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# Likelihood ratio tests for bivariate symmetry against ordered alternatives in a square contingency table

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**Abstract**

Let $(X_1, X_2)$ be a bivariate random variable of the discrete type with joint probability density function $p_{ij} = \mathrm{pr}[X_1 = i, X_2 = j]$, $i, j = 1, \ldots, k$. Based on a random sample from this distribution, we discuss the properties of the likelihood ratio test of the null hypothesis of bivariate symmetry $\mathrm{H}_0$: $p_{ij} = p_{ji}$ $\forall(i,j)$ vs. the alternative $\mathrm{H}_1$: $p_{ij} \geqslant p_{ji}$, $\forall i > j$, in a square contingency table. This is a categorised version of the classical one-sided matched pairs problem. This test is asymptotically distribution-free. We also consider the problem of testing $\mathrm{H}_1$ as a null hypothesis against the alternative $\mathrm{H}_2$ of no restriction on $p_{ij}$'s. The asymptotic null distributions of the test statistics are found to be of the chi-bar square type. Finally, we analyse a data set to demonstrate the use of the proposed tests.

*Keywords:* Chi-bar square distribution; Joint likelihood ratio ordering; Least-favourable configuration; Matched pairs; Ordinal data; Stochastic ordering

## 1. Introduction

Square contingency tables arise frequently in *before and after experiments* in many areas like public health, medicine, psychology and sociology, when a given number of individuals or items are measured before and after treatment to determine its effect. The recorded data are usually ordered on a categorical scale like occupational status, level of injury, grade in an examination, etc. Let the row variable $X_2$ denote the measurement before treatment is given and let the column variable $X_1$ denote the corresponding measurement after treatment. Thus, we have a categorized version of the classical matched pairs problem. The objective in such experiments is to summarise the difference between $X_1$ and $X_2$ as caused by the treatment, taking into account the dependence between $X_1$ and $X_2$. We would like to see whether $X_1$ is greater than $X_2$ according to some *stochastic ordering sense* or not. Thus, our alternatives are usually, directional in such problems.

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Suppose that $X_1$ and $X_2$ take values in a set $\boldsymbol{S}$ of cardinality $k$. Without loss of generality, we assume that $\boldsymbol{S} = \{1,\ldots,k\}$. Let $p_{ij} = \mathrm{pr}[X = i, Y = j]$. We assume that all the $p_{ij}$'s are strictly positive. Take a random sample of size $n$ on $(X_1, X_2)$ and let $n_{ij}$ be the observed frequency of the $(i,j)$th cell for $i,j = 1,\ldots,k$ with $\sum_{i=1}^{k}\sum_{j=1}^{k} n_{ij} = n$. Thus, our data are in the form of a $k \times k$ square contingency table. On the basis of this data, first we consider the problem of testing the null hypothesis of bivariate symmetry,

$$\mathrm{H}_0\colon\ p_{ij} = p_{ji}, \quad \forall (i,j) \in \boldsymbol{S}^2, \tag{1.1}$$

against the alternative

$$\mathrm{H}_1\colon\ p_{ij} \geqslant p_{ji}, \quad \forall i \geqslant j. \tag{1.2}$$

We say that $X_1$ is greater than $X_2$ according to *joint likelihood ratio ordering* ($X_1 \overset{\mathrm{lr:j}}{\geqslant} X_2$) if and only if (1.2) holds (see Shanthikumar and Yao, 1991). It has the following interpretation in terms of expectations of functions of $X_1$ and $X_2$.

Let

$$\mathscr{G}_{\mathrm{lr}} := \{g(x,y) \geqslant g(y,x), \quad \forall x \geqslant y\}. \tag{1.3}$$

Then $X_1 \overset{\mathrm{lr:j}}{\geqslant} X_2$ if and only if

$$E[g(X_1, X_2)] \geqslant E[g(X_2, X_1)], \quad \forall g \in \mathscr{G}_{\mathrm{lr}}. \tag{1.4}$$

It may be mentioned that joint likelihood ratio ordering is an extension of the concept of *ordinary likelihood ratio ordering* (that is, $\sum_{j=1}^{k} p_{ij}/\sum_{j=1}^{k} p_{ji}$ non-decreasing in $i$) to compare two *dependent* random variables. It is easy to show that $X_1 \overset{\mathrm{lr:j}}{\geqslant} X_2$ implies that the marginal distribution of $X_1$ is *stochastically greater* than that of $X_2$ (that is, $\sum_{i=1}^{r}\sum_{j=1}^{k} p_{ij} \leqslant \sum_{j=1}^{r}\sum_{i=1}^{k} p_{ij}$, for $r = 1,\ldots,k$).

