Minimization of Expected Variance of Completion Times on Single Machine for Stochastic Jobs

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This article deals with the problem of scheduling jobs with random processing times on single machine in order to minimize the expected variance of job completion times. Sufficient conditions for the existence of V-shaped optimal sequences are derived separately for general and ordered job processing times. It is shown that when coefficient of variation of random processing times are bounded by a certain value, an optimal sequence is V-shaped.

1. INTRODUCTION

The problem of scheduling n jobs on a single machine in order to minimize the variance of job completion times has been increasingly attracting several researchers since the early 1970s. The problem finds application in computer file organizations, especially in on-line systems where it is desirable to provide uniform response time to the users and is also consistent with the current emphasis on Just-in-time production philosophy. Early results on this problem can be found in Merten and Muller [5], Schrage [6], Eilon and Chowdhury [3] etc. Until now, no efficient method (with polynomial complexity) is available to solve the problem involving job processing times P_j's ($j = 1, 2, \ldots, n$) which are fixed and known. In fact, Kubiak [4] has shown this problem to be NP-hard (also see Cai and Cheng [1]). In real life, the processing times are sometimes random (stochastic).

The stochastic version of the problem has been studied by Chakravarthy [2] and Vani and Raghavachari [7]. Chakravarthy [2] has shown that if all the P_j's have either same means or same variances, an optimal sequence belongs to the set of V-shaped sequences. Also, he has established that any optimal sequence is V-shaped when the P_j's follow exponential distributions. Vani and Raghavachari [7] have dealt with a more general case assuming that for each P_j, the second (raw) moment can be expressed as a quadratic function of the first moment (mean). They have derived, under some assumptions, sufficient conditions for the existence of an optimal sequence which is V-shaped in mean.

In view of the difficulty of finding an optimal sequence, research has been directed towards the nature of optimal sequences. Existence of an optimal sequence in the set of all V-shaped sequences, restricts the search to only 2^{n-1} sequences.

Section 2 of this article contains the basic notations and preliminary results. In Section 3, we derive a sufficient condition for existence of a V-shaped optimal sequence for arbitrary job processing times. Section 4 contains some results for a quadratic relation between mean and variance of processing times. Here, we simplify and improve upon a result of Vani and Raghavachari [7]. In Section 5, a realistic ordering assumption is made on processing times and some results concerning the form of optimal sequence are derived. A relaxed version of the sufficient condition of Section 3 is also given.

2. NOTATION AND PRELIMINARY RESULTS

The processing time P_j of job j, $1 \le j \le n$, is assumed to be random with mean μ_j and variance σ_j^2. Let us assume, without loss of generality, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ and, $\sigma_i^2 \ge \sigma_j^2$ whenever $\mu_i = \mu_j$ and $i < j$. All the P_j's are assumed to be independent. For a sequence $\pi = (\pi_1, \ldots, \pi_n)$, let $C_j(\pi)$ and $E_j(\pi)$ denote the completion time of the job in the jth position in the sequence π and its expected value respectively for $j = 1, 2, \ldots, n$, and $\bar{C}(\pi)$ and $\bar{E}(\pi)$ denote their averages respectively. Also let $V(\pi)$ denote the variance of the completion times and $E[V(\pi)]$ its expectation for the sequence π. Then we have

$$V(\pi) = \frac{1}{n} \sum_{j=1}^{n} [C_j(\pi) - \bar{C}(\pi)]^2.$$

The problem under consideration is to find a sequence π^* that minimizes $E[V(\pi)]$ over all π.

A sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is said to be V-shaped in mean if there exists a position r, $1 \le r \le n$, such that

$$\mu_{\pi_1} \ge \mu_{\pi_2} \ge \cdots \ge \mu_{\pi_{r-1}} \ge \mu_{\pi_r} \le \mu_{\pi_{r+1}} \le \cdots \le \mu_{\pi_n}.$$

Now we shall present some preliminary results which are required later to derive an optimal sequence and to find the form of optimal sequence.

