REPLACEMENT STRATEGIES FOR AGEING ASSETS WITH SPECIFIC REFERENCE TO COCONUT PALMS IN KERALA

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Note on the Convention

Reference to Displays is of the form (x. y) where x is the Chapter number and y is the serial number of the display within the Chapter. For example, (2.3) refers to the third display in the second Chapter.

Reference to Sections is of the form x. y where x is the Chapter number and y is the serial number of the Section within the Chapter. Please note that parentheses are used in the case of display reference.

Reference to Sub-sections is of the form x. y. z where x is the Chapter number, y is the Section number within the Chapter and z is the Serial number of the Sub-section within the Section. For example, 6.2.1 refers to the first Sub-section of the second section in the Sixth Chapter.

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Chapter I

Replacement Problems in the Case of Coconut Palms : An Introduction to the Present Study

1.1 Introduction

1.1.1 Production assets which deteriorate in performance with time (or age) are required to be replaced. In general, a stream of benefits and costs are associated with every productive asset, be it a machine or a tree. Usually, in replacement theory, the benefits and costs are taken as given functions of the age of the asset. These functions provide the criteria for identifying the physical condition of the asset. In simple replacement models, an asset is replaced by an identical one. The objective of a replacement policy is to find a sequence of time points (or alternatively, replacement ages) for replacing successive pieces of the asset that maximises some given objective function based on the stream of benefits and costs over the time-horizon of the investment process. Discounted net returns, average cost over the investment period etc., are the most common type of objective functions used in the literature.

1.1.2 A considerable literature exists on the optimal replacement of assets in the deterministic case. This is a situation where the return from an asset at a given point of time is assumed to be non-random — a given function of time or age of the asset. The objective of these studies was to provide an understanding of the various principles of asset replacement. One of the questions asked is : How is the replacement date related to the annuity formed by the sum of the discounted annual earnings and other such economic variables ? Most of these studies consider a static situation where a piece of an asset is replaced by an identical one over an infinite horizon. See for example, Preinreich (1940), Perin (1972). Etherington (1977) gives a brief bibliography of studies on the economies of replacement. Dean (1961) has summarised various deterministic replacement models developed by different authors for application to industrial problems. In this study also we shall consider a static situation but our focus will be on the analytical characteristics of the replacement age i.e., how the replacement age is related to the parameters that specify the performance of the asset over time etc. We shall further attempt a comparison between the solutions corresponding to the finite and infinite horizon cases.

1.1.3 Studies which include stochastic elements in the replacement models have generally concentrated on 'unintentional' replacement. Here the concern was with an asset that 'dies' unexpectedly. See for example Burt (1965), Jorgenson (1967). A light bulb is a typical example of this situation. The performance of this asset is stable over time but it ceases to perform all of a sudden. Thus, it's life is stochastic in nature. But it is quite raro to find a replacement study in the other kind of stochastic situation where there is no sudden 'death' of the asset but it produces benefits of stochastic nature and deteriorates over time. An example of this is the deterioration in yield of perennial crops like coconut palms, rubber trees etc. At any given point of time (usually year as a unit) the level of yield is stochastic and is dependent on the age of the plant. Etherington (1977) dealt with a similar problem for rubber trees. He made a simulation study of the sensitivity of the net revenue and annuity profiles to different parameters that regulate the stochastic behaviour of the yield and prices. In his model the optimum replacement date was at the point where annual returns plus the change in the salvage value of the asset is equal to the annuity formed by the discounted sum of the annual earnings plus the salvage value.

Ward and Faris (1968) developed a stochastic model for the optimal replacement of plum trees. They employed a Markov Chain formulation to specify the movement from one age-yield state to another. Dynamic programming was used to find out the optimum policy. A general mathematical model for Markov replacement decisions (i.e., decisions at a time point which depend on the age-performance at the previous point) is provided by Howard (1960) and (1971) using principles of Dynamic programming. A good demonstration of the applicability of Howard's model is given by Kao (1973).