We also consider the problem of testing the order restriction as imposed by (1.2) as the null hypothesis against the alternative $\mathrm{H}_2$ of no restrictions on the $p_{ij}$'s.

Bowker (1948) discussed the problem of testing $\mathrm{H}_0$ against the two-sided alternative $p_{ij} \neq p_{ji}$ for all $i \neq j$ and recommended chi-square goodness of fit test for this problem. McCullagh (1977, 1978) proposed some parametric models to study such problems. In this paper we use the nonparametric approach. Compared to parametric tests, minimal assumptions are needed for the validity of the nonparametric tests. If $\mathrm{H}_0$ is rejected using the nonparametric test, one can go in for more detailed analysis of the problem using the appropriate parametric models to obtain estimates of the treatment effects.

In Section 2, we obtain the maximum likelihood estimators of the $p_{ij}$'s under $\mathrm{H}_0$, $\mathrm{H}_1$ and $\mathrm{H}_2$ and then use these to construct the likelihood ratio statistics in Section 3. The asymptotic distributions of the test statistics are obtained and they are seen to be of the chi-bar square type (a mixture of independent chi-square distributions). An attractive feature of our asymptotic test of $\mathrm{H}_0$ against $\mathrm{H}_1$ is that it is a similar test, a property which does not hold very frequently in order-restricted testing problems. The least-favourable configuration in testing $\mathrm{H}_1$ as a null hypothesis has also been obtained and we give an upper bound on the probability of type-one error in this case. In the last section, we report the results of a simulation study on the powers of the proposed tests and also give an example to demonstrate our test procedures.

## 2. Maximum likelihood estimation

Let $N = ((n_{ij}))$, $P = ((p_{ij}))$ be the matrices of the cell frequencies and the cell probabilities. The likelihood function is

$$L \propto \left( \prod_{i=1}^{k} p_{ii}^{n_{ii}} \right) \prod_{i>j} p_{ij}^{n_{ij}} p_{ji}^{n_{ji}}. \tag{2.1}$$

Since $N$ has a multinomial distribution, the unconstrained m.l.e. of $p_{ij}$ is $\hat{p}_{ij} = n_{ij}/n$, $(i,j) \in S^2$. To find the m.l.e's of the $p_{ij}$'s under $H_0$ and $H_1$, we reparametrize the problem as follows.

Let, for $i > j$,

$$\theta_{ij} = p_{ij}/(p_{ij} + p_{ji}), \qquad \phi_{ij} = p_{ij} + p_{ji} \quad \text{and} \quad \psi_i = p_{ii}, \tag{2.2}$$

so that for $i > j$,

$$p_{ij} = \theta_{ij}\phi_{ij}, \quad p_{ji} = \phi_{ij}(1 - \theta_{ij}) \quad \text{and} \quad p_{ii} = \psi_i. \tag{2.3}$$

In terms of the new parameters, the problem reduces to testing the null hypothesis $H_0: \theta_{ij} = \frac{1}{2}$, for all $(i,j)$ against the alternative $H_1: \theta_{ij} \geqslant \frac{1}{2}$ for all $(i,j)$ and with strict inequality for some $(i,j)$.

The likelihood function in terms of the new parameters is

$$L \propto \left( \prod_{i>j} \theta_{ij}^{n_{ij}} (1 - \theta_{ij})^{n_{ji}} \right) \left( \prod_{i>j} \phi_{ij}^{n_{ij}+n_{ji}} \right) \left( \prod_{i=1}^{k} \psi_i^{n_{ii}} \right). \tag{2.4}$$