LEMMA 1 (Vani and Raghavachari [7]): For a sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, the expected variance of completion times under π is given by,

$$E[V(\pi)] = \frac{1}{n} \sum_{r=1}^{n} [E_r(\pi) - \bar{E}(\pi)]^2 + \sum_{r=1}^{n} \frac{(r-1)(n-r+1)}{n^2} \sigma_{\pi_r}^2 \tag{1}$$

where $E_r(\pi) = \sum_{i=1}^{r} \mu_{\pi_i}$ and $\bar{E}(\pi) = (1/n) \sum_{r=1}^{n} E_r(\pi) = (1/n) \sum_{r=1}^{n} (n-r+1)\mu_{\pi_r}$.

It can be seen that the expression $E[V(\pi)]$ is independent of the parameters of first job π_1 in the sequence, but depends on the other jobs through their first two moments only. Note that the first term on the right-hand-side of (1) is the variance of expected job completion times. Let us denote this term by $S(\pi)$.

REMARK 1: It is clear from (1) that the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$ becomes the deterministic version of the problem, and hence the stochastic problem under consideration is also NP-hard.

LEMMA 2: Let $\pi = (\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-1}, \pi_n)$ and let $\pi^D = (\pi_1, \pi_n, \pi_{n-1}, \ldots, \pi_3, \pi_2)$. Then $E[V(\pi)] = E[V(\pi^D)]$.

This lemma has been proved independently by Chakravarthy [2] and Vani and Raghavachari [7]. Note that this is a generalization of the result of Merten and Muller [5] for deterministic case. The lemma proves the existence of at least two optimal sequences.

LEMMA 3 (Chakravarthy [2]): If $\sigma_1^2 = \max_{1 \le i \le n} \sigma_i^2$, then there exists an optimal sequence of the form $(1, \cdots)$.

It can be proved directly from Lemma 1 by observing that for any given sequence π,

(i) each P_i has nonnegative contribution to $E[V(\pi)]$ through its mean (μ_i) and variance (σ_i^2),
(ii) $E[V(\pi)]$ is independent of the parameters of the first job in π.

The following lemma evaluates the change in the expected variance due to interchange of two jobs in a sequence.

LEMMA 4: Let $\pi = (\pi_1, \ldots, \pi_n)$ be any sequence. Let π' be a sequence obtained from π by interchanging the two jobs π_s and π_t, ($s < t$) only. Then

$$n\{E[V(\pi')] - E[V(\pi)]\} = 2(\mu_{\pi_t} - \mu_{\pi_s})\left[\sum_{r=s}^{t-1}\{E_r(\pi) - \bar{E}(\pi)\} + \frac{(t-s)(n-t+s)}{2n}(\mu_{\pi_t} - \mu_{\pi_s})\right] - \frac{(t-s)(n-t-s+2)}{n}(\sigma_{\pi_t}^2 - \sigma_{\pi_s}^2). \quad (2)$$

PROOF: Let $D = n\{E[V(\pi')] - E[V(\pi)]\}$. Using Lemma 1, we can write

$$D = n[S(\pi') - S(\pi)] + \sum_{r=1}^{n} \frac{(r-1)(n-r+1)}{n}(\sigma_{\pi'_r}^2 - \sigma_{\pi_r}^2) \quad (3)$$

where $S(\pi')(S(\pi))$ is the variance of expected job completion times for the sequence $\pi'(\pi)$.

We have

$$E_r(\pi') = E_r(\pi) \qquad \text{for } r = 1, \ldots, s-1, t, t+1, \ldots, n \quad (4)$$

$$\text{and} \qquad E_r(\pi') = E_r(\pi) + (\mu_{\pi_t} - \mu_{\pi_s}) \qquad \text{for } r = s, s+1, \ldots, t-1 \quad (5)$$

and therefore

$$\bar{E}(\pi') = \bar{E}(\pi) + \frac{t-s}{n}(\mu_{\pi_t} - \mu_{\pi_s}). \quad (6)$$

Now, we can write

$$n[S(\pi') - S(\pi)] = \sum_{r=s}^{t-1} [E_r^2(\pi') - E_r^2(\pi)] - n[\bar{E}^2(\pi') - \bar{E}^2(\pi)]$$

$$= (\mu_{\pi_t} - \mu_{\pi_s}) \sum_{r=s}^{t-1} [2E_r(\pi) + (\mu_{\pi_t} - \mu_{\pi_s})]$$

$$- (t-s)(\mu_{\pi_t} - \mu_{\pi_s})\left[2\bar{E}(\pi) + \frac{t-s}{n}(\mu_{\pi_t} - \mu_{\pi_s})\right]$$

(using the equations (4), (5), and (6)).