1.1.4 The Markov Chain approach is developed essentially for the class of assets whose performance at a given point of time depends not only on the age but also on past performance. This is a feature common to a number of perennial crops. Here we deal with replacement problems for one such perennial crop, namely, coconut palms. Although the formulations presented here are in reference to the palm, the results obtained are general enough to be applicable to the case of any asset characterised by the feature mentioned above (with necessary modifications specific to each case). In the next section, we shall introduce the essential features of coconut palms required for the study.

1.2 Characterisation of Coconut Palms

1.2.1 The yield profile of a palm over it's life span can be divided into four phases. A palm starts yielding after a few years of gestation during which period the seedling grows. This period constitutes the first phase of a palm's life — the 'pre-bearing phase'. During the initial years of bearing the annual yield of the palm increases till it stabilises at a certain level. This period of increasing annual yield constitutes the second phase. The annual yield hereafter remains stable for a number of years before it starts declining. This is the third stage — the full bearing phase. During the fourth phase the annual yield declines till the palm dies. A palm is said to be senile in it's fourth phase of life.

The above is, however, a broad profile of the annual yield of a palm over it's life. What follows is a more precise description.

1.2.2 Let $Y(x)$ be the annual yield (number of nuts) of a palm at age x, $x = 1, 2, \ldots, L$, where L is the life of a palm i.e., L is the age at which the palm yields for the last time. In reality, there is a considerable inter-palm variation in the annual yield at a given age. Also, the life of a palm varies from palm to palm. Thus, a general formulation would require the joint distribution of $(Y(x), L)$ where both $Y(x)$ and L are random variables defined for a palm. But, for simplicity, we shall consider throughout this study the conditional distribution $(Y(x) | L)$: in other words, L is assumed to be a given constant. One of the reasons for considering the conditional distribution is the lack of adequate empirical information on thelife of palms.

$$
\begin{aligned}
&E(Y(x) | L) = \mu_x \\
(1.1) \qquad &V(Y(x) | L) = \sigma_x^2 \\
&x = 1, 2, \ldots\ldots, L
\end{aligned}
$$

Both μ_x and σ_x^2 are obviously functions of L but for the purpose of subsequent development we need not specify the functions since L is assumed to be fixed. Hence, throughout this study the results are derived for a given L and we shall accordingly omit further references to variations in L. We shall now define the following parameters to specify the mean yield sequence $\{ \mu_x \}$.

(1.2)

$a_f = \text{Min} \{ x : \mu_x > 0 \}$, the first bearing age i.e., the beginning of second phase of a palm's life.

$a_s = \text{Max} \{ x : \mu_{x-1} < \mu_x \}$, beginning of the full bearing or the third phase.

$a_e = \text{Min} \{ x : \mu_x > \mu_{x+1} \}$, beginning of the senile or the fourth phase.

$a_f \leq a_s \leq a_e \leq L$

On the basis of an examination of the yield data available on sixty palms*, we postulate the following.

* Data were provided by the Central Plantation Crop Research Institute, Kasaragod, Kerala.

(1.3) (i) $V(Y(x)) = \sigma_x^2 = \sigma^2$, a constant, $x \geq a_f$
$= 0$, for $x < a_f$

(ii) $\text{Cov}\,(Y(x),\ Y(x')) = \sigma^2\,\rho_{xx'} > 0;\ x \neq x';\ x,\ x' \geq a_f$

where $\rho_{xx'}$ decreases with $|x - x'|$, a_f is given in (1.2).

(iii) $(Y(1),\ Y(2),\ \ldots.,\ Y(L))$ has a Multivariate Normal distribution.

This assumption is based on the fact that $Y(x)$ is found to have a symmetric distribution for a given x. We may note here that most of the literature dealing with the stochastic nature of asset performance makes a similar assumption. But $Y(x)$ are usually assumed to be stochastically independent.

(iv) We shall further assume that the dispersion matrix of $(Y(a_f)\ \ Y(a_f+1),\ \ldots.,\ Y(L))$ is positive definite.

We shall further assume the following piece-wise linear form for the mean yield profile of a palm.

$$
\begin{aligned}
\mu_x &= 0 && \text{for } 1 \leq x < a_f \\
&= m_1 + s_1\,(x - a_f) && \text{for } a_f \leq x < a_s \qquad (1.4)\\
&= m_2 && \text{for } a_s \leq x < a_e \\
&= m_2 - s_2\,(x - a_e) && \text{for } a_e \leq x \leq L
\end{aligned}
$$

where $m_1,\ m_2,\ s_1,\ s_2$ are positive constants.