The unrestricted m.l.e. of $\theta_{ij}$ is $\hat{\theta}_{ij} = n_{ij}/(n_{ij} + n_{ji})$. The m.l.e.'s under $H_0$ are

$$\hat{\theta}_{ij}^{(0)} = 1/2, \quad \hat{\phi}_{ij}^{(0)} = (n_{ij} + n_{ji})/n, \qquad i > j \tag{2.5}$$

and

$$\hat{\psi}_i^{(0)} = n_{ii}/n, \quad i = 1, \ldots, k$$

Under the alternative $H_1$: $\theta_{ij} \geqslant \frac{1}{2}$ for $i > j$. There are no additional constraints on $\phi_{ij}$'s and $\psi_i$'s so their m.l.e.'s remain unchanged. The m.l.e of $\theta_{ij}$ under $H_1$ is given by

$$\hat{\theta}_{ij}^{(1)} = \left( \frac{n_{ij}}{n_{ij} + n_{ji}} \right) \vee \frac{1}{2}, \quad \text{for } i > j, \tag{2.6}$$

where $a \vee b$ ($a \wedge b$) denotes the maximum (minimum) of $a$ and $b$. Using (2.3) and (2.6), we find that the m.l.e's of $p_{ij}$'s under $H_1$ are

$$\hat{p}_{ij}^{(1)} = \left( \frac{n_{ij} + n_{ji}}{n} \right) \left[ \frac{n_{ij}}{n_{ij} + n_{ji}} \vee \frac{1}{2} \right] \quad \text{for } i > j, \tag{2.7}$$

$$\hat{p}_{ji}^{(1)} = \left( \frac{n_{ij} + n_{ji}}{n} \right) \left[ \frac{n_{ji}}{n_{ij} + n_{ji}} \wedge \frac{1}{2} \right] \quad \text{for } i > j, \tag{2.8}$$

$$\hat{p}_{ii}^{(1)} = \hat{p}_{ii}^{(0)} = n_{ii}/n, \quad i = 1, \ldots, k. \tag{2.9}$$

## 3. Likelihood ratio tests

In this section we derive the likelihood ratio tests for testing $H_0$ against $H_1$ and also for testing $H_1$ against the alternative $H_2$, where $H_2$ puts no restriction on $P$. We see below that both the tests are of chi-bar square type and where as the first test is asymptotically similar over $H_0$, the second one is not. For the second test we find the asymptotic least-favorable configuration and an upper bound on its asymptotic type-one error probability over $H_1$.

### *3.1. Testing* $H_0$ *against* $H_1$

The likelihood ratio test for testing $H_0$ against $H_1$ rejects $H_0$ for small values of

$$\Lambda_{01} = \frac{\prod_{i>j}(\frac{1}{2})^{n_{ij}}(\frac{1}{2})^{n_{ji}}}{\prod_{i>j}(\hat{\theta}_{ij}^{(1)})^{n_{ij}}(1-\hat{\theta}_{ij}^{(1)})^{n_{ji}}}. \tag{3.1}$$

The log-likelihood ratio is

$$\begin{aligned} T_{01} &= -2\ln\Lambda_{01} \\ &= \sum_{i>j}[n_{ij}\ln\hat{\theta}_{ij}^{(1)} + n_{ji}\ln(1-\hat{\theta}_{ij}^{(1)}) - n_{ij}\ln(\tfrac{1}{2}) - n_{ji}\ln(\tfrac{1}{2})]. \end{aligned} \tag{3.2}$$

Expanding $\ln\hat{\theta}_{ij}^{(1)}$ and $\ln(\frac{1}{2})$ about $\hat{\theta}_{ij}$ and expanding $\ln(1-\hat{\theta}_{ij}^{(1)})$ and $\ln(\frac{1}{2})$ about $(1-\hat{\theta}_{ij})$ using Taylor's expansion with a second-degree remainder, we find from the properties of isotonic regression that the linear terms cancel out giving

$$\begin{aligned} T_{01} &= \sum_{i>j} n_{ij}\left\{-\frac{1}{\alpha_{ij}^2}(\hat{\theta}_{ij}^{(1)}-\hat{\theta}_{ij})^2 + \frac{1}{\beta_{ij}^2}(\hat{\theta}_{ij}-\tfrac{1}{2})^2\right\} \\ &\quad + n_{ji}\left\{-\frac{1}{\delta_{ij}^2}(\hat{\theta}_{ij}^{(1)}-\hat{\theta}_{ij})^2 + \frac{1}{\gamma_{ij}^2}(\hat{\theta}_{ij}-\tfrac{1}{2})^2\right\} \\ &= \sum_{i>j}\left[(\hat{\theta}_{ij}-\tfrac{1}{2})^2\left\{\frac{n_{ij}}{\beta_{ij}^2}+\frac{n_{ji}}{\gamma_{ij}^2}\right\} - (\hat{\theta}_{ij}^{(1)}-\hat{\theta}_{ij})^2\left\{\frac{n_{ij}}{\alpha_{ij}^2}+\frac{n_{ji}}{\delta_{ij}^2}\right\}\right], \end{aligned} \tag{3.3}$$

where $\alpha_{ij}$, $\beta_{ij}$, $\delta_{ij}$ and $\gamma_{ij}$ are random variables converging almost surely to $\frac{1}{2}$ under $H_0$.