By simplifying the above terms of right-hand-side, it can be seen that $n[S(\pi') - S(\pi)]$ is same as the first term on right-hand-side of (2). Because $\sigma^2_{\pi'_r} = \sigma^2_{\pi_r}$ for all r except $r = s$ and t, it can be easily verified that the second term on right-hand-side of (3) is same as that of (2).

Hence the lemma holds. ■

3. V-SHAPED PROPERTY OF OPTIMAL SEQUENCES

In this section, we derive sufficient conditions for the V-shaped property under a general assumption on job processing times, that is, they can have any arbitrary means and variances.

Let

$$\Delta = \max\left\{0, \max_{\mu_i \neq \mu_j} \frac{\sigma_i^2 - \sigma_j^2}{\mu_i - \mu_j}\right\}. \tag{7}$$

THEOREM 1: If for any three jobs r, s and t with $\mu_r > \max\{\mu_s, \mu_t\}$, the condition

$$2\mu_r + (n-1)(\mu_s + \mu_t) > 2(n-2)\Delta \tag{8}$$

holds, any optimal sequence $\pi = (\pi_1, \ldots, \pi_n)$ satisfies

i) V-shaped property in mean, and

ii) if $\mu_{\pi_i} = \mu_{\pi_{i+1}}$ then $\sigma^2_{\pi_i} \geq \sigma^2_{\pi_{i+1}}$ ($\sigma^2_{\pi_i} \leq \sigma^2_{\pi_{i+1}}$) for $i \leq n/2$ ($i \geq n/2 + 1$).

PROOF: Suppose the condition (8) holds for any three jobs r, s, and t. Let $\pi = (\pi_1, \ldots, \pi_n)$ be an optimal sequence which is not V-shaped in mean. Then, there exist three successive jobs π_i, π_{i+1}, π_{i+2} such that $\mu_{\pi_{i+1}} > \max\{\mu_{\pi_i}, \mu_{\pi_{i+2}}\}$. Let $\pi^{(1)}$ ($\pi^{(2)}$) be a sequence obtained from π by interchanging the two jobs π_i and π_{i+1} (π_{i+1} and π_{i+2}).

Let $D^{(1)} = n\{E[V(\pi^{(1)})] - E[V(\pi)]\}$ and $D^{(2)} = n\{E[V(\pi^{(2)})] - E[V(\pi)]\}$. The values of $D^{(1)}$ and $D^{(2)}$ are nonnegative since π is optimal. By Lemma 4, we have

$$D^{(1)} = 2(\mu_{\pi_{i+1}} - \mu_{\pi_i})\left[E_i(\pi) - \bar{E}(\pi) + \frac{n-1}{2n}(\mu_{\pi_{i+1}} - \mu_{\pi_i})\right] - \frac{(n-2i+1)}{n}(\sigma^2_{\pi_{i+1}} - \sigma^2_{\pi_i})$$

$$\text{and } D^{(2)} = 2(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\left[E_{i+1}(\pi) - \bar{E}(\pi) + \frac{n-1}{2n}(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\right]$$

$$- \frac{(n-2i-1)}{n}(\sigma^2_{\pi_{i+2}} - \sigma^2_{\pi_{i+1}}).$$

Let

$$Q = D^{(2)}/\{2(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\} - D^{(1)}/\{2(\mu_{\pi_{i+1}} - \mu_{\pi_i})\} \tag{9}$$

Then, we can write

$$Q = \mu_{\pi_{i+1}} + \frac{n-1}{2n}(\mu_{\pi_i} + \mu_{\pi_{i+2}} - 2\mu_{\pi_{i+1}}) - \frac{n-2i-1}{2n}\{(\sigma^2_{\pi_{i+2}} - \sigma^2_{\pi_{i+1}})/(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\}$$

$$+ \frac{n-2i+1}{2n}\{(\sigma^2_{\pi_{i+1}} - \sigma^2_{\pi_i})/(\mu_{\pi_{i+1}} - \mu_{\pi_i})\}$$

i.e., $2nQ = 2\mu_{\pi_{i+1}} + (n-1)(\mu_{\pi_i} + \mu_{\pi_{i+2}}) - (n-2i-1)\{(\sigma^2_{\pi_{i+2}} - \sigma^2_{\pi_{i+1}})/(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\}$

$$+ (n-2i+1)\{(\sigma^2_{\pi_{i+1}} - \sigma^2_{\pi_i})/(\mu_{\pi_{i+1}} - \mu_{\pi_i})\}$$

$$\geq 2\mu_{\pi_{i+1}} + (n-1)(\mu_{\pi_i} + \mu_{\pi_{i+2}}) - (n-3)\Delta - (n-1)\Delta$$