We may, however, note that except in Chapter VII where the rate of decline in the mean yield is estimated, the results which follow do not require any assumption of a specific functional form of μ_x such as (1.4). The above form has been used chiefly for the empirical exercises.

1.2.3 The characterising parameters defined above for a palm, (1.1) through (1.4), vary with the variety of palms. For a given variety of palm, they also depend upon - (i) cultivation practices (manuring, watering etc.) adopted i.e., the level of management of the coconut garden and (ii) the density of plantation i.e., number of palms per unit area. The exact nature of sensitivity of the parameters to the changes in the above two factors is not known.

For the purpose of this study, we shall be concerned with a single variety of palm under given cultivation practices and density of plantation.

1.2.4 There are two ways of replacing a palm by another. One way is to remove the existing palm and plant a new seedling in it's place. The other way is to plant a new seedling somewhere in the proximity of the existing palm to be replaced, and remove the old palm when it becomes necessary to get rid of it in order to allow the new one to grow. We shall refer to these two ways as replacement by 'replanting' and 'underplanting' respectively.

Replacement by underplanting is preferable because it reduces the effective gestation period of the new palm : the poor-yielding old palm can be retained for sometime with marginal benefits. But underplantation requires adequate space around the palm to be replaced, which means it is feasible only for a certain range of the density of plantation.

In any case, we shall consider underplantation as our method of replacement. It will be easy to modify the results for the case of replantation.

1.3 Data

1.3.1 The life of coconut palms being very long (over 80 years for certain varieties) it is difficult to gather adequate amount of data over the whole life of palms of a given variety. The State of Kerala is one of the important coconut growing regions of India with about one fourth of it's cultivated land being under coconut palms. Annual production of nuts in Kerala is of the order of 4,000 millions which is about 62 per cent of all India production. The Central Plantation Crops Research Institute (CPCRI) situated in North Kerala has been conducting experimental research on coconut palms for over two decades. The Institute has generated some data on the yield response to manuring, irrigation etc. Moreover, this Institute has a few plots with old palms of which yield data are available for a major part of their life. We have drawn upon this data source for the purpose of formulating different models and empirical exercises.

1.3.2 The data presented here are for a particular variety of palms traditionally grown in Kerala, namely, West Coast Tall (WCT).

(i) Life of a palm : WCT palms are found to have a very long life. As it is mentioned earlier, the life of a palm varies from palm to palm. There are no data available on the distribution of the life of palms. However, it is generally believed that an average of 80 years of life span should be a reasonable assumption. Thus, we put

$$(1.5) \qquad L = 80$$

In other words, for our purpose, we shall assume that the life of a palm is given to be 80 years.

(ii) <u>Yield profile</u> : As noted earlier, the mean yield profile of a palm depends on the cultivation practice. We have used the data from CPCRI experimental plots which provide adequate information on the annual yields of WCT palms in the second and third phase of life. The palms were under rainfed condition and a specified level of manuring etc. For a given density of 100 palms per acre which is the average stand (of palms) per acre in Kerala, we shall assume the following.

(1.6)

$$a_f = 9 \text{ years} \qquad m_1 = 15 \text{ nuts per annum}$$
$$a_s = 12 \text{ years} \qquad m_2 = 60 \text{ nuts per annum}$$
$$a_e = 60 \text{ years} \qquad s_1 = 15 \text{ nuts per annum}$$
$$s_2 = 3 \text{ nuts per annum}$$

(iii) <u>Variance, Covariance and Correlation</u> : From the CPCRI data we have already referred to, it was found that the correlation between $Y(x)$ and $Y(x+1)$ is more or less a constant for $x \geq a_f$. We have assumed the following.

(1.7)

$$\sigma^2 = 900$$
$$\rho_{xx'} = 0.62 \qquad \text{for } x' = x+1, \quad x \geq a_f$$
$$= \pi^{|x-x'|-1} \, 0.62$$ [redacted] $\text{for } x, x' > a_f$

where π is a constant, $0 < \pi < 1$.

