - q_2 = \sum_{i=1}^{s} d_i + st - 2q_2 .$$

Thus π satisfies C3 and hence it also satisfies condition (3). This finishes Case 1.

<u>Case 2</u>. $\mathscr{C}_p((G,P)) = \emptyset$. We then prove that π satisfies condition (4). By Theorem 5.4, G is connected. Now from Corollary 1.15 it follows that there is an element σ in $\mathscr{C}_m((G,P))$. Also, if τ is any cycle of σ, then

$|\tau| \equiv 0 \pmod 4$ and τ takes vertices alternately from A and B. Thus $\sigma(A) = B$, and so $m = n$ and m is even. Also by Theorem 1.9, $\sigma^2 \varepsilon$ Aut (G). Further, the sequence $(m - e_1, m - e_2, \ldots, m - e_m)$ is a rearrangement of the sequence $(d_1, d_2, \ldots, d_m)$. Since $d_1 \geq d_2 \geq \ldots \geq d_m$ and $m - e_m \geq \ldots \geq m - e_2 \geq m - e_1$, it follows that $d_i + e_{m+1-i} = m$, $1 \leq i \leq m$. Since $\sigma^2 \varepsilon$ Aut (G) and the length of every cycle in σ is a multiple of four it also follows that $d_{2i-1} = d_{2i}$, $1 \leq i \leq t$. Thus π satisfies condition (4). This finishes case 2 and the necessity is proved.

<u>Proof of Sufficiency in Theorem 6.1</u> : The proof of sufficiency is divided into two cases depending on whether π satisfies one of conditions (1), (2), (3) or π satisfies condition (4).

<u>Case 1</u>. π is graphic and satisfies one of conditions (1), (2) and (3). Without loss of generality we assume that if π satisfies (1) then m is odd and if π satisfies (2) then $d_s = d_{s+1} = t$.

We now prove by induction on m that there exists a bipartitioned graph (G,P) with an ordering $S = (u_1, \ldots, u_m | v_1, \ldots, v_n)$ satisfying the following

Condition Q_π : (G,P) with S is a bipsc realisation of π and $\mathcal{C}_p((G,P))$ contains

$$\sigma_1 = \prod_{i=1}^{\left[\frac{m+1}{2}\right]} (u_i \; u_{m+1-i}) \prod_{j=1}^{t} (v_j \; v_{n+1-j})$$

if π satisfies (1) or (3)

$$= \prod_{i=1}^{s-1} (u_i \; u_{m+1-i}) \; (u_s) \; (u_{s+1}) \prod_{j=1}^{t} (v_j \; v_{n+1-j})$$

if π satisfies (2).

If $m = 1$, and π satisfies (1), then by C1, $\pi = (t \mid 1^t, 0^t)$. Let $S = (u_1 \mid v_1, \ldots, v_n)$ and (G,P) be the graph with ordering S defined by :

$$E(G) = \{u_1 \; v_j \mid 1 \leq j \leq t\} .$$

Clearly, (G,P) with S satisfies Q_π .

If $m = 2$ and π satisfies (2), then by C1, $\pi = (t^2 \mid 2^r, 1^{n-2r}, 0^r)$ for some r, $0 \leq r \leq t$. Now let $S = (u_1, u_2 \mid v_1, \ldots, v_n)$ and (G,P) be the graph with ordering S, defined by :

$$E(G) = \{u_1 v_j \mid 1 \leq j \leq t\} \cup \{u_2 v_j \mid 1 \leq j \leq r \text{ or } t+1 \leq j \leq n-r\} .$$

It is easy to check that (G,P) with s satisfies Q_π.

If $m = 2$ and π satisfies (3), then by C1, $\pi = (d_1, n-d_1 | 2^r, 1^{n-2r}, 0^r)$ for some r, $0 \leq r \leq t$. Also,

$$r - d_1 = \sum_{j=1}^{t} (e_j - 1) - d_1$$

is even by C3. Since n is even and π is graphic it follows that $d_1 - r$ and $n-d_1 - r$ are even non-negative integers. Now let $S = (u_1, u_2 | v_1, \ldots, v_n)$ and (G,P) be the graph with ordering S and degree sequence π defined thus : $v_1, \ldots, v_r$ are joined to both u_1 and u_2 ; among $v_{r+1}, \ldots, v_{n-r}$, the first $\frac{d_1 - r}{2}$ and the last $\frac{d_1 - r}{2}$ vertices are joined to u_1 and the remaining vertices are joined to u_2. It is easy to check that (G,P) with S satisfies Q_π.

Now let $m \geq 3$ and assume that if a bipartitioned sequence $\pi^- = (d_1, \ldots, d_{m-2} | e_1, \ldots, e_n)$ satisfies the hypothesis of Case 1 (with $m-2$ replacing m) then there exists a graph (G,P) and an ordering S satisfying Q_{π^-}.

Let $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ satisfy the hypothesis of Case 1. Then define a new bipartitioned sequence

$$\pi^* = (d_1^*, \ldots, d_{m-2}^* | e_1^*, \ldots, e_n^*)$$

where

$$d_i^* = d_{i+1} \quad \text{for} \quad 1 \le i \le m-2$$

and

$$e_j^* = \begin{cases} e_j - 2 & \text{if} \quad 1 \le j \le d_m , \\ e_j - 1 & \text{if} \quad d_m + 1 \le j \le d_1 , \\ e_j & \text{if} \quad d_1 + 1 \le j \le n . \end{cases}$$

We note that $e_1^* \ge \dots \ge e_n^*$ may not hold.

Let $\pi^{**} = (d_1^{**}, \dots, d_{m-2}^{**} | e_1^{**}, \dots, e_n^{**})$ where $d_i^{**} = d_i^*$ for all i, $1 \le i \le m-2$, $e_j^{**} = e_{\alpha(j)}^*$ for all j, $1 \le j \le n$ and $(\alpha(1), \alpha(2), \dots, \alpha(n))$ is a permutation of $(1,2,\dots,n)$ such that [illegible] $\ge e_{\alpha(1)}^* \ge e_{\alpha(2)}^* \ge \dots \ge e_{\alpha(n)}^*$. Note that since $e_j^* + e_{n+1-j}^* = m-2$, $1 \le j \le n$, it follows that $\alpha(n+1-j) = n+1-\alpha(j)$, $1 \le j \le n$.

We will now prove that π^{**} satisfies the hypothesis of Case 1 (with $m-2$ replacing m, d_i^{**} replacing d_i and e_j^{**} replacing e_j for all i and j). This is done in several steps.

<u>Step 1</u>. We show that π^* and hence π^{**} is graphic. This follows from the following lemma since π^* is the degree sequence of the bipartitioned graph obtained from (G^*, P) by deleting the vertices u_1, u_m.

Lemma. Let $\pi = (d_1,\ldots,d_m|e_1,\ldots,e_n)$ be a graphic bipartitioned sequence satisfying C1, where $m \neq 2$, $d_1 \geq \ldots \geq d_m$ and $e_1 \geq \ldots \geq e_n$. Then there exists a bipartitioned graph (G^*,P) and an ordering $S = (u_1,\ldots,u_m|v_1,\ldots,v_n)$ of (G^*,P) such that (G^*,P) with S is a realisation of π and

$$N_{G^*}(u_i) = \{v_j | 1 \leq j \leq d_i\} \text{ for } i = 1 \text{ and } m.$$

Proof of the Lemma : Throughout the proof of this lemma, let S be the fixed ordering $(u_1,\ldots,u_m|v_1,\ldots,v_n)$. If (G,P) with the ordering S is a realisation of π, then we define for $i = 1$, m

$$Z_G(u_i) = \sum_{u_i v_j \in E(G)} j .$$

If (G,P) and (H,P) with the ordering S are realisations of π, then we write $H < G$ if either (i) $Z_H(u_1) < Z_G(u_1)$ or (ii) $Z_H(u_1) = Z_G(u_1)$ and $Z_H(u_m) < Z_G(u_m)$. Clearly the relation $<$ is antisymmetric and transitive. Hence there exists a graph (G^*,P) such that (G^*,P) with S is a realisation of π and if (H,P) with S is any other realisation of π then $H \nless G^*$. We will prove that G^* has the required property.

Suppose first $N_{G^*}(u_1) \neq \{v_j | 1 \leq j \leq d_1\}$. Then there exist integers $a \leq d_1$ and $b > d_1$ such that $u_1 v_a \notin E(G^*)$ but $u_1 v_b \in E(G^*)$. Since $e_a \geq e_b$, for some $c \neq 1$ we have $u_c v_a \in E(G^*)$ but $u_c v_b \notin E(G^*)$. Now if H is the graph obtained from G^* by an interchange along $(u_1, v_b, u_c, v_a, u_1)$, then (H,P) with S is a realisation of π and $H < G^*$, a contradiction to the choice of G^*. Thus $N_{G^*}(u_1) = \{v_j | 1 \leq j \leq d_1\}$. If now $m = 1$, we are done. So let $m \geq 3$.

Suppose next, $N_{G^*}(u_m) \neq \{v_j | 1 \leq j \leq d_m\}$. Let $a \leq d_m$ be the smallest integer such that $u_m v_a \notin E(G^*)$ and $b > a$ be the smallest integer such that $u_m v_b \in E(G^*)$. Clearly $e_a \geq e_b$ and $u_1 v_a \in E(G^*)$. We now consider two cases as follows :

<u>Case (i)</u> : There exists an integer $c \neq 1, m$ such that $u_c v_a \in E(G^*)$ but $u_c v_b \notin E(G^*)$. Then by an interchange along $(u_m, v_b, u_c, v_a, u_m)$ from G^* we arrive at a contradiction.

<u>Case (ii)</u> : Case (i) does not hold. Since $e_a \geq e_b$, $u_m v_a \notin E(G^*)$ and $u_m v_b \in E(G^*)$, it follows that $e_a = e_b$, $u_1 v_b \notin E(G^*)$ and so $b > d_1$. We also have $N_{G^*}(v_a) - \{u_1\} = N_{G^*}(v_b) - \{u_m\} = N$ (say). Now by C1

$$\frac{m}{2} \le e_{d_m} \le e_a = e_b \le e_{d_1} \le \frac{m}{2} .$$

Thus $e_a = e_b = \frac{m}{2}$ and so $|N| = \frac{m}{2} - 1 > 0$ as $m > 2$. Let $u_h \in N$. We will prove that if $k \ge a$ then v_k or v_{n+1-k} is adjacent to u_h in G^*, according as $e_k = \frac{m}{2}$ or not. If $k = a$ then clearly $u_h v_a \in E(G^*)$. So let $k > a$.

Suppose first $e_k = \frac{m}{2}$. If $u_1 v_k, u_m v_k \in E(G^*)$ then, since $e_a = \frac{m}{2}$ and $u_m v_a \notin E(G^*)$, for some $c \ne 1, m$ we have $u_c v_a \in E(G^*)$ but $u_c v_k \notin E(G^*)$. Now by an interchange along $(u_m, v_k, u_c, v_a, u_m)$ from G^* we arrive at a contradiction. Thus v_k is adjacent to at most one of u_1 and u_m. Now if $u_h v_k \notin E(G^*)$ then, since $e_b = \frac{m}{2}$ and $u_m v_b, u_h v_b \in E(G^*)$ it follows that for some $c \ne 1, m$ we have $u_c v_k \in E(G^*)$ but $u_c v_b \notin E(G^*)$. Then by an interchange along $(u_m, v_b, u_c, v_k, u_h, v_a, u_m)$ from G^*, we arrive at a contradiction. Hence $u_h v_k \in E(G^*)$.

Next suppose $e_k \ne \frac{m}{2}$. Then $e_k < \frac{m}{2}$ and so $e_{n+1-k} > \frac{m}{2}$. Now if $u_h v_{n+1-k} \notin E(G^*)$ then, since $e_b = \frac{m}{2}$ and $u_h v_b, u_m v_b \in E(G^*)$, it follows that for some $c \ne 1, m$ we have $u_c v_{n+1-k} \in E(G^*)$ but $u_c v_b \notin E(G^*)$. Then by an interchange along $(u_m, v_b, u_c, v_{n+1-k}, u_h, v_a, u_m)$ from G^*, we arrive at a contradiction. Hence $u_h v_{n+1-k} \in E(G^*)$.

Thus if $k \geq a$ then v_k or v_{n+1-k} is adjacent to u_h in G^* according as $e_k = \frac{m}{2}$ or not. Thus corresponding to each $k \geq a$, there is a distinct vertex adjacent to u_h. It now follows that

$$d_h \geq n-a+1 \geq n-d_m+1 = d_1+1 ,$$

a contradiction, since $d_1 \geq d_h$.

Thus $N_{G^*}(u_m) = \{v_j | 1 \leq j \leq d_m\}$. This proves the lemma.

Step 2. π^{**} satisfies C1. This follows easily from the fact that

$$\begin{aligned} d_i^* + d_{m-1-i}^* &= n, \quad 1 \leq i \leq m-2, \\ e_j^* + e_{n+1-j}^* &= m-2, \quad 1 \leq j \leq n. \end{aligned} \qquad \ldots(6.1)$$

Step 3. If π satisfies (1), (2) or (3), then so does π^{**}.

First let π satisfy (1). Then m and hence $m-2$ is odd and π^{**} satisfies (1).

Next let π satisfy (2). Then m and n are even and $d_s = d_{s+1} = t$. Since $d_i^{**} = d_{i+1}$, it easily follows that π^{**} satisfies (2).

Finally let π satisfy (3). Then m, n are even and

$$\sum_{j=1}^{t} e_j^* - \sum_{i=1}^{s-1} d_i^* - (s-1)t$$

$$= \sum_{j=1}^{t} e_j - \sum_{i=1}^{s} d_i - st + (n-2d_m)$$

is even. Now

$$\sum_{j=1}^{t} e_j^{**} = \sum_{j=1}^{t} \max(e_j^*, e_{n+1-j}^*) \quad \text{by (6.1)}$$

$$\equiv \sum_{j=1}^{t} e_j^* \pmod 2$$

since $e_j^* + e_{n+1-j}^* = m$ is even. Thus π^{**} satisfies C3 and hence (3).

Thus π^{**} satisfies the hypothesis of Case 1 and so by induction hypothesis there exist a bipsc graph (G^{**},P^{**}) and an ordering $S^{**} = (u_2,\ldots,u_{m-1} | v_{\alpha(1)},\ldots,v_{\alpha(n)})$ such that (G^{**},P^{**}) with the ordering S^{**} is a realisation of π^{**} and $\mathcal{C}_p((G^{**},P^{**}))$ contains

$$\sigma_1^{**} = \prod_{i=2}^{[\frac{m+1}{2}]} (u_i\ u_{m+1-i}) \prod_{j=1}^{t} (v_j\ v_{n+1-j})$$

if π^{**} satisfies (1) or (3),

and

$$\sigma_2^{**} = \prod_{i=2}^{s-1} (u_i \; u_{n+1-i}) \; (u_s) \; (u_{s+1}) \prod_{j=1}^{t} (v_j \; v_{n+1-j})$$

if π^{**} satisfies (2).

Notice that in σ_1^{**} and σ_2^{**} we replaced

$\prod_{j=1}^{t} (v_{\alpha(j)} \; v_{\alpha(n+1-j)})$ by $\prod_{j=1}^{t} (v_{\alpha(j)} \; v_{n+1-\alpha(j)})$ since

$\alpha(n+1-j) = n+1-\alpha(j)$, $1 \leq j \leq n$.

Now construct a graph (G,P) with the ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_n)$ from (G^{**},P^{**}) by adding two new vertices u_1, u_n and joining u_i to $v_1, v_2,\ldots,v_{d_i}$ for $i = 1$ and n. Then clearly

$$d_G(u_i) = d_i \; , \; 1 \leq i \leq m$$

and

$$d_G(v_j) = e_j \; , \; 1 \leq j \leq n.$$

Thus (G,P) with S is a realisation of π. Further $G[u_2,\ldots, u_{n-1}|B] = (G^{**},P^{**})$ is bipsc and $\mathcal{C}_p((G^{**},P^{**}))$ contains σ_1^{**} or σ_2^{**} according as π^{**} satisfies (1) or (3), or π^{**} satisfies (2). Also $G[u_1, u_n|B]$ is by construction bipsc with $(u_1 \; u_n) \prod_{j=1}^{t} (v_j \; v_{n+1-j})$ as a bipcp.