(since $1 \leq i \leq n-2$)

$$= 2\mu_{\pi_{i+1}} + (n-1)(\mu_{\pi_i} + \mu_{\pi_{i+2}}) - 2(n-2)\Delta.$$

Using inequality (8), we can see that $Q > 0$, that is,

$$D^{(2)}/\{2(\mu_{\pi_{i+2}} - \mu_{\pi_{i+1}})\} > D^{(1)}/\{2(\mu_{\pi_{i+1}} - \mu_{\pi_i})\} \geq 0$$

since $D^{(1)} \geq 0$. It means $D^{(2)} < 0$ which leads to the contradiction that π is not optimal. Therefore the optimal sequence π is V-shaped.

Next, it can be seen from second term on the right-hand-side of (1) that any optimal sequence π must have the property (*ii*).

Hence the theorem holds. ■

REMARK 2: From the above result, it can be seen that if the variances are homogeneous or small when compared to the means, then optimal sequence is more likely to be V-shaped in mean. However, if variances are very large in relation to means, optimal sequence tends to be V-shaped in variance (by Lemma 1).

The following result shows that when the coefficient of variation of each processing time does not exceed a limit determined by the means of processing times, any optimal sequence is V-shaped.

THEOREM 2: An optimal sequence is V-shaped in mean, if

$$\sigma_j/\mu_j \in [0, \sqrt{\mu_n/\{(\mu_1 + \mu_2) + \gamma\mu_1^2\}}\,] \tag{10}$$

for all j where $\gamma^{-1} = \min_{\mu_i \neq \mu_k} |\mu_i - \mu_k|$.

PROOF: Let $c = \max_j \sigma_j/\mu_j$ and $\sigma_i^2 = c^2\mu_i^2 - \epsilon_i$, $\epsilon_i \geq 0$ for $i = 1, 2, \ldots, n$. For any i and j with $\mu_i \neq \mu_j$, we have

$$\frac{\sigma_i^2 - \sigma_j^2}{\mu_i - \mu_j} = \frac{c^2(\mu_i^2 - \mu_j^2) + (\epsilon_i - \epsilon_j)}{\mu_i - \mu_j}$$

$$\leq c^2(\mu_i + \mu_j) + \left|\frac{\epsilon_j - \epsilon_i}{\mu_i - \mu_j}\right|$$

$$\leq c^2(\mu_1 + \mu_2) + \gamma \max_{1 \leq k \leq n} \epsilon_k$$

$$\leq c^2(\mu_1 + \mu_2) + \gamma c^2 \mu_1^2.$$

Thus

$$\Delta \leq c^2[(\mu_1 + \mu_2) + \gamma \mu_1^2] \leq \mu_n$$

due to (10).

For any r, s and t, we have

$$2\mu_r + (n-1)(\mu_s + \mu_t) \geq 2n\mu_n > 2(n-2)\Delta.$$

Now the result follows from Theorem 1. ■

4. QUADRATIC RELATION BETWEEN μ AND σ^2

Vani and Raghavachari [7] have considered a special case based on a quadratic relation between mean and variance of each processing time. They have assumed that $g_j = A\mu_j^2 + B\mu_j$, $1 \leq j \leq n$ for fixed nonnegative values of A and B, where g_j is the second (raw) moment of P_j, and observed that quite a few standard probability distributions have this property. For example, the distributions (*i*) Uniform (with interval starting from zero), (*ii*) Gamma (with fixed shape parameter), (*iii*) Chi-square, (*iv*) Poisson and (*v*) Binomial (with $A = (n-1)/n$ and $B = 1$) satisfy this property. They have proved that optimal sequence is V-shaped in mean if

$$\delta_i = \frac{A(n+1) - 2(A-1)i - 2}{2n} \geq 0, \qquad i = 1, 2, \ldots, n-2 \tag{11}$$

$$\alpha_j = \frac{(2-A)n + 2(A-1)j - A}{2n} \geq 0, \qquad j = 2, 3, \ldots, n-1 \tag{12}$$

In order to know that there exists an optimal sequence which is V-shaped in mean, we need to verify $2n - 4$ constraints.