By the Central Limit Theorem for multinomial variables, the random matrix $\sqrt{n}(\hat{\boldsymbol{P}}-\boldsymbol{P})$ has a limiting multivariate normal distribution with mean $\boldsymbol{0}$, a $k \times k$ matrix whose all entries are zeros and with dispersion matrix $\sum = (\sigma_{lk,st})$ where

$$\sigma_{lk,st} = p_{lk}(\delta_{lk,st} - p_{st}),$$

and where $\delta_{lk,st} = 1$, if $(l, k) = (s, t)$ and zero otherwise. Using the multivariate delta method (cf. Serfling, 1980, p. 122), it follows that for $i > j$, the $K = k(k-1)/2$ random variables $\sqrt{n}(\hat{\theta}_{ij}-\theta_{ij})$ have asymptotically the same distribution as $K$ independent normally distributed random variables $\{U_{ij}, i > j\}$ each with mean 0 and with variance of $U_{ij}$ as $p_{ij}p_{ji}/(p_{ij}+p_{ji})^3$. Also we know that $(1/n)\{(n_{ij}/\beta_{ij}^2)+(n_{ji}/\gamma_{ij}^2)\}$ and $(1/n)\{(n_{ij}/\alpha_{ij}^2)+(n_{ji}/\delta_{ij}^2)\}$ converge almost surely to $1/\mathrm{var}(U_{ij})$.

Therefore, it follows that under $H_0$,

$$\begin{aligned} T_{01} &\xrightarrow{\mathscr{D}} \sum_{i>j}\frac{1}{\mathrm{var}(U_{ij})}U_{ij}^2 - \sum_{i>j}\frac{1}{\mathrm{var}(U_{ij})}(U_{ij}\vee 0)^2 \\ &= \sum_{i>j}\frac{1}{\mathrm{var}(U_{ij})}(U_{ij}\wedge 0)^2. \end{aligned} \tag{3.4}$$

The proof of the following theorem follows from (3.4) and Theorem 5.3.1 of Robertson et al. (1988).

**Theorem 3.1.** *Under* $\mathrm{H}_0$, *for any real number* $t$, we have

$$\lim_{n\to\infty} \mathrm{pr}(T_{01} \geqslant t) = \sum_{l=0}^{K} \binom{K}{l} \frac{1}{2^K} \mathrm{pr}(\chi_l^2 \geqslant t),$$

*where* $\chi_0^2 \equiv 0$.

Using this result, the $p$-value of the asymptotic test based on $T_{01}$ can be easily obtained. Table 5.3.1 of Robertson et al. (1988) gives the values of some selected percentiles of this asymptotic distribution.

It is clear from this result that the likelihood ratio test based on large values of the statistic $T_{01}$ is asymptotically distribution-free over $\mathrm{H}_0$. Many testing procedures involving inequality constraints do not lead to asymptotically similar tests and hence these tests are often conservative over much of the null hypothesis region as is the case with our next problem.

### 3.2. *Testing* $\mathrm{H}_1$ *against* $\mathrm{H}_2$

Now, we consider the problem of testing $\mathrm{H}_1$ as a null hypothesis against the alternative $\mathrm{H}_2$ of no restrictions on the parameters. The likelihood ratio tests rejects $\mathrm{H}_1$ in favour of $\mathrm{H}_2$ for small values of

$$\Lambda_{12} = \frac{\prod_{i>j}(\hat{\theta}_{ij}^{(1)})^{n_{ij}}(1-\hat{\theta}_{ij}^{(1)})^{n_{ji}}}{\prod_{i>j}(\hat{\theta}_{ij})^{n_{ij}}(1-\hat{\theta}_{ij})^{n_{ji}}} \tag{3.5}$$

and the log-likelihood ratio is

$$\begin{aligned} T_{12} &= -2\ln \Lambda_{12} \\ &= 2\sum_{i>j} [n_{ij}\ln\hat{\theta}_{ij} + n_{ji}\ln(1-\hat{\theta}_{ij}) - n_{ij}\ln\hat{\theta}_{ij}^{(1)} - n_{ji}\ln(1-\hat{\theta}_{ij}^{(1)})]. \end{aligned}$$