It now follows that (G,P) with S satisfies condition Q_π.

This completes the induction and proves the sufficiency in Case 1.

<u>Case 2.</u> π is graphic and satisfies condition (4). By C2, $m = n$ is even. We now define a new sequence

$$\pi' = (f_1, f_2, \ldots, f_{4t})$$

where

$$f_i = \begin{cases} d_i + 2t - 1 & \text{if } 1 \leq i \leq 2t, \\ e_{i-2t} & \text{if } 2t+1 \leq i \leq 4t. \end{cases}$$

Let (G,P) be a realisation of π. Then π' is graphic since it is the degree sequence of the graph obtained from G by joining all $\binom{2t}{2}$ pairs of vertices in A by edges.

We next show that π' is a non-increasing sequence of non-negative integers. Clearly $f_1 \geq \ldots \geq f_{2t}$ and $f_{2t+1} \geq \ldots \geq f_{4t} \geq 0$. Now, if $d_{2t} = 0$ then by C2, $e_1 = 2t$, a contradiction to the graphicness of π. Thus $d_{2t} \geq 1$ and so $e_1 \leq 2t$. It now follows that $f_{2t} \geq f_{2t+1}$. Thus $f_1 \geq f_2 \geq \ldots \geq f_{4t} \geq 0$.

Again by C2, $d_{2i-1} = d_{2i}$, $1 \leq i \leq t$. So we have $f_{2i-1} = f_{2i}$, $1 \leq i \leq t$. Also if $1 \leq i \leq 2t$, then by C2,

$$f_i + f_{4t+1-i} = d_i + 2t - 1 + e_{2t+1-i} = 4t-1 .$$

Thus π' satisfies conditions (i), (ii) and (iii) of Result A (with k replaced by t). Hence by Result A, it follows that π' is the degree sequence of a self-complementary graph G' with a complementing permutation σ given by

$$\sigma = \prod_{i=1}^{t} (w_{2i-1} \; w_{4t+1-2i} \; w_{2i} \; w_{4t+2-2i})$$

where w_i is the vertex having degree f_i in G'. Let $A = \{w_1, \ldots, w_{2t}\}$, $B = \{w_{2t+1}, \ldots, w_{4t}\}$. Then clearly $\sigma(A) = B$.

Since π is graphic, $\sum_{i=1}^{2t} d_i = \sum_{j=1}^{2t} e_j$, and $e_j \leq 2t$ for all j. Thus equality holds when $r = 2t$ in the following Erdős-Gallai criterion (See [3]) :

$$\sum_{i=1}^{r} f_i \leq r(r-1) + \sum_{i=r+1}^{4t} \min(r, f_i), \; 1 \leq r \leq 4t .$$

Hence it follows that in G' any two distinct vertices of A are adjacent and any two distinct vertices of B are nonadjacent. Now let $S = (w_1, \ldots, w_{2t} | w_{2t+1}, \ldots, w_{4t})$ and (G,P) the graph with ordering S be defined by :

$$E(G) = E(G') - \{w_i\, w_j | 1 \leq i < j \leq 2t\} .$$

Then clearly (G,P) with S is a realisation of π. Now since in G', $\sigma(A) = B$, it follows that (G,P) is bipsc with $\sigma \in \mathcal{C}_n((G,P))$. Thus π is potentially bipsc.

This finishes Case II and the sufficiency is proved.

This completes the proof of Theorem 6.1. $\square$

We now list a few corollaries which follow directly from the proof of Theorem 6.1.

<u>COROLLARY 6.2</u>. A graphic bipartitioned sequence $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ is the degree sequence of a bipsc graph (G,P) with $\mathcal{C}_p((G,P)) \neq \emptyset$ iff π satisfies at least one of the conditions (1), (2) and (3) in Theorem 6.1. Also then (G,P) and an ordering $S = (u_1, \ldots, u_m | v_1, \ldots, v_n)$ can be chosen so that (G,P) with S is a bipsc realisation of π and $\mathcal{C}_p((G,P))$ contains

$$\sigma_1 = \prod_{i=1}^{[\frac{m+1}{2}]} (u_i\, u_{m+1-i}) \prod_{j=1}^{[\frac{n+1}{2}]} (v_j\, v_{n+1-j})$$

if π satisfies condition (1) or (3).

$$\sigma_2 = \prod_{i=1}^{s-1} (u_i \; u_{n+1-i}) \; (u_s) \; (u_{s+1}) \; \prod_{j=1}^{t} (v_j \; v_{n+1-j})$$

if π satisfies condition (2) and $d_s = d_{s+1} = t$,

$$\sigma_3 = \prod_{i=1}^{t} (u_i \; u_{n+1-i}) \; \prod_{j=1}^{t-1} (v_j \; v_{n+1-j}) \; (v_t) \; (v_{t+1})$$

if π satisfies condition (2) and $e_t = e_{t+1} = s$.

<u>COROLLARY 6.3.</u> A graphic bipartitioned sequence $\pi = (d_1,\ldots,d_m|e_1,\ldots,e_n)$ is the degree sequence of a bipsc graph (G,P) with a bipcp σ which sends A to B iff π satisfies C2. Also then $m = n$, m is even and (G,P) and an ordering $S = (u_1,\ldots,u_m|v_1,\ldots,v_m)$ can be chosen so that (G,P) with S is a bipsc realisation of π and $\mathcal{C}_m((G,P))$ contains

$$\sigma = \prod_{i=1}^{t} (u_{2i-1} \; v_{m+1-2i} \; u_{2i} \; v_{m+2-2i}) .$$

<u>COROLLARY 6.4.</u> A graphic bipartitioned sequence $\pi = (d_1,\ldots,d_m|e_1,\ldots,e_n)$ is the degree sequence of a connected bipsc graph (G,P) iff

(i) $\min (d_m, e_n) > 0$,

(ii) $\pi \not\in \{(n_1, n-n_1|1^n), (1^m|m_1, m-m_1)\}$,

(iii) π satisfies at least one of conditions (1), (2), (3) and (4) in Theorem 6.1.

PROOF : Necessity follows easily. To prove the sufficiency, let π satisfy (i), (ii) and (iii). By Theorem 6.1, there is a bipsc graph (G,P) and an ordering $S = (u_1,\dots,u_m| v_1,\dots,v_n)$ such that (G,P) with S is a realisation of π. If G is connected we are done. Otherwise, by (i) and Theorem 5.2,

$$\pi = \pi((G,P)) = (n_1^{m_1}, (n-n_1)^{m-m_1} | m_1^{n_1}, (m-m_1)^{n-n_1}) \text{ for some}$$

integers m_1, n_1 with $0 < m-m_1 \leq m_1$, $0 \leq n-n_1 \leq n_1$ and either $m_1 = \frac{m}{2}$ or $n_1 = \frac{n}{2}$. Also by (ii), min $(m_1,n_1) \geq 2$. Now construct a graph (H,P) with ordering S by joining

u_i to v_1 , $v_2,\dots,v_{n_1}$ if $1 \leq i \leq m_1-1$

u_{m_1} to v_1 , $v_2,\dots,v_{n_1-1}$, v_{n_1+1} ,

u_{m_1+1} to v_{n_1}, v_{n_1+2}, $v_{n_1+3},\dots,v_n$, and

u_i to v_{n_1+1}, $v_{n_1+2},\dots,v_n$ if $m_1+1 \leq i \leq m$.

Note that (H,P) with S is a realisation of π. Further since $0 \leq m-m_1 \leq m_1$, $0 < n-n_1 \leq n_1$ and min $(m_1,n_1) \geq 2$, it follows that H is connected. Finally it is easily seen that (H,P) is bipsc and $\mathfrak{G}_p((G,P))$ contains

$$\prod_{i=1}^{s} (u_i \; u_{m+1-i}) \prod_{j=1}^{n} (v_j) \quad \text{if} \quad m_1 = \frac{m}{2} ,$$

$$\prod_{i=1}^{m} (u_i) \prod_{j=1}^{t} (v_j \; v_{n+1-j}) \quad \text{if} \quad n_1 = \frac{n}{2} .$$

This proves the sufficiency and Corollary 6.4 is proved. □

All results in this chapter will appear in [5].

CHAPTER 7

FORCIBLY BIPSC BIPARTITIONED SEQUENCES

7.1 MAIN RESULT

Throughout this chapter π will denote the bipartitioned sequence $(d_1,\dots,d_m|e_1,\dots,e_n)$ where $n \geq d_1 \geq \dots \geq d_m \geq 0$ and $m \geq e_1 \geq \dots \geq e_n \geq 0$.

A bipartitioned sequence π is said to be forcibly bipsc if π is graphic and every realisation of π is bipsc.

In this chapter we characterise when a bipartitioned sequence π is forcibly bipsc. This characterisation is in terms of the conditions C1 and C2 as given on page 101. It also uses the characterisation of forcibly self-complementary sequences as obtained by Rao [12] and the characterisation of unigraphic bipartitioned sequences as obtained by Koren [9].

Henceforth, given $\pi = (d_1,\dots,d_m|e_1,\dots,e_n)$, we denote $\frac{m}{2}$ by s if m is even and $\frac{n}{2}$ by t if n is even. Moreover, if C2 holds then we denote $\frac{m}{2} = \frac{n}{2}$ by t. We note here that if π satisfies C1 and $d_i = d_{m+1-i}$ for some i, then n is even and so t is well-defi..ed. Similarly, if π satisfies C1 and $e_j = e_{n+1-j}$ for some j, then s is well-defined.

In what follows, we assume without loss of generality that π satisfies the conditions (I) - (III) given below since, if any one of (I) - (III) is violated by π then $\tilde{\pi} = (e_1,\ldots,e_n|d_1,\ldots,d_m)$ satisfies all of (I) - (III).

(I) If $d_1 > d_m$ then $e_1 > e_n$.

(II) If some $e_j = \frac{n}{2}$, then some $d_i = \frac{n}{2}$.

(III) If $d_1 > d_m$, $e_1 > e_n$, some $d_i = \frac{n}{2}$ and some $e_j = \frac{m}{2}$, then $d_p - n+q \geq e_q - n+p$ where $p = \max\left\{ i | d_i > t \right\}$ and $q = \max\left\{ j | e_j > s \right\}$.

We are now ready to state the main theorem of this chapter as

THEOREM 7.1. A bipartitioned sequence $\pi = (d_1,\ldots,d_m|e_1,\ldots,e_n)$ (with the above assumptions (I) - (III), which can be made without loss of generality), is forcibly bipsc iff $\sum_{i=1}^{m} d_i = \sum_{j=1}^{n} e_j$ and π satisfies one of the following four conditions :

(1) C2 holds and the sequence $\pi' = (d_1 + 2t-1, \ldots, d_{2t} + 2t-1, e_1, \ldots, e_{2t})$ is forcibly self-complementary.

(2) C1 holds, $d_1 = d_n$, $e_1 = e_n$ and either $\min(s,t) \leq 2$ or, $\min(s,t) = 3$ and $\max(s,t) \leq 4$.

(3) C1 holds, $d_1 = d_n$ and if k is the number of e_j's in π which are equal to zero, then either $t-k \leq 2$, or $\pi^o = ((t-k)^n | e_{k+1}, \ldots, e_{2t-k})$ is one of the following bipartitioned sequences :

$\pi_1 = (3^6|3^6)$, $\pi_2 = (4^6|3^8)$, $\pi_3 = (3^8|4^6)$,

$\pi_4 = ((t-k)^2|1^{2(t-k)})$, $\pi_5 = ((t-k)^4|2^{2(t-k)})$,

$\pi_6 = ((t-k)^n|(n-1)^{t-k}, 1^{t-k})$,

$\pi_7 = ((t-k)^4|3, 2^{2(t-k-1)}, 1)$, $\pi_8 = (3^{2s}|2s-1, s^4, 1)$.

(4) C1 holds and if p is the number of d_i's greater than $\frac{n}{2}$ and q the number of e_j's greater than $\frac{n}{2}$, then $0 < p \leq \frac{n}{2}$ and $0 < q \leq \frac{n}{2}$. Further if h is the number of e_j's in π which are not less than $n-p$, then

(a) n is even,

(b) $\sum_{i=1}^{p} d_i = (n-h)\, p + \sum_{j=n-h+1}^{n} e_j$,

(c) $\sum_{j=1}^{h} e_j = h(n-p) + \sum_{i=n-p+1}^{n} d_i$,

(d) Either $p = \frac{n}{2}$

or $t - h \leq 2$

or $\pi^+ = ((t-h)^{n-2p} | e_{h+1} - p, \ldots, e_{2t-h} - p)$

is one of $\pi_1 - \pi_8$ given in (3) above, with

t replaced by $t-h$ and k replaced by 0.

(e) The bipartitioned sequence $\pi^* = (d_1 - n + h, \ldots, d_p - n + h \mid e_{n-h+1}, \ldots, e_n)$ is unigraphic.

The proof of Theorem 7.1 is lengthy and we split it up into several sections. In Section 7.2, we prove certain preliminary lemmas which will be frequently used in the main body of the proof. In Section 7.3, we prove the necessity part of the theorem and finally, the sufficiency part of the theorem is proved in Section 7.4.

In the diagrams which will be used in the course of proving the theorem, we will frequently represent sets of vertices by single vertices for convenience with the following understanding :

If xy is an edge in the diagram then x (or every vertex of x in case x is a set) is adjacent to y (or every vertex of y in case y is a set).

7.2 PRELIMINARIES

In this section we prove a few preliminary lemmas which will be used frequently in the course of proving Theorem 7.1.

LEMMA 7.2. If π is a bipartioned sequence not satisfying C2, and (G,P) is a bipsc realisation of π, then $\mathcal{C}_p((G,P)) \neq \emptyset$.

PROOF : Suppose $\mathcal{C}_p((G,P)) = \emptyset$. Then by Theorem 5.4, G is connected, and so by Corollary 1.15, there is an element σ in $\mathcal{C}((G,P))$ such that $\sigma(A) = B$. Now by Corollary 6.3 it follows that π satisfies C2, a contradiction which proves the Lemma. □

LEMMA 7.3. Let $\pi = (d_1,\ldots,d_m|e_1,\ldots,e_n)$ be a forcibly bipsc bipartitioned sequence not satisfying C2 and let (G,P) with the ordering $S = (u_1,\ldots,u_m|v_1,\ldots,v_n)$ be a realisation of π. Let i,j be integers such that $1 \leq i \leq [\frac{m+1}{2}]$ and $1 \leq j \leq [\frac{n+1}{2}]$ and let $A_1 = \{u_i, u_{i+1},\ldots, u_{m+1-i}\}$, $B_1 = \{v_j, v_{j+1},\ldots,v_{n+1-j}\}$. If (1) $i = 1$ or $d_{i-1} > d_i$ and (2) $j = 1$ or $e_{j-1} > e_j$, then the bipartitioned sequence $\pi^* = \pi(G[A_1|B_1])$ is forcibly bipsc.

PROOF : Let (G^*, P^*) with the ordering $S^* = (u_i, \ldots, u_{m+1-i} | v_j, \ldots, v_{n+1-j})$ be any realisation of π^*. Construct a graph (H,P) from (G,P) by replacing $G[A_1 | B_1]$ by (G^*, P^*). Then (H,P) with the ordering S is a realisation of π. Since π is forcibly bipsc and does not satisfy C2, it follows by Lemma 7.2 that $\mathcal{C}_p((H,P))$ has an element σ. Then by (1) we get $\sigma(A_1) = A_1$ and by (2), $\sigma(B_1) = B_1$. It now follows that σ restricted to $A_1 \bigcup B_1$ is an element of $\mathcal{C}_p((G^*, P^*))$. Thus (G^*, P^*) is bipsc and π^* is forcibly bipsc. This proves the lemma. □

LEMMA 7.4. If π satisfies C1 and $e_1 = e_n$ then π is graphic.