We simplify this result and show that these constraints are either redundant or can be weakened depending upon the value of the parameter A.

THEOREM 3:

i) For $0 \leq A \leq 2$ and $B \geq 0$, optimal sequence is V-shaped in mean.
ii) For $A > 2$ and $B \geq 0$, optimal sequence is V-shaped in mean when $n < 5(A-1)/(A-2)$.

PROOF: Suppose there is an optimal sequence $\pi = (\pi_1, \ldots, \pi_n)$ that is not V-shaped in mean. Then there exists three successive jobs in π, say, π_i, π_{i+1} and π_{i+2} such that $\mu_{\pi_{i+1}} > \max\{\mu_{\pi_i}, \mu_{\pi_{i+2}}\}$.

We show that by interchanging π_{i+1} with either of the other two jobs yields a better sequence than π.

We use the same arguments as in the proof of Theorem 1. Here we can write

$$Q = \delta_i\mu_{\pi_i} + \frac{A}{n}\mu_{\pi_{i+1}} + \alpha_{i+1}\mu_{\pi_{i+2}} + \frac{B}{n} = \delta_i\mu_{\pi_i} + \left(\frac{1}{2n} + \frac{2A-1}{2n}\right)\mu_{\pi_{i+1}} + \alpha_{i+1}\mu_{\pi_{i+2}} + \frac{B}{n}$$

where α_i and δ_i are given in (11) and (12).

Since $\mu_{\pi_{i+1}} > \max\{\mu_{\pi_i}, \mu_{\pi_{i+2}}\}$, we can write

$$Q > \left(\delta_i + \frac{1}{2n}\right)\mu_{\pi_i} + \left(\alpha_{i+1} + \frac{2A-1}{2n}\right)\mu_{\pi_{i+2}} + \frac{B}{n} = \delta_i^*\mu_{\pi_i} + \alpha_{i+1}^*\mu_{\pi_{i+2}} + \frac{B}{n}$$

where $\delta_i^* = \delta_i + 1/2n$ and $\alpha_{i+1}^* = \alpha_{i+1} + (2A-1)/2n$. It is obvious that $\delta_i^* = \alpha_{n-i}^*$, for $i = 1, 2, \ldots, n-2$. We observe that

$$\delta_{n-2}^* \geq \cdots \geq \delta_1^* = \frac{1}{2n}\{A(n-1)+1\}$$

$$\alpha_2^* \geq \cdots \geq \alpha_{n-1}^* = \delta_1^*$$

for $0 \leq A \leq 1$ and

$$\delta_1^* \geq \cdots \geq \delta_{n-2}^* = \frac{1}{2n}\left[(2-A)n + 5(A-1)\right]$$

$$\alpha_{n-1}^* \geq \cdots \geq \alpha_2^* = \delta_{n-2}^*$$

for $A > 1$.

It implies that $Q > 0$ for $0 \leq A \leq 2$. Moreover, $Q > 0$ for $A > 2$ provided $n < 5(A-1)/(A-2)$. Hence the theorem holds. ■

5. ORDERED PROCESSING TIMES

In real-life situations, job processing time having larger mean is generally expected to have larger variance also. For this reason, we now assume that

$$\mu_r < \mu_s \rightarrow \sigma_r^2 < \sigma_s^2 \qquad \text{for any } r \text{ and } s. \tag{13}$$

The processing times are said to be *ordered* if the condition (13) is satisfied.

There may not exist a V-shaped optimal sequence even if the processing times are ordered. For instance, consider the following numerical example of nine-job problem with ordered processing times.

Job	1	2	3	4	5	6	7	8	9
μ	200	200	7	6	5	4	2	2	1
σ	50	50	42	41	40	36	35	3	2

The best among V-shaped sequences is (1, 3, 4, 5, 8, 9, 7, 6, 2) which gives 6020.9136 as expected CTV (variance of completion times). However, the sequence (1, 3, 4, 5, 8, 9, 6, 7, 2) is the best among all 9! sequences but not V-shaped. It gives 6017.4568 as the minimum expected CTV.