We shall assume different values for π in our empirical exercises.

(iv) <u>Cost (annual) of Cultivation and Price of Nuts</u> : Again CPCRI provided the following data at 1974-75 prices.

(a) Price of a coconut (with husk) Rs. 0.90.

(b)

Year (Age)	Annual Cost (Rs.) per palm
1	7.14
2	6.43
3	7.43
4	10.29
5	10.57
6	10.57
7	10.86
8	11.14
9 onwards	11.63

The cost for the first year includes the cost for the seedling and planting also. Annual cost consists of labour charges for cultivating (digging pit, clearing crown etc.), cost of manuring and harvesting.

1.3.3 In order to examine the sensitivity of the of the empirical results to the above specification of different parameters we have tried different values for certain parameters (namely, a_e, s_2, L, π, $\rho_{x,\ x+1}$, σ^2 etc.) in various exercises.

1.4 A Brief Description of the Contents of the Study

1.4.1 In Chapter II, we have considered the deterministic case where optimal replacement age has been derived on the basis of the mean yield profile $\{\mu_x,\ x = 1, 2, \ldots, L\}$. The objective function has been taken as the discounted sum of annual net returns over a given time-horizon. Both infinite and finite horizon cases are considered. The analytical properties of the optimal replacement age are studied separately for the above two cases. Then the optimal solutions corresponding to the two cases are compared.

1.4.2 In Chapter III, we consider an alternative formulation in the deterministic version. Instead of the discounted value as the objective function the time-path (or the trajectory) of the annual net return is considered as the basis for choosing a replacement rule. As the annual net return is a function of the age distribution of the palms, the relationship between a replacement rule and the corresponding limiting stationary age distribution is studied. The case of replacement in phases is also considered as a modification of the optimal solution obtained in Chapter II.

1.4.3 In Chapter IV, we make an attempt to modify the deterministic rule obtained in Chapter II by incorporating the dependence of future yield stream of a palm on it's past yield record. This is done by replacing the future mean yield profile (μ_i, i = x, x+1, ..., L) by (μ_i', i = x, x+1, ..., L) where μ_i' is the conditional mean yield at age i given the past yield record Y(x-1), Y(x-2), ...,Y(x-n), $n < x$, x is the current age of the palm. Conditions under which the modified rule performs better than the deterministic rule are obtained for a certain given criterion.

1.4.4 In Chapter V, we consider stochastic replacement rules in a general set up. Replacement rules are considered on the basis of the corresponding risks of retaining a low yielding palm and removing a satisfactorily yielding one. Efficiency of a rule is defined as the probability that the rule recommends replacement given that the future yields are low. We have confined ourselves to only linear forms of objective and decision functions. Rules are derived on the basis of the decision functions given the objective function. The decision function which provides the most efficient rule is investigated for a given objective function. Further, the relationship between the efficiency of a rule and the common variance σ^2 of Y(i) is studied.

1.4.5 In Chapter VI, we consider a special case of the stochastic version where the yield vector (Y(1), Y(2), ..., Y(L)) is a Markov Chain. A Markov Reward Process formulation is adopted to develop a procedure for deriving a Markov Replacement Rule i.e., replacing or retaining a palm in the current year depending on the yield in the previous year. The choice of rule is based on the extent of improvement in the discounted value over the optimal level obtained in the deterministic case discussed in Chapter II.

1.4.6 In Chapter VII, we consider the problem of estimating the future mean yield profile of a palm i.e., $\{ \mu_i, i = x, x + 1, \ldots, L\}$ where x is the current age of the palm. As the replacement rules are particularly based on the above sequence of mean yield, it becomes necessary to estimate them when the current age of the palm is unknown. A few methods are developed for the estimation and their relative merits are studied by simulation.

1.4.7 In Chapter VIII, we summarise the study.

Chapter II

Deterministic Replacement Policy Based On Discounted Sum Of Annual Net Returns

2.1 Introduction

2.1.1 In this chapter, we shall develop replacement models on the basis of the mean annual yield profile given by μ_a (see 1.2). Since the replacement rules will be based on mean yields and not actual yields (which are random variables) the models in this chapter can be regarded as deterministic. As mentioned earlier, a considerable literature on replacement problems in economics dealt with the deterministic case. The exercises presented here will resemble, in some respects, the earlier work.