PROOF : By hypothesis, n is even and $\pi = (d_1, \ldots, d_s, n-d_s, \ldots, n-d_1 | s^n)$. Let (G,P) be the bipartitioned graph with $A = \{u_1, \ldots, u_{2s}\}$, $B = \{v_1, \ldots, v_n\}$ and

$$E(G) = \{u_i v_j | 1 \leq j \leq d_i, 1 \leq i \leq s\}$$
$$\bigcup \{u_i v_j | d_{2s+1-i} + 1 \leq j \leq n, s+1 \leq i \leq 2s\}.$$

Clearly then (G,P) with the ordering $S = (u_1, \ldots, u_{2s} | v_1, \ldots, v_n)$ is a realisation of π and π is graphic. This proves the lemma. □

LEMMA 7.5. If (G,P) with the ordering $S = (u_1,\ldots,u_n | v_1,\ldots,v_4)$ is any realisation of $\pi = (2^n | m-\alpha,\ m-\beta,\ \beta,\ \alpha)$ where $1 \leq \alpha \leq \beta \leq \frac{m}{2}$, then $\mathcal{C}_p((G,P))$ contains an element σ such that $\sigma(v_j) = v_{5-j}$, $1 \leq j \leq 4$.

PROOF : Let A_{ij} be the set of all vertices adjacent to both v_i and v_j in G and $n_{ij} = |A_{ij}|$, $1 \leq i \neq j \leq 4$. Then since every u_i has degree 2 in G, we have

$$\sum_{j\neq 1} n_{1j} + \sum_{j\neq 4} n_{4j} = d_G(v_1) + d_G(v_4) = m \qquad \ldots(7.1)$$

Also since $n_{ij} = n_{ji}$ for all i,j, we get

$$\sum_{i=1}^{4} \sum_{j=i+1}^{4} n_{ij} = \frac{1}{2} \sum_{j=1}^{4} d_G(v_j) = n \qquad \ldots(7.2)$$

Subtracting (7.1) from (7.2) we obtain $n_{14} = n_{23}$. Now any permutation σ of $V(G)$ satisfying

$$\sigma(A_{14}) = A_{23},$$

$$\sigma(u) = u \text{ if } u \in A_{12} \cup A_{13} \cup A_{24} \cup A_{34},$$

$$\sigma(v_j) = v_{5-j}, \quad 1 \leq j \leq 4,$$

has the properties required in the lemma and the lemma is proved. □

LEMMA 7.6. If (G,P) with the ordering $S = (u_1,\ldots,u_{2s}|v_1,\ldots,v_4)$ is any realisation of $\pi = (2^{2s}|s^4)$, then $\mathcal{C}_p((G,P))$ contains an element σ satisfying $\sigma(v_j) = v_j$ for all j.

PROOF : Let A_{ij} be the set of all vertices of G adjacent to both v_i and v_j and $n_{ij} = |A_{ij}|$, $1 \leq i \neq j \leq 4$. Then we have

$$\sum_{i\neq j} n_{ij} = d_G(v_j) = s \ , \ j = 1,\ldots,4 \qquad \ldots(7.3)$$

Summing (7.3) over all j and using the fact that $n_{ij} = n_{ji}$ we get

$$\sum_{j=1}^{4} \sum_{i<j} n_{ij} = 2s \qquad \ldots(7.4)$$

Subtracting the equations (7.3) corresponding to $j = 1$ and $j = 2$ from the equation (7.4), we get $n_{12} = n_{34}$. By symmetry, $n_{13} = n_{24}$ and $n_{14} = n_{23}$. Now it easily follows that any permutation σ satisfying

$$\sigma(A_{ij}) = A_{k\ell} \ , \ 1 \leq i \neq j \leq 4 \text{ and } \{k, \ell\} = \{1,2,3,4\} - \{i,j\}$$

and

$$\sigma(v_j) = v_j \ , \quad j = 1,\ldots,4$$

is an element of $\mathcal{C}_p((G,P))$. This proves the lemma. □

LEMMA 7.7. If (G,P) with the ordering $S = (u_1,\dots,u_{2s}|v_1,\dots,v_4)$ is any realisation of $\pi = (3, 2^{2s-2}, 1|s^4)$ then $\mathcal{C}_p((G,P))$ contains an element σ such that $\sigma(u_1) = u_{2s}$ and $\sigma(u_{2s}) = u_1$.

PROOF : Let $(H,Q) = G[u_2,\dots,u_{2s-1}|v_1,\dots,v_4]$. Suppose first u_1 and u_{2s} have disjoint neighbourhoods in G. Then $\pi((H,Q)) = (2^{2s-2}|(s-1)^4)$. By Lemma 7.6, it now follows that $\mathcal{C}_p((H,Q))$ has an element σ^* such that $\sigma^*(v_j) = v_j$, $1 \le j \le 4$. Clearly then $\sigma = \sigma^*(u_1\ u_{2s})$ is an element of $\mathcal{C}_p((G,P))$ having the required properties. Next suppose that some v_j is adjacent to both u_1 and u_{2s}. We assume without loss of generality that $u_1 v_1$ is not an edge and $u_{2s} v_4$ is an edge in G. Then $\pi((H,Q)) = (2^{2s-2}|s, (s-1)^2, s-2)$. By Lemma 7.5, it now follows that $\mathcal{C}_p((H,Q))$ has an element σ^* such that $\sigma^*(v_1) = v_4$ and $\sigma^*(v_4) = v_1$. Clearly then $\sigma = \sigma^*(u_1\ u_s)$ is an element of $\mathcal{C}_p((G,P))$ having the required properties. This proves the lemma. □

LEMMA 7.8. Let $\pi = (d_1,\dots,d_n|e_1,\dots,e_n)$ be graphic and i any integer such that $1 \le i \le n$. Then there is a bipartitioned graph (G,P) and an ordering $S = (u_1,\dots,u_n|v_1,\dots,v_n)$ such that (G,P) with the ordering S is a realisation of π and u_i is adjacent to $v_1,\dots,v_{d_i}$ in G.

PROOF : For any bipartitioned graph (H,P) such that (H,P) with the ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_n)$ is a realisation of π, define $\alpha(H)$ to be the number of vertices among $v_1,\ldots,v_{d_i}$ which are adjacent to u_i in H. Let (G,P) be such a bipartitioned graph with the maximum value of α. Then we show that u_i is adjacent to $v_1,\ldots,v_{d_i}$ in G. Otherwise u_i is not adjacent in G to v_j for some j, $1 \leq j \leq d_i$. Then u_i is adjacent to v_k for some k, $d_i + 1 \leq k \leq n$. Since $e_j \geq e_k$ it follows that there is an $r \neq i$ such that u_r is adjacent to v_j but not adjacent to v_k. Let (H,P) be the graph obtained from (G,P) by an interchange along $(u_i, v_k, u_r, v_j, u_i)$. Then (H,P) with the ordering S is a realisation of π and $\alpha(H) = \alpha(G) + 1$, a contradiction which proves the lemma. □

LEMMA 7.9. If n is even and $\pi = (d_1,\ldots,d_n|e_1,\ldots,e_n)$ is a graphic bipartitioned sequence satisfying C1, then there is a bipartitioned graph (G,P) and an ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_{2t})$ such that (G,P) with S is a realisation of π and $u_i v_j$ is an edge of G for all i,j, $1 \leq i \leq \lceil \frac{m+1}{2} \rceil$ and $1 \leq j \leq t$.

PROOF : We prove the lemma by induction on n.

If $n = 1$, then $\pi = (t|1^t\ 0^t)$ and any realisation (G,P) of π proves the theorem.

If $n = 2$, then $\pi = (d_1,\ 2t-d_1|2^r,\ 1^{2t-2r},\ 0^r)$, where $0 \leq r \leq t$. Let (G,P) be the bipartitioned graph with $A = \{u_1,\ u_2\}$, $B = \{v_1,\ldots,v_{2t}\}$ and

$$E(G) = \{u_1\ v_j|1 \leq j \leq d_1\} \cup \{u_2\ v_j|1 \leq j \leq r \text{ or } d_1 + 1 \leq j \leq 2t-r\}.$$

Clearly then (G,P) with the ordering $S = (u_1,u_2|v_1,\ldots,v_{2t})$ is the required realisation of π.

We now assume the lemma for $n-2$ and prove it for n when $n \geq 3$. For convenience we will take $e_o = n$ and $e_{2t+1} = 0$ in what follows. Let r be the number of e_j's, in $\{e_1,\ldots,e_{2t}\}$ such that $e_j - e_{2t+1-j} \geq 2$. Then since $e_1 \geq \ldots \geq e_{2t}$, it follows that $0 \leq r \leq t$ Also by C1 we have $e_r > e_{r+1}$. Now let

$$\pi^o = (d_2,\ldots,d_n|e_1^o\ ,\ldots,e_{2t}^o)$$

where

$$e_j^o = \begin{cases} e_j - 1 & \text{if } 1 \leq j \leq d_1 \\ e_j & \text{Otherwise.} \end{cases}$$

By Lemma 7.8 (with $i=1$) it easily follows that π^o is graphic.

Now let $k = \min \{r, d_n\}$. We will then show that $C = \{e_1^o, \ldots, e_k^o, e_{d_1+1}^o, \ldots, e_{2t-k}^o\}$ is the set of the largest d_n elements in $D = \{e_1^o, \ldots, e_{2t}^o\}$. By C1, $|C| = d_n$. So let $\alpha = \min C$ and $\beta = \max (D-C)$. We will then prove that $\alpha \geq \beta$.

First let $k < d_n$. Then $\alpha = \min \{e_k - 1, e_{2t-k}\}$ and $\beta = \max \{e_{k+1}-1, e_{2t-k+1}\}$. Also $k = r$ and by the definition of r, we have $e_k - e_{2t-k+1} \geq 2$ and $e_{k+1} - e_{2t-k} \leq 1$. It easily follows now that $\alpha \geq \beta$.

Next let $k = d_n$. Then $C = \{e_1^o, \ldots, e_{d_n}^o\}$ and $\alpha = e_{d_n} - 1$. Also $\beta = \max \{e_{d_n+1}-1, e_{d_1+1}\}$. Since $r \geq d_n$ we have $e_{d_n} - e_{d_1+1} \geq 2$ and it easily follows that $\alpha \geq \beta$.

Thus C is the set of the d_n largest elements in D. Since π^o is graphic, it follows from Lemma 7.8 (applied to π^o with $e_1^o, \ldots, e_{2t}^o$ rearranged in non-increasing order) that

$$\pi^* = (d_2, \ldots, d_{n-1} | e_1^*, \ldots, e_{2t}^*)$$

is graphic, where

$$e_j^* = \begin{cases} e_j^o - 1 = e_j - 2 & \text{if } 1 \leq j \leq k \\ e_j^o = e_j - 1 & \text{if } k+1 \leq j \leq d_1 \\ e_j^o - 1 = e_j - 1 & \text{if } d_1+1 \leq j \leq 2t-k \\ e_j^o = e_j & \text{if } 2t-k+1 \leq j \leq 2t. \end{cases}$$

Clearly then

$$e_j^* + e_{2t+1-j}^* = n-2 \quad \text{for} \quad 1 \leq j \leq t \qquad \ldots(7.5)$$

We next show that $\gamma \geq \delta$ where $\gamma = \min \{e_1^*, \ldots, e_t^*\}$ and $\delta = \max \{e_{t+1}^*, \ldots, e_{2t}^*\}$. Now $\gamma = \min \{e_k-2, e_t-1\}$ and $\delta = \max \{e_{t+1}-1, e_{2t-k+1}\}$. Since $e_r > e_{r+1}$, we have

$$e_k-2 \geq e_r-2 \geq e_{r+1}-1 \geq e_{t+1}-1 .$$

Also $e_{2t-r} > e_{2t-r+1}$ and so

$$e_t-1 \geq e_{2t-r}-1 \geq e_{2t-r+1} \geq e_{2t-k+1} .$$

Further $e_k - 2 \geq e_{2t-k+1}$ since $r \geq k$. It now easily follows that $\gamma \geq \delta$.

Now let θ be a permutation of $\{1,2,\ldots,t\}$ such that $e_{\theta(1)}^* \geq \ldots \geq e_{\theta(t)}^*$. Extend θ to a permutation $\emptyset$ of $\{1,\ldots,2t\}$ by defining $\emptyset(j) = 2t+1-\theta(2t+1-j)$ if

$t + 1 \leq j \leq 2t$. Let

$$\pi^{**} = (d_2, \ldots, d_{n-1} | e_1^{**}, \ldots, e_{2t}^{**})$$

where

$$e_j^{**} = e^{*}_{\emptyset(j)} \ , \ 1 \leq j \leq 2t.$$

Then clearly π^{**} is a rearrangement of π^{*} and so is graphic. Also $e_1^{**} \geq \ldots \geq e_t^{**}$ by definition of θ, $e_t^{**} \geq e_{t+1}^{**}$ since $\gamma \geq \delta$, and $e_{t+1}^{**} \geq \ldots \geq e_{2t}^{**}$ by (7.5). Thus $e_1^{**} \geq \ldots \geq e_{2t}^{**}$. Since π satisfies C1, it follows from (7.5) that π^{**} also satisfies C1. Hence by induction hypothesis, there exists a bipartitioned graph (H,Q) and an ordering $S' = (u_2, \ldots, u_{n-1} | v_{\emptyset(1)}, \ldots, v_{\emptyset(2t)})$ such that (H,Q) with S' is a realisation of π^{**} and $u_i \, v_{\emptyset(j)}$ is an edge in H whenever $2 \leq i \leq [\frac{n+1}{2}]$ and $1 \leq j \leq t$. Then (H,Q) with the ordering $(u_2, \ldots, u_{n-1} | v_1, \ldots, v_{2t})$ is a realisation of π^{*}. Also since $\{\emptyset(1), \ldots, \emptyset(t)\} = \{1, \ldots, t\}$, it follows that $u_i \, v_j$ is an edge in H whenever $2 \leq i \leq [\frac{n+1}{2}]$ and $1 \leq j \leq t$.

Now construct a bipartitioned graph (G,P) from (H,Q) by adding two new vertices u_1 and u_n and joining

$$u_1 \text{ to } v_1, \ldots, v_{d_1},$$

$$u_n \text{ to } v_1, \ldots, v_k, \ v_{d_1+1}, \ldots, v_{2t-k}.$$

Then clearly (G,P) with the ordering $S = (u_1,\ldots,u_n|$ $v_1,\ldots,v_{2t})$ is a realisation of π and $u_i\, v_j$ is an edge in G whenever $1 \leq i \leq [\frac{n+1}{2}]$ and $1 \leq j \leq t$. This completes the induction and the lemma is proved. □

LEMMA 7.10. If π is graphic, $m = n$ and $d_i = e_i$ for all i, then there exists a bipartitioned graph (G,P) and an ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_n)$ such that each non-trivial component G_h of G has an automorphism σ_h with $\sigma_h\ (A \cap V(G_h)) = B \cap V(G_h)$.

PROOF : We will actually prove, by induction on n, the following claim : there exists a bipartitioned graph (G,P) and an ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_n)$ such that

(i) (G,P) with the ordering S is a realisation of π,

(ii) $u_i\, v_j$ is an edge of G iff $u_j\, v_i$ is an edge of G,

(iii) u_i and v_i belong to the same component of G if $d_i > 0$.

It then easily follows that $\sigma_h = \prod (u_i\ v_i)$, where the product is taken over all u_i in $V(G_h)$, serves as the required automorphism of G_h provided G_h is non-trivial.