THEOREM 4: There exists an optimal sequence of the form $(1, \ldots, 2)$.

PROOF: It can be seen from Lemma 3 that there exists an optimal sequence of the form $(1, \cdots)$. Suppose $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is an optimal sequence with $\pi_1 = 1$. Let $\pi_n \neq 2$. Then $\pi_k = 2$ for some $2 \le k \le n - 1$.

For $k = 2$, we can see by Lemma 2 that the sequence $(\pi_1, \pi_n, \pi_{n-1}, \ldots, \pi_3, 2)$ is optimal. Let $3 \le k < n - 1$.

Suppose $\mu_2 > \max\{\mu_{\pi_2}, \mu_{\pi_n}\}$. Obtain $\pi^{(1)}$ $(\pi^{(2)})$ from π by interchanging only two jobs π_2 and π_k (π_k and π_n). Let $D^{(1)} = n\{E[V(\pi^{(1)})] - E[V(\pi)]\}$ and $D^{(2)} = n\{E[V(\pi^{(2)})] - E[V(\pi)]\}$. Then by Lemma 4, we have

$$D^{(1)} = 2(\mu_2 - \mu_{\pi_2})X - \frac{(k-2)(n-k)}{n}(\sigma_2^2 - \sigma_{\pi_2}^2)$$

$$D^{(2)} = 2(\mu_{\pi_n} - \mu_2)Y - \frac{(k-2)(n-k)}{n}(\sigma_2^2 - \sigma_{\pi_n}^2)$$

where

$$X = \sum_{r=2}^{k-1} [E_r(\pi) - \bar{E}(\pi)] + \frac{(k-2)(n-k+2)}{2n}(\mu_2 - \mu_{\pi_2})$$

$$Y = \sum_{r=k}^{n-1} [E_r(\pi) - \bar{E}(\pi)] + \frac{k(n-k)}{2n}(\mu_{\pi_n} - \mu_2).$$

Since, $\mu_2 > \max\{\mu_{\pi_2}, \mu_{\pi_n}\}$, we have $\sigma_2^2 > \max\{\sigma_{\pi_2}^2, \sigma_{\pi_n}^2\}$. We also have $D^{(1)} \ge 0$, $D^{(2)} \ge 0$ as π is optimal. It now follows that $Y < 0 < X$.

If $\bar{E}(\pi) \ge E_k(\pi)$, then

$$X < -\sum_{r=2}^{k-1} \mu_{\pi_k} + \frac{(k-2)(n-k+2)}{2n}(\mu_2 - \mu_{\pi_2})$$
$$= (k-2)\left[\left\{\frac{n-k+2}{2n} - 1\right\}\mu_2 - \frac{n-k+2}{2n}\mu_{\pi_2}\right] < 0$$

which contradicts $X > 0$. Similarly, if $\bar{E}(\pi) \le E_{k-1}(\pi)$, we arrive at a contradiction that $Y > 0$.

Therefore $E_{k-1}(\pi) < \bar{E}(\pi) < E_k(\pi)$. Let $\bar{E}(\pi) = E_k(\pi) - \epsilon$, for some $\epsilon > 0$. Now, we have

$$Y \ge (n-k)\left[\epsilon - \frac{k}{2n}\mu_2\right] \qquad \text{and} \qquad X \le (k-2)\left[\epsilon - \frac{n+k-2}{2n}\mu_2\right].$$

Then, $Y < 0 \rightarrow \epsilon < (k/2n)\mu_2 \rightarrow X < 0$ which is a contradiction.

Therefore, if π is optimal with $\pi_k = 2$ and $3 \le k \le n-1$, then $\mu_2 \not> \max\{\mu_{\pi_2}, \mu_{\pi_n}\}$.

Suppose $\mu_{\pi_2} = \mu_{\pi_k}$. If jobs π_2 and π_k are interchanged, the value of first term on the right-hand-side in (1) remains same whereas the second term does not increase. That is, interchange of jobs π_2 and π_k does not increase expected variance. Now it follows from Lemma 2 that $(\pi_1, \pi_n, \pi_{n-1}, \ldots, \pi_{k+1}, \pi_2, \pi_{k-1}, \ldots, \pi_3, 2)$ is at least as good as π.