2.1.2 In the deterministic version, a replacement rule is defined by specifying the age upto which a palm can be retained before it is replaced. The replacement age is to be determined by optimising some specified objective function. The proposed objective function is the discounted sum of the annual net returns (value of the annual yield minus the annual cost incurred) over a given time-horizon. This is one of the objective functions most commonly used in replacement theory.

We shall consider replacement by underplantation. This means that if we decide to retain a palm upto age A, a seedling is underplanted when the palm reaches age (A+1) and the old palm is removed after a few years when it cannot be retained any more without hampering the growth of the underplanted seedling (see 1.2.4). Let u be the number of years a palm can be retained after a seedling has been underplanted to replace it. Hence, the old palm will be removed when it reaches age (A+u+1). We shall define age (A+1) as the 'replacement age'.

Note that if we put u=0, it becomes equivalent to replacement by replantation; Since, in this case, the old palm will be removed at age (A+1) itself.

2.2 Formulation of a Replacement Model with Infinite Horizon

2.2.1 Let us consider a palm of current age x. At some point of time this palm is to be replaced by another new palm (of the same variety) which will subsequently be replaced by another and so on. This defines an infinite sequence of palms beginning with the existing palm of current age x, followed by the subsequent palms that will be planted in it's place over time. Let $(1+A_i)$ be the replacement age of the ith palm in the above sequence, i = 1, 2, 3, ... This means when the existing palm reaches age $(1+A_1)$ a seedling is underplanted and the palm is removed when it reaches age $(1+u+A_1)$. Similarly, when the palm that has replaced the existing palm, reaches age $(1+A_2)$ another seedling is underplanted and the palm is removed when it reaches age $(1+u+A_2)$, and so on. We assume that all underplanting and removal are performed at the beginning of the age specified.

Thus, for every age x (x = 1, 2, ..., L) there is now defined an age sequence $(A_1, A_2, ...)$. By a replacement rule we would mean a specified age sequence $(A_1, A_2, ...)$ for a given palm of current age x.

2.2.2 Let us use the following notations :

p : Price of a coconut, a constant price.

$s(x)$: Annual cost of maintaining a palm at age x, $s(1)$ will include the price of the seedling and the cost of planting besides the cost of maintenance which consists of cost of manuring etc.

d : Discount factor

$r(x) = p\,\mu_x - s(x)$: Annual expected net return from a palm at age x.

Throughout this chapter, by actual return from a palm we would mean the expected return since we are dealing with the mean annual yield profile.

For any palm of current age x if the replacement age is $(1+A)$ where $x \leq A \leq L$, we define the following :

$$
\begin{aligned}
Z(x, A) &= \sum_{n=x}^{A+u} r(n) d^{n-x}, \qquad x \leq A < L-u \\
&= \sum_{n=x}^{L} r(n) d^{n-x}, \qquad L-u \leq A \leq L
\end{aligned} \tag{2.1}
$$

It is easy to see that $Z(x, A)$ is the discounted sum of net returns from a palm of current age x if it's replacement age is given to be $(1+A)$. The summation runs from age x to age $\mathrm{Min}(L, A+u)$ since the palm will be retained till age $\mathrm{Min}(L, A+u)$ given the replacement age to be $(1+A)$.

Let $V(A_1, A_2, \ldots)$ be the discounted sum of net returns from the infinite sequence of palms (corresponding to a single palm at current age x) given the replacement rule $(A_1, A_2, \ldots)$.

Let us now consider a replacement rule $(A_1, A_2, \ldots)$. We shall treat $V(A_1, A_2, \ldots)$ itself as the objective function. Obviously, from (2.1), $Z(x, A_1)$ is the contribution of the existing palm to the discounted sum V.