The claim holds trivially for $n = 1$ since then $\pi = (1|1)$ or $(0|0)$. So we assume the claim for $n-1$ and prove it for n, where $n \geq 2$. If $d_1 \leq 1$, the claim is trivial, so let $d_1 \geq 2$. By using Lemma 7.8 twice we see that the bipartitioned sequence

$$\pi^- = (d_2-1,\ldots,d_{d_1}-1,d_{d_1+1},\ldots,d_n|d_2-1,\ldots,d_{d_1}-1,d_{d_1+1},\ldots,d_n)$$

is graphic. By induction hypothesis, there exists a bipartitioned graph (G^-, P^-) and an ordering $S^- = (u_2,\ldots,u_n|v_2,\ldots,v_n)$ satisfying (i) - (iii) for π^-. Let (G,P) be the bipartitioned graph obtained from (G^-, P^-) by adding two new vertices u_1 and v_1 and joining u_1 to $v_1,\ldots,v_{d_1}$ and v_1 to $u_2,\ldots,u_{d_1}$. Then clearly (G,P) with the ordering S satisfies (i) - (iii) and the lemma is proved. □

Finally, in the following lemma we show that unigraphicness in bipartitioned sequences as defined by Koren [9] (See also page 100) is equivalent to an apparently weaker condition.

<u>LEMMA 7.11</u>. A graphic bipartitioned sequence $\pi = (d_1,\ldots,d_n|e_1,\ldots,e_n)$ is unigraphic iff for any two realisations (G,P) and (H,P) of π, G is isomorphic to H.

PROOF : The 'only if part' of the lemma follows directly from the definition of unigraphicness.

To prove the 'if part', it suffices to consider bipartitioned sequences π with $d_n > 0$ and $e_n > 0$. So let $d_n > 0$, $e_n > 0$ and π satisfy the condition stated in the lemma. Let (G,P) and (H,P) be any two realisations of π. We will then show that there exists an isomorphism σ from G onto H such that $\sigma(B) = B$, where A and B are the sets of P.

By hypothesis, G is isomorphic to H. Let $G_1, G_2, \ldots, G_k$ (respectively $H_1, H_2, \ldots, H_k$) be the connected components of G (respectively H). Then we may assume without loss of generality that $G_i \simeq H_i$, $1 \leq i \leq k$. Define $A_i = A \cap V(G_i)$, $B_i = B \cap V(G_i)$, $C_i = A \cap V(H_i)$ and $D_i = B \cap V(H_i)$. Since G_i and H_i are connected bipartite graphs it follows that there exists an isomorphism σ_i from G_i to H_i such that either (a) $\sigma_i(A_i) = D_i$ or (b) $\sigma_i(A_i) = C_i$. Without loss of generality let (a) hold for $i = 1, \ldots, r$ and (b) hold for $i = r+1, \ldots, k$. Let $A^* = \bigcup_{i=1}^{r} A_i$, $B^* = \bigcup_{i=1}^{r} B_i$, $C^* = \bigcup_{i=1}^{r} C_i$ and $D^* = \bigcup_{i=1}^{r} D_i$. Now by (b),

$\pi(G[A-A^* | B-B^*]) = \pi(H[A\ C^* | B-D^*])$. Also $\pi((G,P)) = \pi((H,P)) = \pi$. Hence

$$\pi^* \overset{\text{def}}{=} \pi(H[C^* | D^*]) = \pi(G[A^* | B^*])$$

$$= \pi(H[D^* | C^*]) \qquad \text{by (a).}$$

Since π and so π^* has no zero-degrees, it follows by Lemma 7.10 that there exists a realisation (H^*, P^*) of π^* such that each component H_i^* of H^* has an automorphism $\emptyset_i$ with $\emptyset_i(C^* \cap V(H_i^*)) = D^* \cap V(H_i^*)$ where we take the sets of P^* to be C^* and D^*. Now let $(\tilde{H}, P)$ be the bipartitioned graph obtained from (H,P) by replacing $H[C^* | D^*]$ by (H^*, P^*). Then $(\tilde{H}, P)$ is a realisation of π and by hypothesis H is isomorphic to $\tilde{H}$. Hence

$$H[C^* | D^*] \simeq (H^*, P^*).$$

Now since the components of $H[C^* | D^*]$ are $H_1, \ldots, H_r$, we may take without loss of generality the components of H^* to be $H_1^*, \ldots, H_r^*$ with $H_i \simeq H_i^*$, $1 \leq i \leq r$. Let θ_i be an isomorphism from H_i onto H_i^*. Define now

$$\sigma_i^* = \theta_i^{-1} \emptyset_i \theta_i .$$

Then σ_i^* is an automorphism of H_i mapping C_i onto D_i since either $\theta_i(C_i) = C^* \cap V(H_i^*)$ or $\theta_i(C_i) = D^* \cap V(H_i^*)$. Now define a permutation σ of $A \cup B$ by :

$$\sigma = \begin{cases} \sigma_i^* \sigma_i & \text{on } A^* \cup B^* \\ \sigma_i & \text{on } (A-A^*) \cup (B-B^*). \end{cases}$$

It is easy to see that σ is an isomorphism from G to H and $\sigma(B) = B$. This completes the proof of the lemma. □

7.3 PROOF OF NECESSITY

In this section we establish the necessity in Theorem 7.1. So let π be forcibly bipsc. Then π is graphic and so $\sum_{i=1}^{n} d_i = \sum_{j=1}^{n} e_j$. We now prove the necessity by showing that if π does not satisfy (1), then π satisfies one of conditions (2) - (4). So let π not satisfy (1). Then either π does not satisfy C2 or $\pi' = (d_1 + 2t - 1, \ldots, d_{2t} + 2t - 1, e_1, \ldots, e_{2t})$ is not forcibly self-complementary.

We first prove that π satisfies C1. If π does not satisfy C2, then since π is also potentially bipsc, it follows by Theorem 6.1 that π satisfies C1. Suppose now π satisfies C2 and π' is not forcibly self-complementary. Let G' be a non-self-complementary realisation of π'. Let u_i be the vertex of G' having degree $d_i + 2t - 1$ and v_j be the vertex of G' having degree e_j, $1 \leq i,j \leq 2t$. Let $A = \{u_1, \ldots, u_{2t}\}$ and $B = \{v_1, \ldots, v_{2t}\}$. Then clearly

$$\sum_{i=1}^{2t} d_{G'}(u_i) = 2t\,(2t - 1) + \sum_{j=1}^{2t} e_j .$$

Hence, it follows that $G'[A] = K$ and $G'[B] = \overline{K}$. Consider the bipartitioned graph (G,P) where G is the graph obtained from G' by removing all edges within A, and the sets of P are A and B. Clearly (G,P) with the ordering $S = (u_1, \ldots, u_{2t} | v_1, \ldots, v_{2t})$ is a realisation of π. Since π is forcibly bipsc, it follows that (G,P) is bipsc. Suppose now $\mathcal{C}_p((G,P)) = \emptyset$. Then by Theorem 5.4, G is connected, and so by Corollary 1.15, there is an element σ in $\mathcal{C}((G,P))$ such that $\sigma(A) = B$. It can be easily verified that σ also acts as an isomorphism between G' and $\bar{G}'$ and so G' is self-complementary, a contradiction. Hence $\mathcal{C}_p((G,P)) \neq \emptyset$. By Corollary 6.2, it now follows that π satisfies C1.

We now consider three cases as follows :

Case 1. $d_1 = d_n$ and $e_1 = e_n$.

Case 2. $d_1 = d_n$ and $e_1 > e_n$.

Case 3. $d_1 > d_n$.

Clearly, these three cases are exclusive and exhaustive.

We will now prove that if Case (x) holds, then π satisfies condition (x+1) of Theorem 7.1, x = 1, 2, 3.

Case 1. $d_1 = d_n$ and $e_1 = e_n$. Then $\pi = (t^{2s} | s^{2t})$. We assume without loss of generality that $s \leq t$. We will then prove that π satisfies (2) by constructing a non-bipsc realisation of π if $s \geq 3$ and $t \geq 5$ or $s = t = 4$.

First let $s \geq 3$ and $t \geq 5$. Let (G,P) be the bipartitioned graph given in Figure 7.1, where

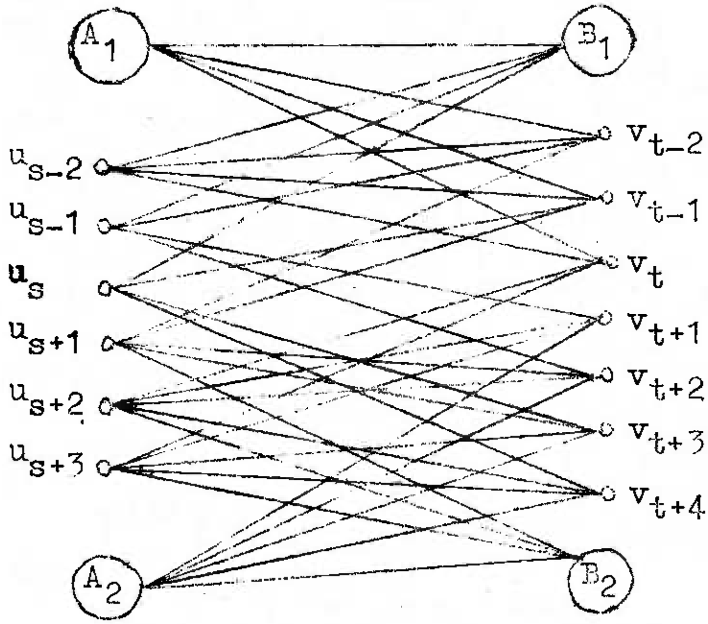


FIGURE 7.1

$A_1 = \{u_1, \ldots, u_{s-3}\}$, $A_2 = \{u_{s+4}, \ldots, u_{2s}\}$, $B_1 = \{v_1, \ldots, v_{t-3}\}$ and $B_2 = \{v_{t+5}, \ldots, v_{2t}\}$. It is easy to check that (G,P) is a realisation of π. We now show that (G,P) is not bipsc.

We first show that there is a unique $K_{s,t-3}$ in G (with the s vertices coming from A). For convenience we write $A_3 = A_1 \bigcup \{u_{s-2}, u_{s-1}, u_s\}$ and $A_4 = A - A_3$. Clearly now $G[A_3|B_1] = K_{s,t-3}$. To show the uniqueness suppose $G[C|D] = K_{s,t-3}$ where $C \subseteq A$, $D \subseteq B$. Then note that $N_G(y) = C$ for all y in D. If now B_2 intersects D then $\{u_{s+1}, u_{s+2}, u_{s+3}\} \subseteq C$, but the number of vertices joined to all of $u_{s+1}, u_{s+2}, u_{s+3}$ is only $t-4$, a contradiction. So $B_2 \bigcap D = \emptyset$. If now $B_1 \bigcap D = \emptyset$, then $D \subseteq \{v_{t-2}, v_{t-1}, \ldots, v_{t+4}\}$ and it can be easily checked that $N(x) \neq N(y)$ for any two distinct vertices x, y in D (note that they exist since $t \geq 5$), a contradiction. So B_1 intersects D, hence $C = A_3$ and $D = B_1$. This proves that G has a unique $K_{s,t-3}$.

Suppose now (G,P) is bipsc. If $\mathcal{C}_p((G,P))$ contains an element σ then since $\overline{G}[A_4|B_1] = K_{s,t-3}$ it follows that $\sigma(A_3) = A_4$. Now the only vertices y in B such that $|N_G(y) \bigcap A_3| = s-1$ are v_{t-2} and v_{t-1}. Also the only vertices y in B such that $|N_{\overline{G}(P)}(y) \bigcap \sigma(A_3)| = s-1$ are v_{t-2} and v_{t-1}. Hence $\{v_{t-2}, v_{t-1}\}$ is invariant under σ. Now u_{s-1} is adjacent in G to only one of v_{t-2}, v_{t-1} and hence $\sigma(u_{s-1})$ is also adjacent in $\overline{G}(P)$ to exactly one of v_{t-2}, v_{t-1}. This is a contradiction since $\sigma(u_{s-1}) \varepsilon A_4$. Thus $\mathcal{C}_p((G,P)) = \emptyset$. Since G is connected, it follows by

Corollary 1.15 that $\mathcal{C}_n((G,F))$ contains an element σ such that $\sigma(A) = B$ and $\sigma(B) = A$. Clearly now $s = t$, G has a unique $K_{t,t-3}$ (with the t vertices coming from A), but $K_{t,t-3}$ occurs twice in $\overline{G}(F)$ (with the t vertices coming from $\sigma(A)$), viz. $\overline{G}[A_1|v_{t+1},\ldots,v_{2t}]$ and $\overline{G}[A_2|v_1,\ldots,v_t]$. This contradiction proves that (G,F) is not bipsc.

Next let $s = t = 4$. Then consider the bipartitioned graph (G,F) given in Figure 7.2. Clearly (G,F) is a

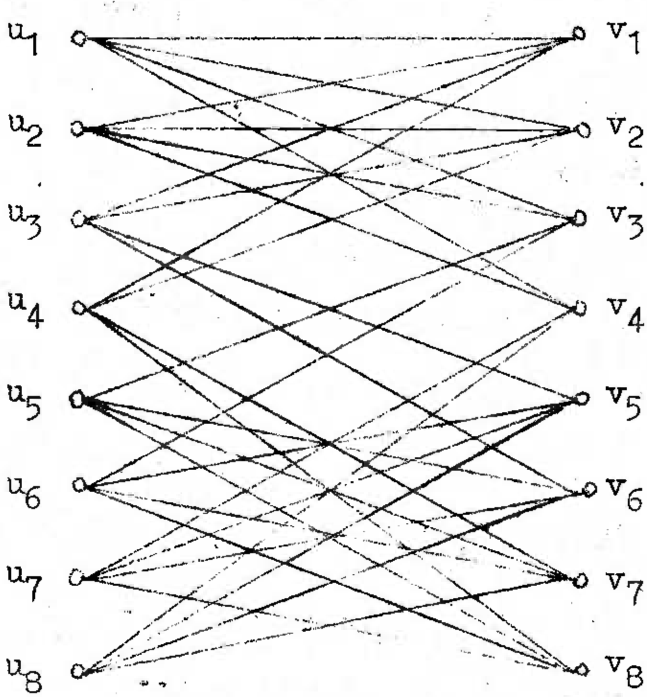


FIGURE 7.2

realisation of $\pi = (4^8|4^8)$. It is easy to see that u_1, u_2 are the only two vertices in A with the same neighbourhood in G, hence $G[u_1, u_2 | v_1, v_2, v_3, v_4]$ is the unique $K_{2,4}$ in G (with the two vertices coming from A). Similarly $G[u_1, u_2, u_3, u_4 | v_1, v_2]$ is the unique $K_{4,2}$ in G (with the four vertices coming from A). Also the union of these two subgraphs has only 8 vertices. Now $\overline{G}[u_1, u_2 | v_5, v_6, v_7, v_8] = K_{2,4}$, $\overline{G}[u_5, u_6, u_7, u_8 | v_1, v_2] = K_{4,2}$ and the union of these two subgraphs of $\overline{G}(P)$ has 12 vertices. Hence (G,P) is not bipsc.

This proves that in Case 1, π satisfies (2).

<u>Case 2</u>. $d_1 = d_n$ and $e_1 > e_n$. In this case we will prove that π satisfies (3).

So let k be the number of e_j's in π which are equal to zero. We first prove the following :

1°. Either $t - k \leq 2$ or $e_{k+1} = e_t$ or $e_{k+2} = \frac{n}{2}$.
Suppose not, then we obtain a contradiction by constructing a non-bipsc realisation (G,P) of π.