Again expanding $\ln\hat{\theta}_{ij}^{(1)}$ about $\hat{\theta}_{ij}$ and $\ln(1-\hat{\theta}_{ij}^{(1)})$ about $1-\hat{\theta}_{ij}$, and using Taylor's expansion with a second-degree remainder, one obtains

$$\begin{aligned} T_{12} &= \sum_{i>j}\left[\frac{n_{ij}}{\alpha_{ij}^2} + \frac{n_{ji}}{\beta_{ij}^2}\right](\hat{\theta}_{ij} - \hat{\theta}_{ij}^{(1)})^2 \\ &= \sum_{i>j}\left[\frac{n_{ij}}{\alpha_{ij}^2} + \frac{n_{ji}}{\beta_{ij}^2}\right]\left((\hat{\theta}_{ij} - \tfrac{1}{2}) \wedge 0\right)^2, \end{aligned} \tag{3.6}$$

where $\alpha_{ij}$ and $\beta_{ij}$ are random variables converging almost surely to $p_{ij}/(p_{ij}+p_{ji})$ and $p_{ji}/(p_{ij}+p_{ji})$, respectively. Since under $\mathrm{H}_1$, $p_{ij}/(p_{ij}+p_{ji}) \geqslant \frac{1}{2}$, one can conclude that

$$T_{12} \xrightarrow{\mathscr{D}} \sum_{i>j,\, p_{ij}=p_{ji}} \frac{(U_{ij}\wedge 0)^2}{\mathrm{var}(U_{ij})}, \tag{3.7}$$

where $U_{ij}$'s are as defined earlier.

The power function of the $T_{12}$ is not constant as a function of $\boldsymbol{P}$ as $\boldsymbol{P}$ ranges over $\mathrm{H}_1$. Thus, the significance level of the test which rejects $\mathrm{H}_1$ in favour of $\mathrm{H}_2$ for values of $T_{12}$ at least as large as $t$ would be given by $\sup_{\boldsymbol{P}\in H_1}\mathrm{pr}_{\boldsymbol{P}}(T_{12}\geqslant t)$, where $\mathrm{pr}_{\boldsymbol{P}}(T_{12}\geqslant t)$ is the probability that $T_{12}\geqslant t$ when $\boldsymbol{P}$ is the *true* matrix of parameter values. Finding this supremum is difficult for finite samples. However, we see from (3.7) that the configuration $\mathrm{H}_0: p_{ij} = p_{ji}, \forall i > j$ is asymptotically least favourable for this test. This result is formally stated in the following theorem.

**Theorem 3.2.** *If $\boldsymbol{P} \in H_1$, then for any real number $t$,*

$$\lim_{n\to\infty} \mathrm{pr}_{\boldsymbol{P}}(T_{12} \geqslant t) = \sum_{l=0}^{M} \binom{M}{l} \frac{1}{2^M} \mathrm{pr}(\chi_l^2 \geqslant t),$$

*where $\chi_0^2 \equiv 0$ and $M = \mathrm{cardinal}\{(i,j), i > j\colon p_{ij} = p_{ji}\}$. Moreover, when $\boldsymbol{P} \in H_1$,*

$$\lim_{n\to\infty} \mathrm{pr}_{\boldsymbol{P}}(T_{12} \geqslant t) \leqslant \sum_{l=0}^{K} \binom{K}{l} \frac{1}{2^K} \mathrm{pr}(\chi_l^2 \geqslant t). \tag{3.8}$$

In addition to providing an upper bound on the type 1 error probability, Theorem 3.2 gives a method for investigating the behaviour of $\lim_{n\to\infty} \mathrm{pr}_{\boldsymbol{P}}(T_{12} \geqslant t)$ for various values of $\boldsymbol{P}$ satisfying $\mathrm{H}_1$. For example, for $\boldsymbol{P}$ such that $p_{ij} > p_{ji}$, for all $i > j$, we have $\lim_{n\to\infty} \mathrm{pr}_{\boldsymbol{P}}(T_{12} \geqslant t) = 0$.