Similarly, if $\mu_{\pi_n} = \mu_{\pi_k}$, we can show that $(\pi_1, \pi_2, \ldots, \pi_{k-1}, \pi_n, \pi_{k+1}, \ldots, \pi_{n-1}, 2)$ is at least as good as π. Hence the theorem holds. ■

The following theorem restricts the position of job n in any optimal sequence.

THEOREM 5: Let $\pi = (\pi_1, \ldots, \pi_n)$ be any optimal sequence with $(\pi_1, \pi_n) = (1, 2)$. Then $\mu_{\pi_2} \ge \mu_{\pi_3} \ge \mu_{\pi_4}$ for $n > 5$.

PROOF: Suppose $\mu_{\pi_4} > \mu_{\pi_3}$. Consider the sequence π' obtained from π by interchanging the jobs π_3 and π_4. By Lemma 4, we have

$$n\{E[V(\pi')] - E[V(\pi)]\} = 2(\mu_{\pi_4} - \mu_{\pi_3})\left[E_3(\pi) - \bar{E}(\pi) + \frac{n-1}{2n}(\mu_{\pi_4} - \mu_{\pi_3})\right] - \frac{(n-5)}{n}(\sigma^2_{\pi_4} - \sigma^2_{\pi_3}). \tag{14}$$

Note that

$$E_3(\pi) - \bar{E}(\pi) + \frac{n-1}{2n}(\mu_{\pi_4} - \mu_{\pi_3}) = \frac{1}{n}\left[(\mu_{\pi_2} - \mu_2) - \left\{\frac{n-5}{2}(\mu_{\pi_3} + \mu_{\pi_4}) + \sum_{r=5}^{n-1}(n-r+1)\mu_{\pi_r}\right\}\right] < 0.$$

Now, from (14) it is obvious that $E[V(\pi')] < E[V(\pi)]$, that is, π' is better than π which contradicts the optimality of π. Hence $\mu_{\pi_3} \ge \mu_{\pi_4}$.

We can argue similarly that $\mu_{\pi_2} \ge \mu_{\pi_3}$. ■

COROLLARY 1: There exists an optimal sequence of the form $(1, \ldots, 2)$ with job n in one of the positions $4, 5, \ldots, n-1$.

COROLLARY 2: If $\pi = (\pi_1, \ldots, \pi_n)$ is an optimal sequence with $(\pi_1, \pi_2) = (1, 2)$, then $\mu_{\pi_n} \geq \mu_{\pi_{n-1}} \geq \mu_{\pi_{n-2}}$ for $n > 5$.

This follows from Theorem 5 and Lemma 2.

COROLLARY 3: For $n = 5$, the sequence (1, 3, 4, 5, 2) is optimal.

This can be proved following the arguments given in Theorem 4 and using Lemma 2.

LEMMA 5: In any optimal sequence $\pi = (\pi_1, \ldots, \pi_n)$, $\mu_{\pi_k} \leq \max\{\mu_{\pi_{k-1}}, \mu_{\pi_{k+1}}\}$ for $k = \lfloor n/2 \rfloor + 1$ when n is even and for $k = \lfloor n/2 \rfloor + 1$ and $\lfloor n/2 \rfloor + 2$ when n is odd.

The proof is given in Appendix.

COROLLARY 4: For $n = 6$ and 7, there exists a V-shaped (in mean) optimal sequence of the form $(1, \ldots, 2)$.

PROOF: If $n = 6$, it follows trivially from Theorem 5 that any optimal sequence of the form $(1, \ldots, 2)$ is V-shaped in mean. For $n = 7$, one can easily verify that any optimal sequence of the form $(1, \ldots, 2)$ is V-shaped in mean due to Theorem 5 and Lemma 5.

REMARK 3: Further, it can be seen from the proof of the result of Vani and Raghavachari [7] concerning the position of the third largest job (for deterministic case) and Lemma 5 that for $n = 6$ and 7, there exists a V-shaped (in mean) optimal sequence of the form $(1, 3, \ldots, 2)$.

We now derive a sufficient condition for V-shapedness of optimal sequence for ordered processing times.