Let $e_{2t-k} = x$ and $e_{t+1} = y$. Then $0 < x < y \leq \frac{n}{2}$ and so $x + y < n$. Now let $A_1 = \{u_1, \ldots, u_x\}$, $A_2 = \{u_{x+1}, \ldots, u_{n-y}\}$, $A_3 = \{u_{n-y+1}, \ldots, u_{n-x}\}$,

$A_4 = \{u_{n-x+1}, \ldots, u_n\}$ and $A = \bigcup_{i=1}^{4} A_i$. Also let

$B_1 = \{v_1, \ldots, v_k\}$, $B_2 = \{v_{k+2}, \ldots, v_{t-1}\}$, $B_3 = \{v_{t+2}, \ldots, v_{2t-k-1}\}$

$B_4 = \{v_{2t-k+1}, \ldots, v_{2t}\}$ and $B = \{v_1, \ldots, v_{2t}\}$. Note that $B_2 \neq \emptyset$ since $t-k \geq 3$. Now take (G,F) to be the graph given in Figure 7.3. Here if $v_j \in B_2$ then

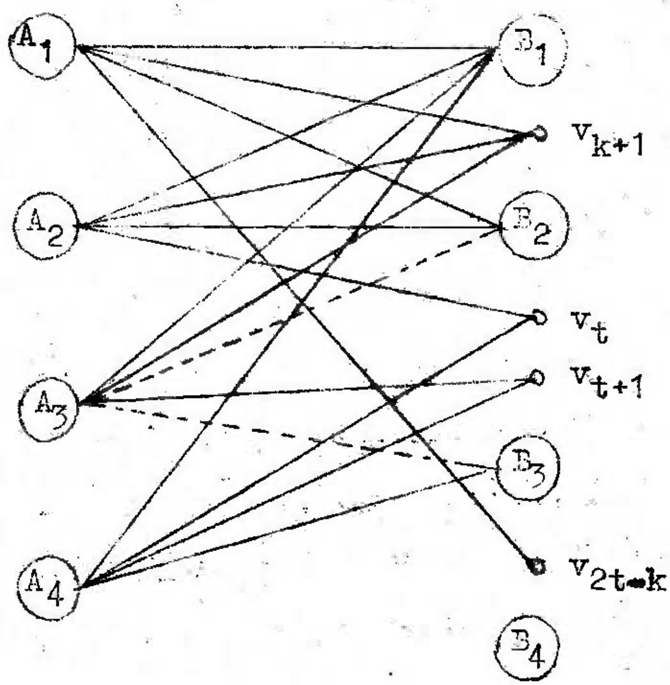


FIGURE 7.3

v_j is joined to the first $e_j-(n-y)$ vertices of A_3 and the corresponding vertex v_{2t+1-j} of B_3 is joined to the remaining vertices of A_3.

Clearly (G,P) is a realisation of π. Also π does not satisfy C2 since $d_1 = d_n$ and $e_1 > e_n$. So if (G,P) is bipsc, then by Lemma 7.2, $\mathfrak{S}_p((G,P)) \neq \emptyset$. But there is no isomorphism σ from G to $\overline{G}(P)$ such that $\sigma(B) = B$, since G has a vertex v^* (namely v_{k+1}) satisfying

(i) v^* has degree $n - x$,

(ii) if v is a vertex in B with degree $n - x$ then $N(v) \subseteqq N(v^*)$,

(iii) there is a vertex $v \neq v^*$ in B (namely v_{k+2}) such that $d(v) > \frac{n}{2}$ and $N(v) \subseteqq N(v^*)$,

but $\overline{G}(P)$ has no such vertex in B. Thus (G,P) is a non-bipsc realisation of π and 1^o is proved.

Now to prove that π satisfies (3), let $t - k \geq 3$ and $\pi^o = ((t-k))^n | e_{k+1}, \ldots, e_{2t-k})$. If (G,P) with the ordering $S = (u_1, \ldots, u_n | v_1, \ldots, v_{2t})$ is any realisation of π then we note that $\pi^o = \pi(G[A | v_{k+1}, \ldots, v_{2t-k}])$. Hence by Lemma 7.3, π^o is forcibly bipsc. We now consider several cases.

<u>Case 2(a)</u>. $e_{k+1} = e_t = \frac{n}{2}$. Then $\pi^o = ((t-k)^{2s} | s^{2(t-k)})$. Now since π^o is forcibly bipsc and $t - k \geq 3$, we have by Case 1 that π^o is one of $\pi_1, \pi_2, \ldots, \pi_5$.

<u>Case 2(b)</u>. $e_{k+1} = e_t > \frac{n}{2}$. Then $e_{k+1} = n-x$ for some x, $1 \leq x < \frac{n}{2}$. We now prove that $\pi^o = \pi_6$ by constructing a non-bipsc realisation (G,P) of π if $x > 1$.

Thus let $x > 1$. Let $A_1 = \{u_1, \ldots, u_{m-2x}\}$, $A_2 = \{u_{m-2x+1}, \ldots, u_{m-x-1}\}$, $A_3 = \{u_{m-x+1}, \ldots, u_{m-1}\}$ and $A = \{u_1, \ldots, u_m\}$. Also let $B_1 = \{v_1, \ldots, v_k\}$, $B_2 = \{v_{k+1}, \ldots, v_{t-1}\}$, $B_3 = \{v_{t+3}, \ldots, v_{2t-k}\}$, $B_4 = \{v_{2t-k+1}, \ldots, v_{2t}\}$ and $B = \{v_1, \ldots, v_{2t}\}$. Then take (G,P) to be the graph given in Figure 7.4.

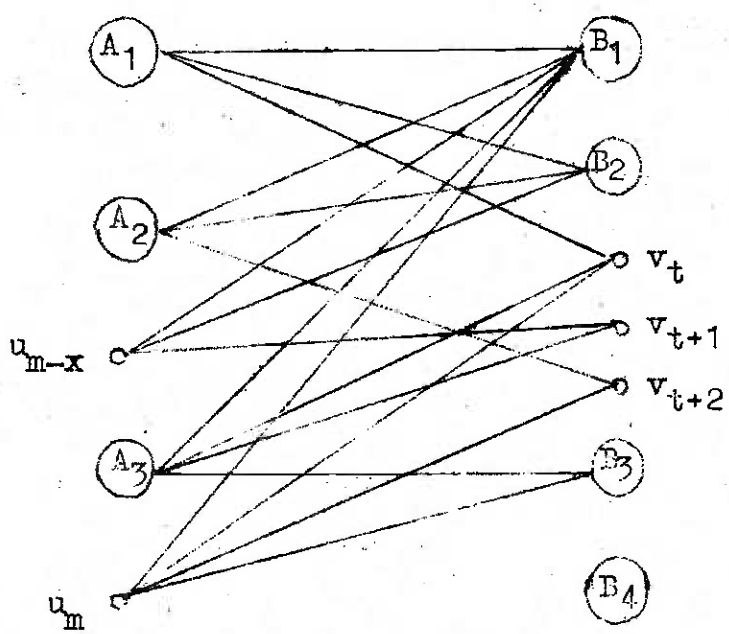


FIGURE 7.4

Clearly (G,P) is a realisation of π. If (G,P) is bipsc, then since π does not satisfy C2, it follows by Lemma 7.2 that $\mathcal{C}_p((G,P))$ has an element σ. Now $\sigma(\{v_{t+1},\ldots,v_{2t-k}\}) = \{v_{k+1},\ldots,v_t\}$ and so $G[\bigcup_{j=t+1}^{2t-k} N_G(v_j)|v_{t+1},\ldots,v_{2t-k}]$ and $\bar{G}[\bigcup_{j=k+1}^{t} N_{\bar{G}(P)}(v_j)|v_{k+1},\ldots,v_t]$ are isomorphic, but the first is connected and the second is not. This contradiction proves that (G,P) is a non-bipsc realisation of π. This is a contradiction since π is forcibly bipsc. Hence it follows that $x = 1$ and so $\pi^o = \pi_6$.

<u>Case 2(c)</u>. $e_{k+1} > e_t$. Then by 1^o, we have $e_{k+2} = \frac{n}{2}$ and so $\pi^o = ((t-k)^{2s}|2s-\alpha,\ s^{2(t-k-1)},\ \alpha)$ for some α, $1 \leq \alpha \leq s-1$.

We will now prove that π^o is π_7 or π_8. For this define

$$\pi^* = ((t-k)^{\alpha},\ (t-k-1)^{2s-2\alpha},\ (t-k-2)^{\alpha}|s^{2(t-k-1)}).$$

Then by Lemma 7.4, π^* is graphic. Let (H,Q) with the ordering $S^* = (u_1,\ldots,u_{2s}|v_2,\ldots,v_{2(t-k)-1})$ be a realisation of π^*. Get a new bipartitioned graph (G,P) from (H,Q) by adding two new vertices, say v_1 and $v_{2(t-k)}$, and joining

v_1 to $u_{\alpha+1},\ldots,u_{2s}$ and joining $v_{2(t-k)}$ to $u_{2s-\alpha+1},\ldots,u_{2s}$. Clearly now (G,P) with the ordering $(u_1,\ldots,u_{2s}|v_1,\ldots,v_{2(t-k)})$ is a realisation of π^0. Now π^0 is forcibly bipsc and does not satisfy C2, $2s-\alpha > s$, and $\pi^* = \pi(G[u_1,\ldots,u_{2s}|v_2,\ldots,v_{2(t-k)-1}]$, hence by Lemma 7.3 π^* is forcibly bipsc. Hence

$$(s^{2(t-k-1)}|(t-k)^{\alpha},\ (t-k-1)^{2s-2\alpha},\ (t-k-2)^{\alpha})$$

is also forcibly bipsc, and so by 1^0 applied to this sequence we have : either $s \leq 2$ or $\alpha = 1$ (note that the number of terms on the right is $2s$ and none of these is zero). Now $s \leq 2$ implies $s = 2$ and $\alpha = 1$, hence we always have $\alpha = 1$. Thus $\pi^* = (t-k,\ (t-k-1)^{2s-2},\ t-k-2|s^{2(t-k-1)})$.

If now $s = 2$, then $\pi^0 = \pi_7$. So let $s \geq 3$. Then define

$$\pi^{**} = ((t-k-1)^{2s-2}|s^{t-k-2},\ (s-1)^2,\ (s-2)^{t-k-2}).$$

Then it follows that π^{**} is forcibly bipsc by arguments similar to those used above for π^*. Now the number of terms on the right of π^{**} is $2(t-k-1)$ and none of these is equal to zero. Since $t-k \geq 3$, it follows from 1^0 applied to π^{**} that $t - k = 3$ and $\pi^0 = \pi_8$.

This proves that in Case 2, π satisfies (3).

Case 3. $d_1 > d_n$. Then by assumption (I), $e_1 > e_n$. We now prove that π satisfies (4). Let p be the number of d_i's greater than $\frac{n}{2}$ and q the number of e_j's greater than $\frac{n}{2}$. Since π satisfies C1, it follows that $0 < p \leq \frac{n}{2}$ and $0 < q \leq \frac{n}{2}$. Also let h be the number of e_j's which are not less than $n-p$. Then we will prove that π satisfies the conditions (a) - (e) of (4).

If n is odd then by C1, $e_{\frac{n+1}{2}} = \frac{n}{2}$ and so by assumption (II), some $d_i = \frac{n}{2}$, a contradiction. This proves (a).

Next we prove (b) and (c) together. This is done in several steps as follows :

Let (G,P) with the ordering $S = (u_1,\ldots,u_n|v_1,\ldots,v_{2t})$ be any realisation of π. Since π is forcibly bipsc, (G,P) is bipsc. Define

$$A_1 = \{u_i|1 \leq i \leq p\}, \quad A_2 = \{u_i|p + 1 \leq i \leq [\tfrac{n+1}{2}]\},$$

$$A_3 = \{u_i|\ [\tfrac{n+1}{2}] + 1 \leq i \leq n-p\},$$

$$A_4 = \{u_i|n - p + 1 \leq i \leq n\},$$

$$B_1 = \{v_j|1 \leq j \leq q\}, \quad B_2 = \{v_j|q + 1 \leq j \leq t\},$$

$$B_3 = \{v_j|t + 1 \leq j \leq 2t - q\} \text{ and}$$

$$B_4 = \{v_j|2t - q + 1 \leq j \leq 2t\}.$$

Also let $B_{11} = \{v_j \mid 1 \leq j \leq h\}$ and $B_{12} = B_1 - B_{11}$, $B_{41} = \{v_j \mid 2t - q + 1 \leq j \leq 2t - h\}$ and $B_{42} = B_4 - B_{41}$. We note that if $B_2 \neq \emptyset$ then by assumption (ii), $|A_2| = |A_3| > 0$ and so $p < \frac{n}{2}$. We will now show that $B_{11} \subseteq B_1$. Let $v \in B_{11}$. If $B_2 \neq \emptyset$, then $p < \frac{n}{2}$ and $d(v) \geq n - p > \frac{n}{2}$. If $B_2 = \emptyset$, then $d(v) \neq \frac{n}{2}$, but $d(v) \geq n - p \geq \frac{n}{2}$. Thus, in either case $d(v) > \frac{n}{2}$ and $v \in B_1$. Hence $B_{11} \subseteq B_1$. Similarly it can be proved that $B_{42} \subseteq B_4$.

We now show that there exists an element σ of $\mathcal{G}((G,P))$ such that $\sigma(A_1 \cup B_1) = A_4 \cup B_4$ and $\sigma(A_2 \cup A_3 \cup B_2 \cup B_3) = A_2 \cup A_3 \cup B_2 \cup B_3$. If $n = n$ then any element of $\mathcal{G}((G,P))$ will do. If $n \neq n$ then π does not satisfy C2, hence by Lemma 7.2, $\mathcal{G}_p((G,P)) \neq \emptyset$ and we take σ to be any element of $\mathcal{G}_p((G,P))$.

From the result proved in the preceding paragraph it follows that $G[A_1|B_1] \simeq \overline{G}[A_4|B_4]$ and $G[A_2 \cup A_3| B_2 \cup B_3] \simeq \overline{G}[A_2 \cup A_3|B_2 \cup B_3]$. Now by Lemma 7.9, we can choose (G,P) such that $G[A_1 \cup A_2|B_1 \cup B_2] = K$. Throughout the rest of Case 3, we let (G,P) be such a graph. Since every vertex in A_2 (resp. B_2) has degree t (resp. $\frac{n}{2}$), it follows that $G[A_2|B_3 \cup B_4] = \overline{K}$ and $G[A_3 \cup A_4|B_2] = \overline{K}$.

Choose now a σ in $\mathscr{C}((G,P))$ with the properties given above. Then it follows that $G[A_4|B_4] = \overline{K}$ since $G[A_1|B_1] = K$. Also if $B_2 \neq \emptyset$, then since $G[A_2 \cup A_3|B_2 \cup B_3] \simeq \overline{G}[A_2 \cup A_3|B_2 \cup B_3]$ and the former has at most $2(s-p)(t-q)$ edges whereas the latter has at least $2(s-p)(t-q)$ edges, it follows that $G[A_3|B_3] = K$ and $\sigma(A_2 \cup B_2)$ is either $A_2 \cup B_3$ or $A_3 \cup B_2$.

We now prove that $G[A_1|B_3] = K$. We may take $B_3 \neq \emptyset$ (hence B_2, A_2 and A_3 non-empty) since otherwise the claim is vacuously true. If $\sigma(A_2 \cup B_2) = A_2 \cup B_3$, then $\overline{G}[A_2 \cup A_4|B_3 \cup B_4] \simeq G[A_1 \cup A_2|B_1 \cup B_2] = K$, so $G[A_4|B_3] = \overline{K}$. Since every vertex of B_2 has degree s it follows that $G[A_1|B_3] = K$. So let $\sigma(A_2 \cup B_2) = A_3 \cup B_2$. Then we can prove as above that $G[A_3|B_1] = K$. Thus v_q is joined to all vertices in $A - A_4$, hence $c_q \geq 2s - p$, so by assumption (III), $d_p \geq 2t - q$. If possible, let $u \in A_1$ and $v \in B_3$ be non-adjacent. Since $d(u) \geq 2t - q$ it follows that there exists $v' \in B_4$ adjacent to u and since $d(v) = s$ it follows that there exists $u' \in A_4$ adjacent to v. Now if H is the graph obtained from G by an interchange along (u, v', u', v, u), then (H,P) with the ordering S is a realisation of π , $H[A_1|B_1] = K$ and $H[A_4|B_4] \neq \overline{K}$, a contradiction. This proves that $G[A_1|B_3] = K$.

Since the degree of every vertex in B_3 is s, it follows that $G[A_4|B_3] = \overline{K}$.

Next we show that $G[A_3|B_{11}] = K$. If possible, let $u \in A_3$ and $v \in B_{11}$ be non-adjacent. Since $d(u) = t$, it follows that there exists $v' \in B_4$ adjacent to u and since $d(v) \geq n-p$, it follows that there exists $u' \in A_4$ adjacent to v. Now by an interchange along (u, v', u', v, u) from G, we arrive at a contradiction. Hence $G[A_3|B_{11}] = K$.