Similarly, d^{1-x+A_1} $Z(1, A_2)$ is the contribution of the second palm in the sequence to V, where the multiplier is to adjust for the discounting over time. The contribution of the third palm in the sequence will be, $d^{1-x+A_1+A_2}$ $Z(1, A_3)$. And so on. Thus, we can write,

$$V(A_1, A_2, \ldots\ldots) = Z(x,A_1)+d^{1-x+A_1} Z(1,A_2) + d^{1-x+A_1+A_2} Z(1,A_3) + \ldots\ldots\ldots \tag{2.2}$$

$$= Z(x,A_1) + d^{1-x+A_1} \sum_{k=2}^{\infty} d^{a(k)} Z(1, A_k)$$

where $a(k) = 0$ for $k = 2$

$= A_2 + A_3 + \ldots\ldots + A_{k-1}$ for $k > 2$.

Let us define

$$Z'(A_1) = Z(x, A_1) \tag{2.3}$$

$$Z''(A_2, A_3, \ldots\ldots) = \sum_{k=2}^{\infty} d^{a(k)} Z(1, A_k)$$

Then we have from (2.2),

$$V(A_1, A_2, \ldots\ldots) = Z'(A_1) + d^{1-x+A_1} Z''(A_2, A_3, \ldots\ldots) \tag{2.4}$$

In (2.4) above, we have divided the total discounted sum into two parts. The first part $Z'(A_1)$ accounts for the contribution due to the initial palm and the second part $Z''(A_2, A_3, \ldots\ldots)$ accounts for the contribution due to the subsequent palms. The multiplier to the second term adjusts for the discounting.

It is easy to see that

$$\max_{A_1, A_2, \ldots} V(A_1, A_2, \ldots\ldots)$$

(2.5)

$$= \max_{A_1} \{ Z'(A_1) + d^{1-x+A_1} \max_{A_2, A_3, \ldots} Z''(A_2, A_3, \ldots\ldots) \}$$

From (2.5) it is clear that we can find out optimal A_1 and the optimal A_2, A_3, ... separately. We shall refer to the replacement of the existing (initial) palm as the first cycle of replacement, replacement of the palm which replaces the existing palm as the second cycle of replacement and so on.

2.3 Determining the Optimal A_1 i.e., Optimal Replacement Age for the First Cycle

2.3.1 From (2.4) we have

(2.6)

$$\max_{A_1, A_2, \ldots} V(A_1, A_2, \ldots\ldots)$$

$$= \max_{A_1} \{Z'(A_1) + d^{1-x+A_1} \max_{A_2, A_3, \ldots} Z''(A_2, A_3, \ldots\ldots)\}$$

Let us define,

(2.6a)

$$E = \max_{A_2, A_3, \ldots} Z''(A_2, A_3, \ldots\ldots)$$

$$Z'''(A_1) = Z'(A_1) + d^{1-x+A_1} E$$

Note that E is independent of A_1. Therefore, from (2.6) and (2.6a) we have,

(2.6b)

$$\max_{A_1, A_2, A_3, \ldots} V(A_1, A_2, A_3, \ldots\ldots) = \max_{A_1} Z'''(A_1).$$

Thus, the optimal A_1 is the one which maximises $Z'''(A_1)$ given in (2.6a). Before determining A_1 let us note that in the deterministic version we need to consider replacement of a palm only in it's declining yield phase i.e., when the palm is senile. Again, since we are considering replacement by underplantation, a palm should be retained till the age (L-u) at the latest so that the effective gestation period between removal of the old palm and the first bearing of the new palm can be minimised. Thus, assuming that current age $x < a_e$, we shall restrict our choice of the optimal A_1 within the range $a_e \leq A_1 < L-u$ where a_e is the age at which the declining yield phase sets in (see 1.4).

2.3.2 Theorem : If $r(L) \leq (1 - d)d^{-u}$ E then the optimal A_1 satisfies

(2.7)
$$r(A+u) > (1-d)d^{-u} E \qquad \text{for } A = a_e, \ldots, A_1$$
$$r(A+u) \leq (1-d)d^{-u} E \qquad \text{for } A = 1+A_1, \ldots, L-u$$

If $r(L) > (1-d)d^{-u}E$ then optimal A_1 is equal to L-u. E is defined in (2.6a).

Proof : From (2.6b) we already know that optimal A_1 is the one that maximises $Z'''(A)$, $a_e \leq A \leq L-u$.