THEOREM 6: If for any three jobs r, s and t with $\mu_r > \max\{\mu_s, \mu_t\}$,

$$2\mu_r + (n-1)(\mu_s + \mu_t) > (n-5)\Delta \tag{15}$$

where Δ is as defined in (7), there exists an optimal sequence $\pi = (\pi_1, \ldots, \pi_n)$ which satisfies

i) $\pi_1 = 1$,
ii) V-shaped property in mean, and
iii) if $\mu_{\pi_i} = \mu_{\pi_{i+1}}$ then $\sigma^2_{\pi_i} \geq \sigma^2_{\pi_{i+1}}$ ($\sigma^2_{\pi_i} \leq \sigma^2_{\pi_{i-1}}$) for $i \leq n/2$ ($i \geq n/2 + 1$).

PROOF: Since the processing times are ordered, (*i*) follows directly from Lemma 3. (*ii*) and (*iii*) can be proved using the same arguments as in the proof of Theorem 1. ■

Note that the condition (15) is weaker than the condition (8).

6. DISCUSSION

We have dealt with the problem of sequencing jobs having random processing times on a single machine to minimize expected variance of job completion times.

We have derived sufficient conditions for V-shaped property of optimal sequences for general random processing times with known means and variances. Theorem 2 enables to confine our search for optimal sequence to V-shaped sequences only when the coefficient of variation of each job processing time is bounded by a specified value. It happens in most of the real-life situations since standard deviation is very small when compared to the mean. We have made another realistic assumption that the job processing times are ordered, that is, if $\mu_i > \mu_j$ for some i and j, then $\sigma_i > \sigma_j$. A numerical example has been provided to show that even under this assumption, there may not exist a V-shaped optimal sequence. However, simple sufficient conditions have been derived for the existence of V-shaped optimal sequence. We have also shown that V-shaped optimal sequences exist for $n \leq 7$ if the processing times are ordered.

APPENDIX

PROOF OF LEMMA 5: Suppose $\mu_{\pi_k} > \max\{\mu_{\pi_{k-1}}, \mu_{\pi_{k+1}}\}$. Obtain $\pi^{(1)}$ ($\pi^{(2)}$) from π by interchanging the jobs π_{k-1} and π_k (π_k and π_{k+1}). We shall show that $\pi^{(1)}$ is better than π.

Let $D^{(i)} = n\{E[V(\pi^{(i)})] - E[V(\pi)]\}$, $i = 1, 2$. From Lemma 4, we have

$$D^{(1)} = 2(\mu_{\pi_k} - \mu_{\pi_{k-1}})X - \frac{n - 2k + 3}{n}(\sigma^2_{\pi_k} - \sigma^2_{\pi_{k-1}})$$

$$D^{(2)} = 2(\mu_{\pi_{k+1}} - \mu_{\pi_k})Y - \frac{n - 2k + 1}{n}(\sigma^2_{\pi_{k+1}} - \sigma^2_{\pi_k})$$

where

$$X = E_{k-1}(\pi) - \bar{E}(\pi) + \frac{n - 1}{2n}(\mu_{\pi_k} - \mu_{\pi_{k-1}})$$

$$Y = E_k(\pi) - \bar{E}(\pi) + \frac{n - 1}{2n}(\mu_{\pi_{k+1}} - \mu_{\pi_k}).$$

It is obvious that $Y > X$.

Note that for $k = \lfloor n/2 \rfloor + 1$ and $\lfloor n/2 \rfloor + 2$ with odd n and $k = \lfloor n/2 \rfloor + 1$ with even n, we have $n - 2k + 1 \leq 0 \leq n - 2k + 3$. Since π is optimal, we have $D^{(2)} \geq 0$, that is, $2(\mu_{\pi_{k+1}} - \mu_{\pi_k})Y \geq 0$ which implies $Y < 0$. As $X < Y$, it now follows that

$$D^{(1)} < 2(\mu_{\pi_k} - \mu_{\pi_{k-1}})Y - \frac{n - 2k + 3}{n}(\sigma^2_{\pi_k} - \sigma^2_{\pi_{k-1}}) < 0$$

since $n - 2k + 3 \geq 0$. It means that the sequence $\pi^{(1)}$ is better than π which contradicts the optimality of π.

Hence the lemma holds. ■

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