Next we show that $G[A_3|B_{42}] = \overline{K}$. If possible, let $u \in A_3$ and $v \in B_{42}$ be adjacent. Since $d(u) = t$, it follows that there exists $v' \in B_{12}$ non-adjacent to u and since $d(v) \leq p$, it follows that there exists $u' \in A_1$ non-adjacent to v. Now by an interchange along (u, v, u', v', u) from G we arrive at a contradiction. Hence $G[A_3|B_{42}] = \overline{K}$.

Next we show that $G[A_1|B_{41}] = K$. If possible, let $u \in A_1$ and $v \in B_{41}$ be non-adjacent. Since $d(v) > p$, it follows that there exists $u' \in A_3$ adjacent to v. Since $d(u') = t$, it follows that there exists $v' \in B_{12}$ non-adjacent to u'. Now by an interchange along (u, v', u', v, u) from G, we arrive at a contradiction. Hence $G[A_1|B_{41}] = K$.

Next we show that $G[A_4|B_{12}] = \bar{K}$. If possible, let $u \in A_4$ and $v \in B_{12}$ be adjacent. Since $d(v) < n-p$, it follows that there exists $u' \in A_3$ non-adjacent to v. Since $d(u') = t$, it follows that there exists $v' \in B_{41}$ adjacent to u'. Now by an interchange along (u, v, u', v', u) from G, we arrive at a contradiction. Hence $G[A_4|B_{12}] = \bar{K}$.

Summing up, we obtain that $G[A_1|B-B_{42}] = K$, $G[A_4|B-B_{11}] = \bar{K}$, $G[A-A_4|B_{11}] = K$ and $G[A-A_1|B_{42}] = \bar{K}$ From this we immediately have

$$\sum_{i=1}^{p} d_i = (2t-h)\,p + \sum_{j=2t-h+1}^{2t} e_j \,,$$

$$\sum_{j=1}^{h} e_j = (n-p)h + \sum_{i=n-p+1}^{n} d_i \,.$$

This proves (b) and (c).

We now prove that (d) holds. If $p = \frac{n}{2}$, then we are done. So let $p < \frac{n}{2}$. If $t - h \leq 2$ then (d) holds. So let $t - h \geq 3$ and

$$\pi^{+} = ((t-h)^{n-2p}|e_{h+1} - p, \ldots, e_{2t-h} - p)\,.$$

Clearly $\pi^{+} = \pi(G[A_2 \cup A_3|B - B_{11} - B_{42}])$. We now prove that π^{+} is forcibly bipsc. If π does not satisfy C2, then this claim follows by Lemma 7.3. So let π satisfy C2. Then $m = n$ and $d_i = e_i$ for all i, so $p = q$. Since $G[A_1|B - B_{42}] = K$, it follows that $d_p \geq n - q = n - p$, hence $e_q \geq n - p$ and $B_{11} = B_1$. Thus $\pi^+ = \pi(G[A_2 \cup A_3| B_2 \cup B_3])$. Let (G^+, Q) with the ordering $S^+ = (u_{p+1},\ldots,u_{m-p}|v_{q+1},\ldots,v_{2t-q})$ be any realisation of π^+. Let (H,P) be the graph obtained from (G,P) by replacing $G[A_2 \cup A_3|B_2 \cup B_3]$ by (G^+, Q). Then (H,P) with the ordering S is a realisation of π and hence (H,P) is bipsc. Hence, as shown on page 153 , there is a $\sigma \in \mathcal{C}((H,P))$ such that $\sigma(A_2 \cup A_3 \cup B_2 \cup B_3) = A_2 \cup A_3 \cup B_2 \cup B_3$. It now follows that (G^+, Q) is bipsc with the restriction of σ to $A_2 \cup A_3 \cup B_2 \cup B_3$ as a bipcp. Thus π^+ is forcibly bipsc. Since $t - h \geq 3$, it now follows from Cases 1 and 2 (applied to π^+) that π^+ is one of $\pi_1 - \pi_8$ with t replaced by $t-h$ and k replaced by zero. This proves that (d) holds.

Finally, to prove that (e) holds, let

$$\pi^* = (d_1-n+h,\ldots,d_p-n+h|e_{n-h+1},\ldots,e_n).$$

Note that $\pi^* = \pi(G[A_1|B_{42}]) = \pi(\bar{G}[A_4|B_{11}])$. Since π does not satisfy condition (1) of Theorem 7.1, it follows that either π does not satisfy C2 or, π satisfies C2 and $\pi' = (d_1 + 2t - 1, \ldots, d_{2t} + 2t - 1, e_1, \ldots, e_{2t})$ is not forcibly self-complementary. We accordingly consider two cases.

<u>Case 3 (a)</u>. π does not satisfy C2. Let (G_1, P_1) and (G_2, P_2) be two realisations of π^*. Let (H,P) be the graph obtained from (G,P) by replacing $G[A_1|B_{42}]$ by (G_1, P_1) and $G[A_4|B_{11}]$ by $(\bar{G}_2(P_2), P_2)$. Then (H,P) is a realisation of π and so is bipsc. But π does not satisfy C2, so $\mathcal{C}_P((H,P))$ contains an element σ. Clearly $\sigma(A_1) = A_4$ and $\sigma(B_{42}) = B_{11}$. Hence

$$G_1 = H[A_1|B_{42}] \simeq \bar{H}[A_4|B_{11}] = G_2 .$$

Thus any two realisations of π^* are isomorphic, hence by Lemma 7.11, it follows that π^* is unigraphic. Thus (e) holds in this case.

<u>Case 3(b)</u>. π satisfies C2 and $\pi' = (d_1 + 2t - 1, \ldots, d_{2t} + 2t - 1, e_1, \ldots, e_{2t})$ is not forcibly self-complementary. We now prove that (e) holds by assuming that π^* is not unigraphic and obtaining a contradiction.

We first show that if $(G^*; P^*)$ is a realisation of π^* then there is a $\sigma^* \in \mathcal{C}((G^*, P^*))$ such that $\sigma^*(A^*) = D^*$

where A^* and B^* are the sets of P^*. Since π^* is not unigraphic, by Lemma 7.11, there exists another realisation (H^*, P^*) of π^* such that $G^* \not\simeq H^*$. Now let (H,P) be the graph obtained from (G,P) by replacing $G[A_1|B_{42}]$ by (G^*, P^*) and $G[A_4|B_{11}]$ by $(\bar{H}^*(P^*), P^*)$. Clearly (H,P) is a realisation of π and so is bipsc. If now $\mathfrak{G}_p((H,P))$ contains an element σ then $\sigma(A_1) = A_4$ and $\sigma(B_{42}) = B_{11}$, hence $G^* \simeq H^*$, a contradiction. Thus $\mathfrak{G}_p((H,P)) = \emptyset$. Hence by Theorem 5.4 and Corollary 1.15, it follows that $\mathfrak{G}((H,P))$ contains an element σ such that $\sigma(A) = B$. Now since π satisfies C2 it follows that $m = n$, $d_i = e_i$ for all i, and so as in page 157 we have $B_{11} = B_1$ and $B_{42} = B_4$. Since $m = n$, it also follows that $\sigma(A_1) = B_4$ and $\sigma(B_4) = A_1$. Now the restriction of σ to $A_1 \cup B_4$ serves as the required σ^*.

Let now G' be any realisation of π'. Let u_i (resp. v_i) be the vertex with degree $d_i + 2t - 1$ (resp. e_i), $i = 1,\dots,2t$. Also define $A_1,\dots,A_4$, $B_1,\dots,B_4$ as before and let $A = \bigcup_{i=1}^{4} A_i$, $B = \bigcup_{j=1}^{4} B_j$. If now $G'[A] \neq K$ or $G'[B] \neq \bar{K}$, then

$$\sum_{i=1}^{2t} (d_i + 2t - 1) < 2t\,(2t - 1) + \sum_{i=1}^{2t} e_i ,$$

a contradiction since $d_i = e_i$ for all i. Thus $G'[A] = K$ and $G'[B] = \bar{K}$. Let $P = \{A, B\}$ and (G_1, P) the bipartitioned graph obtained from G' by deleting the edges in A. Then (G_1, P) is a realisation of π, and it follows from (b) and (c) that

$$G_1[A_1 | B - B_4] = K \ , \ G_1[A - A_1 | B_4] = \bar{K} \ ,$$

$$G_1[A - A_4 | B_1] = K \ , \ G_1[A_4 | B - B_1] = \bar{K} \ .$$

Hence $\pi(G_1[A_1|B_4]) = \pi^* = \pi(\bar{G}_1[A_4|B_1])$. So by the result proved in the preceding paragraph, there exist $\sigma_1 \in \mathcal{C}(G_1[A_1|B_4])$ such that $\sigma_1(A_1) = B_4$ and $\sigma_2 \in \mathcal{C}(\bar{G}_1[A_4|B_1])$ such that $\sigma_2(A_4) = B_1$. Now consider the permutation σ of $A \cup B$ defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in A_1 \cup B_4 \ , \\ \sigma_2^{-1}(x) & \text{if } x \in A_4 \cup B_1 \ , \\ v_{2t+1-i} & \text{if } x = u_i \in A_2 \cup A_3 \ , \\ u_j & \text{if } x = v_j \in B_2 \cup B_3 \end{cases}$$

It is easy to see that σ is an isomorphism between G' and $\bar{G'}$. Hence G' is self-complementary and π' is forcibly self-complementary. This contradiction proves that (e) holds in this case.

Thus in Case 3, π satisfies (4) and the proof of necessity is complete.

7.4 PROOF OF SUFFICIENCY

In this section we establish the sufficiency in Theorem 7.1. So let $\pi = (d_1,\ldots,d_n|e_1,\ldots,e_n)$ be a bipartitioned sequence satisfying $\sum_{i=1}^{n} d_i = \sum_{j=1}^{n} e_j$ and at least one of conditions (1) – (4). We will prove that π is forcibly bipsc. We divide this proof into four cases.

Case 1. π satisfies (1). As π satifies C2, we have $n = n = 2t$. We first prove that π is graphic and $d_{2t} > 0$. Since π' is forcibly self-complementary, it is also graphic. Let G' be a realisation of π'. Let $A = \{u_1,\ldots,u_{2t}\}$ and $B = \{v_1,\ldots,v_{2t}\}$, where u_i has degree $d_i + 2t - 1$ and v_i has degree e_i in G', $1 \leq i \leq 2t$. Then clearly

$$\sum_{i=1}^{2t} d_{G'}(u_i) = 2t\,(2t-1) + \sum_{i=1}^{2t} e_i \,.$$

Hence it follows that $G'[A] = K$ and $G'[B] = \overline{K}$. Consider the bipartitioned graph (G,P) where G is the graph obtained from G' by deleting all edges within A and the sets of P are A and B. Then (G,P) with the ordering $S = (u_1,\ldots,u_{2t}|v_1,\ldots,v_{2t})$ is a realisation of π and so π is graphic. If now $d_{2t} = 0$, then by C2, $e_1 = 2t$ and so π is not graphic, a contradiction. Hence $d_{2t} > 0$.

We next prove that any realisation of π is bipsc. Let (G,P) with the ordering $S = (u_1,\ldots,u_{2t}|v_1,\ldots,v_{2t})$ be any realisation of π. Let G' be the graph obtained from G by joining every pair of distinct vertices in A by an edge. Then G' is a realisation of π'. Since π' is forcibly self-complementary, it follows that G' is self-complementary. Let σ be a complementing permutation of G'. Now if $\sigma(u_i) = u_j$, then since $d_{2t} > 0$, it follows that

$$4t \leq d_i + d_j + 4t - 2 = d_{G'}(u_i) + d_{G'}(u_j) = 4t - 1,$$

a contradiction. Hence $\sigma(A) = B$ and $\sigma(B) = A$. It now follows that σ is also an isomorphism between G and $\overline{G}(P)$. Thus (G,P) is bipsc and π is forcibly bipsc.

Case 2. π satisfies (2). By **Lemma** 7.4 , it follows that π is graphic. Without loss of generality we assume that $s \leq t$. It then follows that π is one of $(t^2|1^{2t})$, $(t^4|2^{2t})$, $(3^6|3^6)$, $(4^6|3^8)$.

If $\pi = (t^2|1^{2t})$ and (G,P) with the ordering $S = (u_1,u_2|v_1,\ldots,v_{2t})$ is a realisation of π, then without loss of generality one can take

$$E(G) = \left\{ u_1 v_j \ , \ u_2 v_{t+j} \middle| 1 \le j \le t \right\}.$$

Clearly (G,P) is bipsc and $\sigma = (u_1 u_2) \prod_{j=1}^{2t} (v_j) \in \mathcal{C}_p((G,P))$.

If $\pi = (t^4 | 2^{2t})$, then it follows by Lemma 7.6 that every realisation (G,P) of π is bipsc with $\mathcal{C}_p((G,P)) \neq \emptyset$.

If $\pi = (3^6 | 3^6)$, then one can verify that π has exactly six non-isomorphic realisations (G,P), and each of these is bipsc with $\mathcal{C}_p((G,P)) \neq \emptyset$. We give these realisations and a complementing permutation for each of these in Appendix I.

If $\pi = (4^6 | 3^8)$, then one can verify that π has exactly twenty non-isomorphic realisations (G,P), and each of these is bipsc with $\mathcal{C}_p((G,P)) \neq \emptyset$. We give these realisations and a complementing permutation for each of these in Appendix II.

This proves that π is forcibly bipsc in this case.

Case 3. π satisfies (3). Then $\pi = (t^m | m^k, e_{k+1}, \ldots, e_{2t-k}, 0^k)$. By Lemma 7.4 (applied to π with $d_1, \ldots, d_m$ and $e_1, \ldots, e_{2t}$ interchanged) we get that π is graphic. Let (G,P) with the ordering $S = (u_1, \ldots, u_m | v_1, \ldots, v_{2t})$ be any realisation of π. Then clearly $u_i v_j$ is an edge whenever

$1 \leq i \leq m$ and $1 \leq j \leq k$. If now $t - k = 0$ then

$$\sigma = \prod_{i=1}^{m} (u_i) \prod_{j=1}^{t} (v_j \ v_{2t+1-j}) \ \varepsilon \ \mathcal{C}_p((G,P))$$ and π is forcibly bipsc. So let $t - k \geq 1$. Then let $(G^o, P^o) = G[u_1, \ldots, u_m | v_{k+1}, \ldots, v_{2t-k}]$. If we now prove that (G^o, P^o) is bipsc with $\sigma^o \ \varepsilon \ \mathcal{C}_p((G^o, P^o))$, then $\sigma = \sigma_o \prod_{j=1}^{k} (v_j \ v_{2t+1-j}) \ \varepsilon \ \mathcal{C}_p((G,P))$ and it follows that π is forcibly bipsc. Thus it remains to prove that $\mathcal{C}_p((G^o, P^o)) \neq \emptyset$.

First let $t - k = 1$. Then without loss of generality we may take $E(G^o) = \{u_i v_t | 1 \leq i \leq e_t\} \sqcup \{u_i \ v_{t+1} | e_t + 1 \leq i \leq m\}$. Clearly then $\sigma^o = \prod_{i=1}^{m} (u_i) \ (v_t \ v_{t+1}) \ \varepsilon \ \mathcal{C}_p((G^o, P^o))$.

Next let $t - k = 2$. Then $\pi((G^o, P^o)) = (2^m | e_{t-1}, e_t, m - e_t, m - e_{t-1})$. Hence by Lemma 7.5, $\mathcal{C}_p((G^o, P^o)) \neq \emptyset$.

Finally let $t - k \geq 3$. Then $\pi((G^o, P^o)) = \pi^o$ and by (3), π^o is one of $\pi_1 - \pi_8$. If π^o is one of $\pi_1 - \pi_5$, then as proved in Case 2, $\mathcal{C}_p((G^o, P^o)) \neq \emptyset$.