## 4. A simulation study and an example

### 4.1. Simulations

In this section, with the help of a simulation study, we compare the power of our restricted likelihood ratio test based on $T_{01}$ with the usual unrestricted chi-square test as proposed by Bowker (1948). For this purpose, we consider the following model:

$$p_{ij} = \begin{cases} \theta\pi_{ij}, & i > j, \\ \pi_{ii}, & i = j, \\ (2-\theta)\pi_{ij}, & i < j, \end{cases}$$

where $0 < \theta < 2$ and

$$\pi_{ij} = \pi_{ji} = \binom{k}{i}\binom{k}{j} p^{i+j}(1-p)^{2k-i-j};$$

$0 \leqslant i, j \leqslant k$, $0 < p < 1$. The case $\theta = 1$ corresponds to $\mathrm{H}_0$ and $\mathrm{H}_1$ holds if and only if $\theta > 1$. Based on 5000 samples each of size 500 generated from the above distribution with $k = 3$, $p = 0.5$, and using the asymptotic critical values, we report in Table 1, the simulated powers of these two tests.

It is clear from the above table that the restricted likelihood ratio test performs better than the unrestricted chi-square test. We expect similar results for the other alternatives too.

**Example.** To illustrate our testing procedures with a real life problem, we consider some rather famous data from Stuart (1953) concerning the unaided distance vision of 7477 women aged 30–39 employed in Royal Ordnance factories from 1943 to 1946. The column variable $X_1$ is the right eye grade and the row variable $X_2$ is the left eye grade. The categories are ordered from lowest to highest. The data are represented in Table 2.

We would like to test the null hypothesis $\mathrm{H}_0$ of bivariate symmetry that the vision of both the eyes is the same against the alternative that the right eye has better vision than the left eye (better in the sense of joint likelihood ratio ordering). For testing $\mathrm{H}_0$ against $\mathrm{H}_1$, the value of the test statistic $T_{01}$ is 19.1492 giving an asymptotic $p$-value of less than 0.001. For testing $\mathrm{H}_1$ vs. $\mathrm{H}_2$, the $p$-value of the test statistic $T_{12}$ (which is 0.1) is 0.9468. Thus, we have a strong evidence to conclude that the vision of right eye is significantly better than of left eye. We also computed the value of the usual log-likelihood ratio statistic for testing $\mathrm{H}_0$ against $\mathrm{H}_2$. Its value of 19.2492 which gives us a $p$-value of 0.0038 using chi-square distribution with 6 degrees of freedom. A similar conclusion was reached by McCullagh (1978).

Table 1
Simulated powers of $T_{01}$ and the usual chi-square test

| $\theta$ | 1 | 1.03 | 1.06 | 1.09 | 1.12 | 1.15 | 1.18 | 1.21 | 1.24 | 1.27 | 1.30 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{01}$ | 0.048 | 0.118 | 0.223 | 0.374 | 0.565 | 0.763 | 0.886 | 0.960 | 0.990 | 0.991 | 1.00 |
| Chi-square | 0.046 | 0.065 | 0.105 | 0.172 | 0.324 | 0.504 | 0.703 | 0.845 | 0.943 | 0.983 | 0.997 |

Table 2
Unaided distance vision of 7477 women aged 30–39 employed in Royal Ordnance factories from 1943 to 1946

| | Right eye | | | | |
|---|---|---|---|---|---|
| Left eye | Lowest grade | Third grade | Second grade | Highest grade | Totals |
| Lowest grade | 492 | 205 | 78 | 66 | 841 |
| Third grade | 179 | 1772 | 432 | 124 | 2507 |
| Second grade | 82 | 362 | 1512 | 266 | 2222 |
| Higest grade | 36 | 117 | 234 | 1520 | 1907 |
| Totals | 789 | 2456 | 2256 | 1976 | 7444 |

## 5. Conclusions

In this paper we have developed an asymptotically distribution-free test for testing bivariate symmetry against a one-sided alternative in a square contingency table. Tables for the asymptotic critical values already exist in the literature and it is also easy to find the $p$-values of the proposed test. Like other nonparametric tests, the tests proposed here can be meaningfully used for preliminary analysis. If the null hypothesis of bivariate symmetry is rejected against one-sided alternative, one can go in for a more detailed analysis of the problem using appropriate parametric models as developed by McCullagh (1977) and others.

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