Let now $\pi^o = \pi_6 = ((t-k)^m | (m-1)^{t-k}, 1^{t-k})$. Let $B_1 = \{v_{k+1}, \ldots, v_t\}$ and $B_2 = \{v_{t+1}, \ldots, v_{2t-k}\}$. For

$1 \leq i \leq m$, let B_{1i} be the set of all vertices of B_1 not adjacent to u_i and B_{2i} the set of all vertices of B_2 adjacent to u_i. Then since $|B_1| = |B_2| = t - k$ and $d_{G^o}(u_i) = t - k$, it follows that $|B_{1i}| = |B_{2i}|$. Also since the degree of every vertex of B_1 is $m - 1$ in G^o, it follows that B_{1i} and B_{1h} are disjoint if $i \neq h$. Similarly, B_{2i} and B_{2h} are disjoint if $i \neq h$. Further $B_1 = \bigcup_{i=1}^{m} B_{1i}$ and $B_2 = \bigcup_{i=1}^{m} B_{2i}$. Now if σ^o is any permutation such that $\sigma^o(u_i) = u_i$, $\sigma^o(B_{1i}) = B_{2i}$ and $\sigma^o(B_{2i}) = B_{1i}$ for $i = 1,\ldots,m$, then $\sigma^o \in \mathcal{G}_p((G^o, P^o))$.

Next let $\pi^o = \pi_7 = ((t-k)^4|3,2^{2(t-k-1)}, 1)$. Then by Lemma 7.7, $\mathcal{G}_p((G^o, P^o)) \neq \emptyset$.

Finally let $\pi^o = \pi_8 = (3^{2s}|2s-1, s^4, 1)$. Let $(H,Q) = G^o[u_1,\ldots,u_{2s}|v_{k+2},\ldots,v_{2t-k-1}]$. Note that $t-k = 3$ and so $v_{k+2} = v_{t-1}$ and $v_{2t-k-1} = v_{t+2}$.

First let v_{t-2} and v_{t+3} have disjoint neighbourhoods in G^o. Then $\pi((H,Q)) = (2^{2s}|s^4)$. Now let A_{ij} be the set of all vertices adjacent to both v_{t-2+i} and v_{t-2+j} in H and $n_{ij} = |A_{ij}|$, $1 \leq i \neq j \leq 4$. Without loss of generality we also assume that the vertex adjacent to v_{t+3} in G^o belongs to A_{34}. Now,

$$\sum_{i \neq j} n_{ij} = d_H(v_{t-2+j}) = s, \ j = 1,\ldots,4 \qquad \ldots(7.6)$$

Summing (7.6) over all j and using the fact that $n_{ij} = n_{ji}$, we get

$$\sum_{j=1}^{4} \sum_{i<j} n_{ij} = 2s \qquad \ldots(7.7)$$

Subtracting the equations (7.6) corresponding to $j = 1$ and $j = 4$ from the equation (7.7) we get $n_{14} = n_{23}$. Now any permutation σ^o of $V(G^o)$ satisfying

$$\sigma^o(A_{14}) = A_{23},$$

$$\sigma^o(u) = u \text{ if } u \in A_{12} \cup A_{13} \cup A_{24} \cup A_{34},$$

$$\sigma^o(v_j) = v_{2t+1-j}, \quad t-2 \leq j \leq t+3,$$

is an element of $\mathcal{C}_p((G^o, P^o))$.

Next let some u_i be adjacent to both v_{t-2} and v_{t+3} in G^o. Without loss of generality we assume that in G^o, u_1 is not adjacent to v_{t-2} and u_{2s} is adjacent to v_{t+3}. Then $\pi((H,Q)) = (3, 2^{2s-2}, 1|s^4)$. Now by Lemma 7.7, $\mathcal{C}_p((H,Q))$ contains an element σ such that $\sigma(u_1) = u_{2s}$

and $\sigma(u_{2s}) = u_1$. It now follows that $\sigma(v_{t-2}\, v_{t+3}) \in \mathcal{C}_p((G^o, P^o))$.

Thus we have shown that $\mathcal{C}_p((G^o, P^o)) \neq \emptyset$ if $t - k \geq 1$. As explained before, this proves that π is forcibly bipsc in Case 3.

<u>Case 4</u>. π satisfies (4). Then $n = 2t$. Let $A_1 = \{u_1, \ldots, u_p\}$, $A_2 = \{u_{m-p+1}, \ldots, u_m\}$, $B_1 = \{v_1, \ldots, v_h\}$ and $B_2 = \{v_{2t-h+1}, \ldots, v_{2t}\}$. We then prove the following

<u>Claim</u> : (G,P) with the ordering $S = (u_1, \ldots, u_m | v_1, \ldots, v_{2t})$ is a realisation of π iff

(i) $G[A_1 | B-B_2] = K$, $G[A-A_1 | B_2] = \overline{K}$,

(ii) $G[A-A_2 | B_1] = K$, $G[A_2 | B-B_1] = \overline{K}$.

(iii) $G[A_1 | B_2]$ with the ordering $(u_1, \ldots, u_p | v_{2t-h+1}, \ldots, v_{2t})$ as well as $\overline{G}[A_2 | B_1]$ with the ordering $(u_m, \ldots, u_{m-p+1} | v_h, \ldots, v_1)$ is a realisation of π^*,

(iv) $G[A-A_1-A_2 | B-B_1-B_2]$ with the ordering $(u_{p+1}, \ldots, u_{m-p} | v_{h+1}, \ldots, v_{2t-h})$ is a realisation of π^+.

The 'if part' of the claim is trivial. To prove the 'only if part', let (G,P) with the ordering $S = (u_1,\dots,u_n \mid v_1,\dots,v_{2t})$ be a realisation of π. Then (i) follows by (b), (ii) follows by (c), and (iii) and (iv) follow from (i) and (ii). This proves the claim.

Now a graph (G,P) satisfying (i) - (iv) above exists since by (e), π^* is graphic and by Lemma 7.4, π^+ is graphic. By the claim proved above, such a graph is a realisation of π and so π is graphic.

Next let (G,P) with the ordering $S = (u_1,\dots,u_n \mid v_1,\dots,v_{2t})$ be a realisation of π. Then (i) - (iv) of the above claim hold. Also since π^* is unigraphic it follows from (iii) that there exists an isomorphism σ^* from $G[A_1 \mid B_2]$ to $\bar{G}[A_2 \mid B_1]$ such that $\sigma^*(A_1) = A_2$ and $\sigma^*(B_2) = B_1$.

If now $p = \frac{m}{2}$ then no d_i is $\frac{n}{2}$, hence by assumption (II), no e_j is $\frac{m}{2}$, so $q = \frac{n}{2} = h$. Thus $A = A_1 \bigcup A_2$ and $B = B_1 \bigcup B_2$. It is easy to see that the permutation σ defined by

$$\sigma = \begin{cases} \sigma^* & \text{on } A_1 \bigcup B_2 \\ \sigma^{*\,-1} & \text{on } A_2 \bigcup B_1 \end{cases}$$

is an element of $\mathfrak{S}_p((G,P))$.

Next let $p < \frac{m}{2}$. If now $t - h = 0$ then $B = B_1 \bigcup B_2$ and the permutation σ defined by

$$\sigma(x) = \begin{cases} \sigma^*(x) & \text{if } x \in A_1 \bigcup B_2 \\ \sigma^{*-1}(x) & \text{if } x \in A_2 \bigcup B_1 \\ x & \text{if } x \in A - A_1 - A_2 \end{cases}$$

is an element of $\mathcal{C}_p((G,P))$. So let $p < \frac{m}{2}$ and $t - h > 0$. Then by (d), $\pi(G[A-A_1-A_2|B-B_1-B_2]) = ((t-h)^{m-2p}|$ $e_{h+1}-p,\ldots,e_{2t-h}-p)$ satisfies condition (3) of Theorem 7.1 with t replaced by $t - h$ and k replaced by 0. Hence by Case 3, it follows that $\mathcal{C}_p(G[A-A_1-A_2|B-B_1-B_2])$ contains an element σ^+. Now the permutation σ defined by

$$\sigma = \begin{cases} \sigma^* & \text{on } A_1 \bigcup B_2 \\ \sigma^{*-1} & \text{on } A_2 \bigcup B_1 \\ \sigma^+ & \text{on } (A-A_1-A_2) \bigcup (B-B_1-B_2) \end{cases}$$

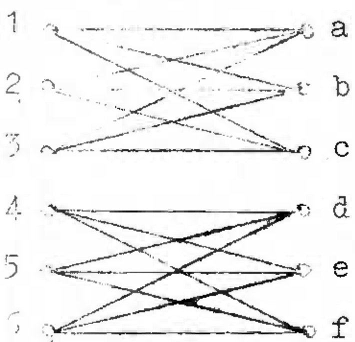
is an element of $\mathcal{C}_p((G,P))$.

Thus π is forcibly bipsc in Case 4, and sufficiency is established.

The main result of Chapter 7 is included in [4]

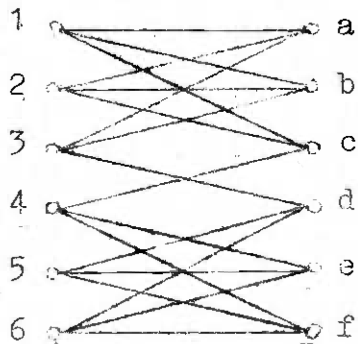
APPENDIX I

Below we verify that the bipartitioned sequence $\pi = (3^6|3^6)$ has exactly six nonisomorphic realisations (G,P), and each of these is bipsc with $\mathcal{C}_p((G,P)) \neq \emptyset$. We label these realisations as (G_1,P_1), (G_2,P_2), ..., (G_6,P_6), and exhibit a complementing permutation σ_i below each graph (G_i,P_i).



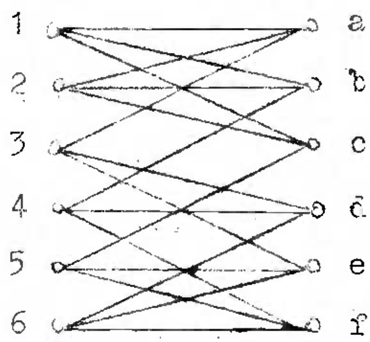
A (G_1,P_1) B

$\sigma_1 = (1)(2)(3)(4)(5)(6)(af)(be)(cd)$



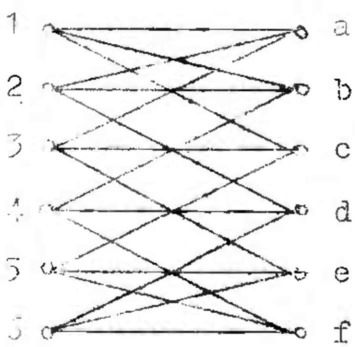
A (G_2,P_2) B

$\sigma_2 = \sigma_1$



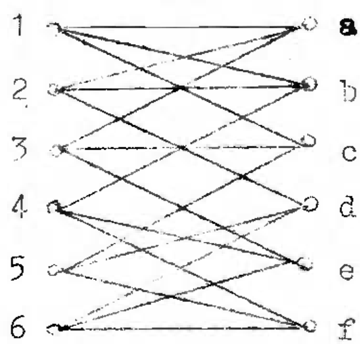
A (G_3,P_3) B

$\sigma_3 = \sigma_1$



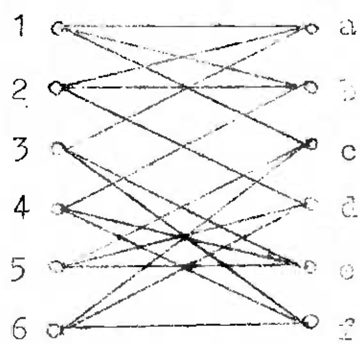
A (G_4,P_4) B

$\sigma_4 = \sigma_1$



A (G_5,P_5) B

$\sigma_5 = (1)(2)(3)(4\,5)(6)(af)(be)(cd)$

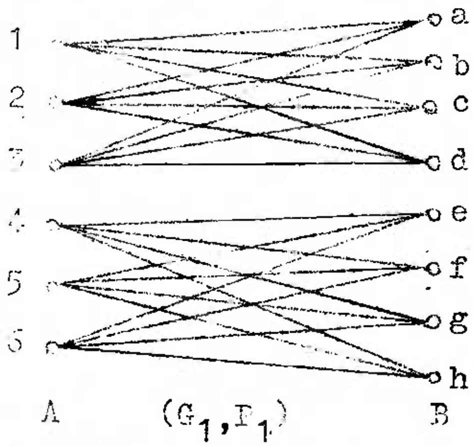


A (G_6,P_6) B

$\sigma_6 = (1)(2)(35)(46)(af)(be)(cd)$

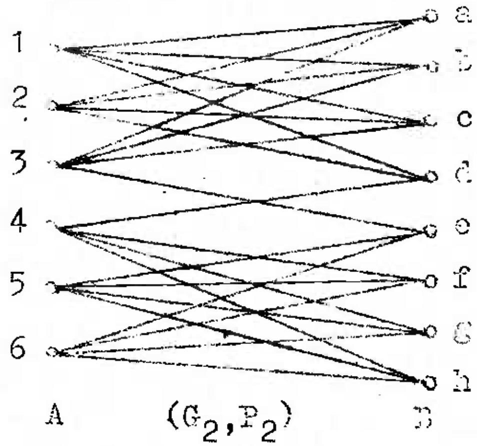
APPENDIX II

Below we verify that the bipartitioned sequence $\pi = (4^6|3^8)$ has exactly twenty nonisomorphic realisations (G,P), and each of these is bipsc with $\mathfrak{G}_p((G,P)) \neq \emptyset$. We label these realisations as (G_1,P_1), (G_2,P_2), ..., (G_{20},P_{20}), and exhibit a complementing permutation σ_i below each graph (G_i,P_i).



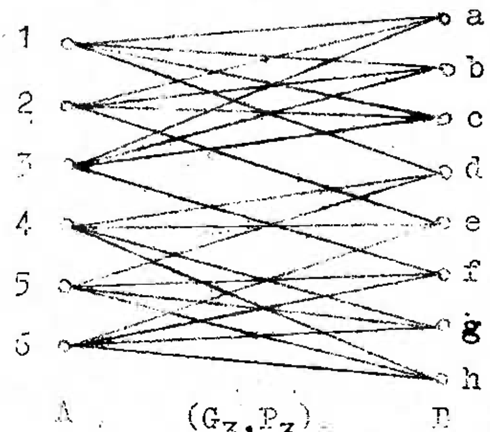
(G_1,P_1)

$\sigma_1 = (1)(2)(3)(4)(5)(6)$
$(ah)(bg)(cf)(de)$



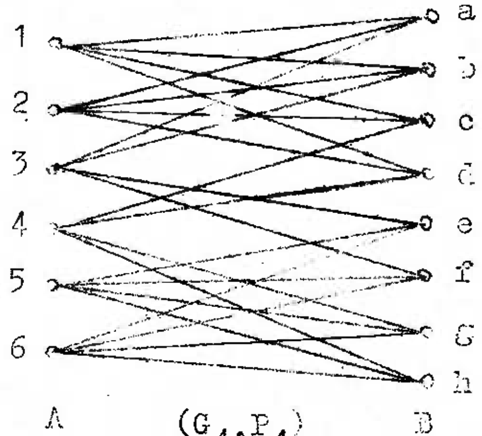
(G_2,P_2)

$\sigma_2 = \sigma_1$



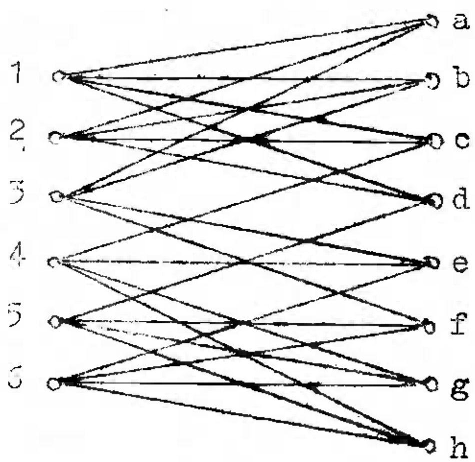
(G_3,P_3)

$\sigma_3 = (16)(25)(34)(a)(b)$
$(c)(d)(e)(f)(g)(h)$



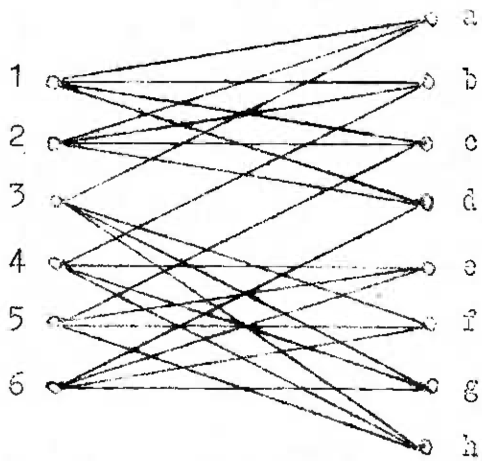
(G_4,P_4)

$\sigma_4 = \sigma_1$



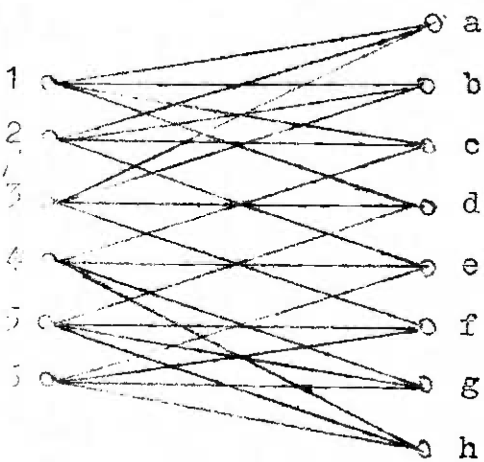
A (G_5, P_5) B

$\sigma_5 = \sigma_1$



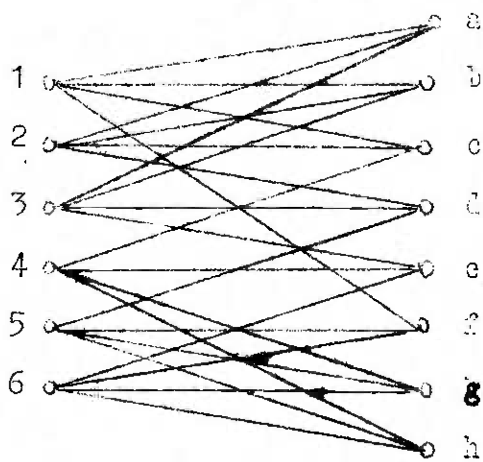
A (G_6, P_6) B

$\sigma_6 =$ (1) (2) (3) (4) (5) (6)
(ae) (bf) (cg) (dh)



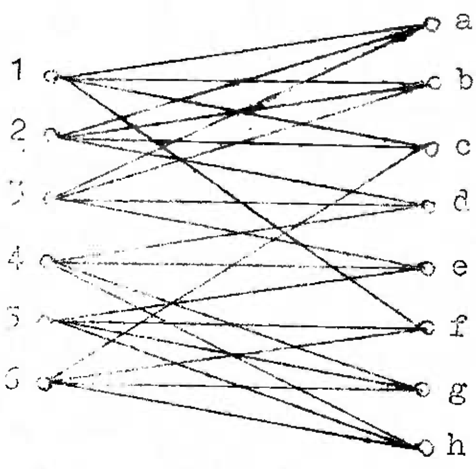
A (G_7, P_7) B

$\sigma_7 = \sigma_1$



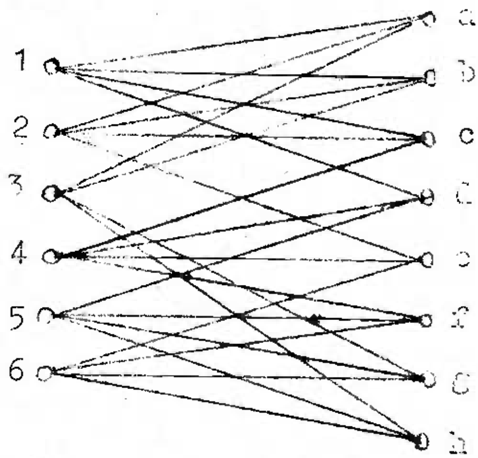
A (G_8, P_8) B

$\sigma_8 =$ (1) (2) (3) (45) (6)
(ah) (bg) (ce) (df)



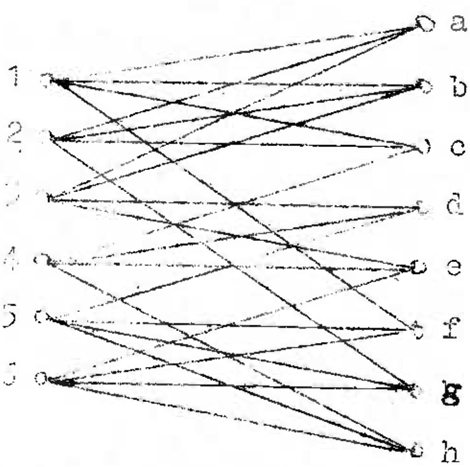
A (G_9, P_9) B

$\sigma_9 = (14)(25)(36)(a)(b)$
$(c)(d)(e)(f)(g)(h)$



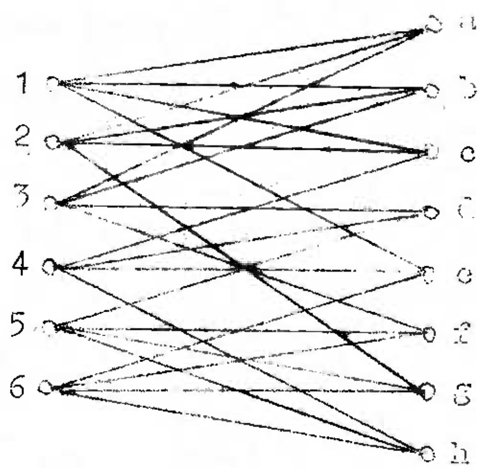
A (G_{10}, P_{10}) B

$\sigma_{10} = \sigma_3$



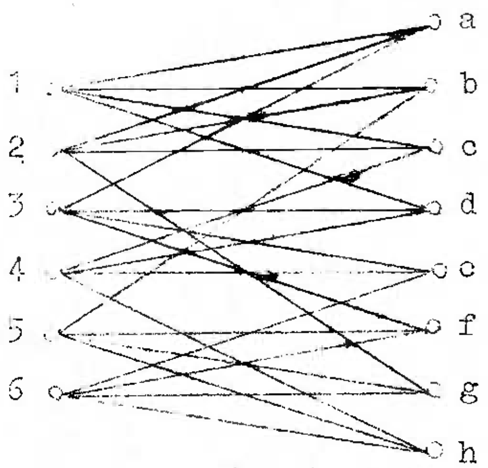
A (G_{11}, P_{11}) B

$\sigma_{11} = (16)(25)(34)(a)(b)$
$(c)(df)(eg)(h)$



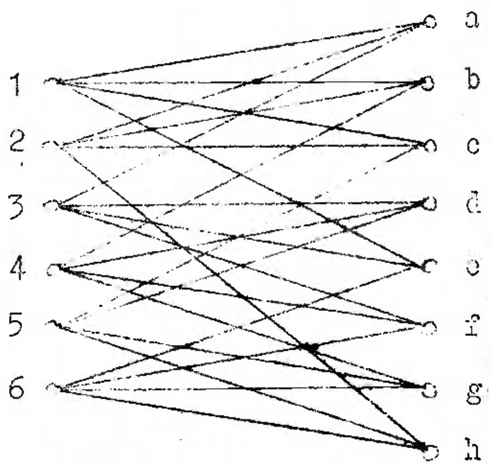
A (G_{12}, P_{12}) B

$\sigma_{12} = (15)(26)(34)(a)(b)$
$(c)(dg)(e)(f)(h)$



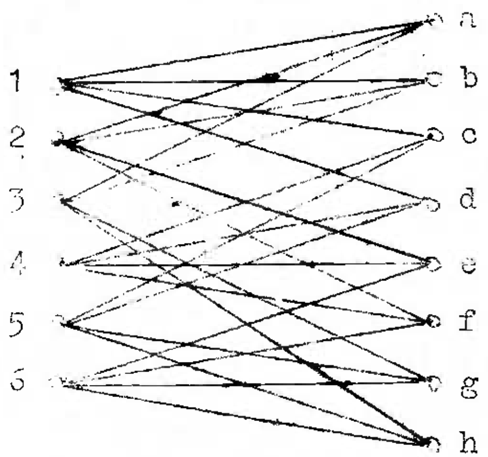
A (G_{13}, P_{13}) B

$\sigma_{13} =$ (1) (2) (3) (4) (5) (6)
(ah) (be) (cf) (dg)



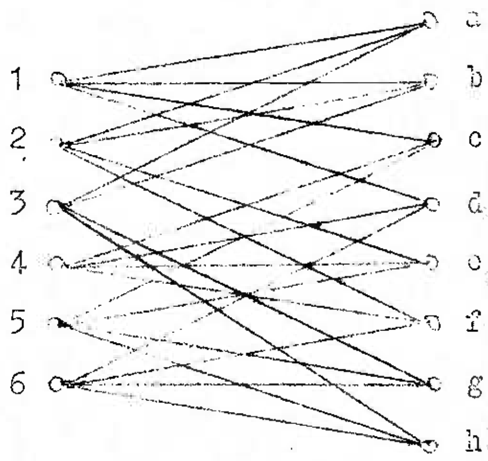
A (G_{14}, P_{14}) B

$\sigma_{14} =$ (1) (2) (3) (46) (5)
(ag) (bd) (cf) (eh)



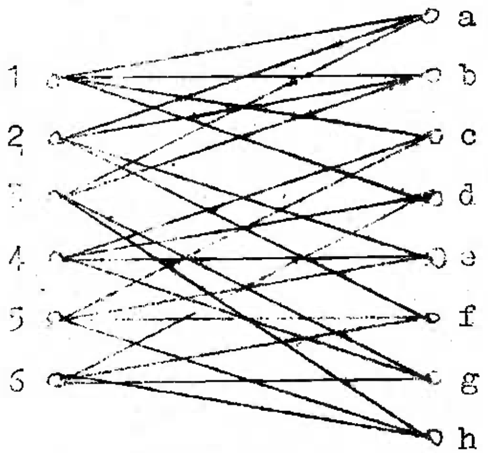
A (G_{15}, P_{15}) B

$\sigma_{15} = \sigma_3$



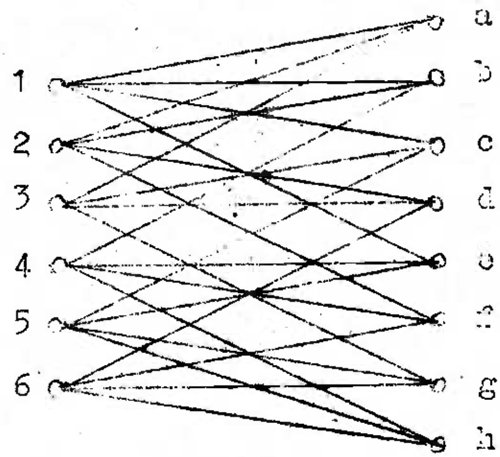
A (G_{16}, P_{16}) B

$\sigma_{16} =$ (1) (2) (34) (5) (6)
(ah) (bg) (cf) (d



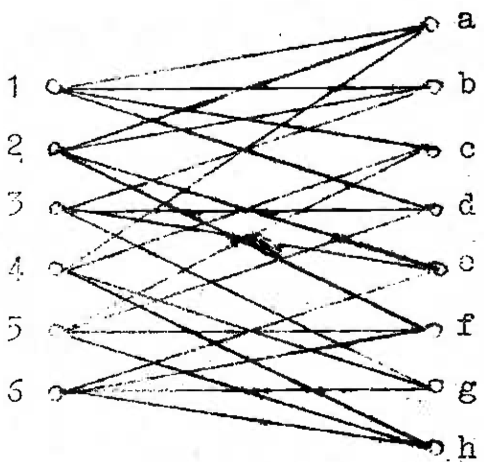
A (G_{17}, P_{17}) B

$\sigma_{17} = (16)(25)(34)(a)(b)$
$(c)(de)(fg)(h)$



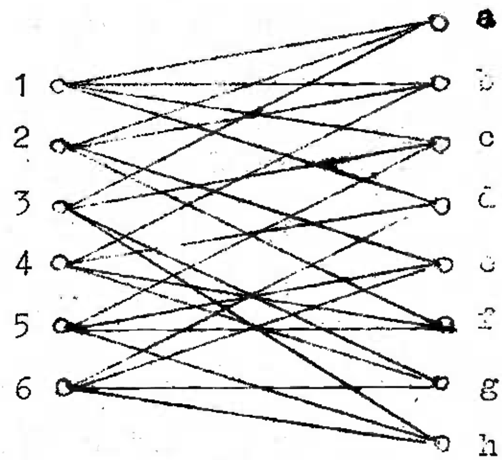
A (G_{18}, P_{18}) B

$\sigma_{18} = \sigma_3$



A (G_{19}, P_{19}) B

$\sigma_{19} = (16)(25)(34)(a)(b)$
$(c)(d)(e)(fg)(h)$



A (G_{20}, P_{20}) B

$\sigma_{20} = \sigma_{11}$

REFERENCES

[1] C.R.J. CLAPHAM : Hamiltonian arcs in self-complementary graphs, Discrete Maths., 8 (1974), pp. 251-255.

[2] C.R.J. CLAPHAM and D.J. KLEITMAN : The degree sequences of self-complementary graphs, J. Comb. Theory (B), 20 (1976), pp. 67-74.

[3] P. ERDÖS and T. GALLAI : Graphs with prescribed degrees of vertices, Mat. Lopok, 11 (1960), pp. 264-274.

[4] T. GANGOPADHYAY : Characterisation of forcibly bipartite self-complementary bipartitioned sequences, presented to the Symposium on Combinatorics and Graph Theory, Indian Statistical Institute, Calcutta, February 25-29, 1980.

[5] T. GANGOPADHYAY : Characterisation of potentially bipartite self-complementary bipartitioned sequences, Discrete Maths., (submitted for publication).

[6] T. GANGOPADHYAY and S.P. RAO HEBBARE : r-partite self-complementary graphs - Diameters, Discrete Maths., (to appear).

[7] T. GANGOPADHYAY and S.P. RAO HEBBARE : Paths in r-partite self-complementary graphs, Discrete Maths., (to appear).

[8] S.P. RAO HEBBARE : A generalisation of self-complementary graphs, Tech. Report No.15 (1977), ISI, Calcutta.

[9] M. KOREN : Pairs of sequences with a unique realisation by bipartite graphs, J. Comb. Theory (B), 21 (1976), pp. 224-234.

[10] E.A. NORDHAUS and J.W. GADDUM : On complementary graphs, Amer. Math. Monthly, 63 (1956), pp. 175-177.

[11] A. RAMACHANDRA RAO : Degree sequences of Cacti, presented to the Symposium on Combinatorics and Graph Theory, Indian Statistical Institute, Calcutta, February 25-29, 1980.

[12] S.B. RAO : Characterisation of forcibly self-complementary degree sequences, Discrete Maths., (submitted for publication).

[13] S.B. RAO : Solution of the Hamiltonian problem for self-complementary graphs, J. Comb. Theory (B), 27 (1979), pp. 13-41.

[14] S.B. RAO : Cycles in self-complementary graphs, J. Comb. Theory (B), 22 (1977), pp. 1-10.

[15] S.B. RAO : Characterisation of self-complementary graphs with 2-factors, Discrete Maths. 17 (1977), pp. 225-233.

[16] RÉDEI : Ein Kombinatorischer Satz. Acta Litt. Szeged, 7 (1934), pp. 39-43.

[17] G. RINGEL : Selbskomplementäre Graphen, Arch. Math., 14 (1963), pp. 354-358.

[18] H. SACHS : Über Selbskomplementäre Graphen, Pub. Math. Debrecen, 9 (1962), pp. 270-288.

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