Now, from (2.6a), (2.6b), (2.3) and (2.1) we can write

(2.8)
$$Z'''(1+A) - Z'''(A) > 0 \qquad \text{if and only if}$$
$$r(1+u+A) > (1-d)d^{-u} E \qquad \text{for } a_e \leq A < L-u$$

But $r(x) = p\ \mu_x - s(x)$ (from 2.2.2) and from (1.4) it can be seen that μ_x is decreasing for $x \geq a_e$. Therefore, $r(1+u+A)$ is decreasing in A for $a_e \leq A < L-u$.

Hence, if $r(L) \leq (1-d)d^{-u} E$, it easily follows from (2.8) that there exists A_1 satisfying (2.7) which maximises $Z'''(A)$ i.e., $Z'''(A_1) \geq Z'''(A)$ for $a_e \leq A \leq L-u$. If $r(L) > (1-d)d^{-u} E$ then it trivially follows from (2.8) that $A_1 = L-u$ maximises $Z'''(A)$ since $r(1+u+A)$ is decreasing in A.

Hence the theorem.

2.3.3 Let us note that the above theorem in 2.3.2 does not ensure the existence of a unique optimal A_1. It can be easily seen from (2.7) that if there there exists A' such that $r(1+u+A') = (1-d)d^{-u} E$, $a_e \leq A' < L-u$, then $Z'''(A') = Z'''(1+A') \geq Z'''(A)$ for $a_e \leq A \leq L-u$. This means that both A' and $(1+A')$ will be optimal. However, for our purpose we shall characterise A_1 in the following way.

$$A_1 = \text{Max } \{ A;\ r(A+u) > (1-d)d^{-u} E, \quad a_e \leq A \leq L-u \}$$

(2.9) $$\text{if } r(a_e+u) > (1-d)d^{-u} E$$

$$= a_e \quad \text{if } r(a_e+u) \leq (1-d)d^{-u} E$$ where E is defined in (2.6a)

Obviously, A_1 thus defined is unique, maximises $Z'''(A)$, $a_e \leq a \leq L-u$, and hence is the optimal age in the first cycle of replacement.

While determining A_1 we started with the assumption that current age $x < a_e$. Now it easily follows from the above theorem and discussion that if $a_e \leq x \leq A_1$ then optimal replacement age for the palm will be A_1 (defined in (2.9)), and if $x > A_1$ then the optimal age will be x itself.

2.4 Determining Optimal A_2, A_3, ..., i.e., Optimal Replacement Ages for the Second Cycle, Third Cycle and so on.

2.4.1 In the previous section 2.3 we have characterised the optimal replacement age for the first cycle of replacement. We shall now do the same for the subsequent cycles.

From (2.1) through (2.5) it is clear that optimal A_2, A_3, ... are the ones that maximise $Z''(A_2, A_3, ...)$. Now, from (2.3) and (2.6) we have

$$
\begin{aligned}
E &= \max_{A_2,A_3,\ldots} Z''(A_2, A_3, \ldots) \\
&= \max_{A_2,A_3,\ldots} \sum_{k=2}^{\infty} d^{a(k)} Z(1, A_k) \\
&= \max_{A_2,A_3,\ldots} \{Z(1, A_2) + d^{A_2} Z(1, A_3) + \ldots\ldots\} \\
&= \max_{A_2} \left[Z(1, A_2) + d^{A_2} \max_{A_3,A_4,\ldots} \{Z(1,A_3) + d^{A_3} Z(1,A_4) + \ldots\} \right] \\
&= \max_{A_2} \{ Z(1, A_2) + d^{A_2} E \}
\end{aligned}
\tag{2.10}
$$

Obviously, if A_2, A_3, ... maximise $Z''(A_2, A_3, ...)$ then $A_2 = A_3 = ...$

So, let us denote the common optimal replacement age for the second and subsequent cycles by $(1+A_2)$. Then, we have,

$$
\begin{aligned}
E &= \sum_{k=2}^{\infty} p^{(k-2)A_2} Z(1, A_2) \\
&= Z(1, A_2) / (1 - d^{A_2})
\end{aligned}
\tag{2.11}
$$

Thus, A_2 is the one that maximises $Z(1,A)/(1-d^A)$, $a_e \leq A \leq L-u$.

2.4.2 We shall now characterise A_2 in the same fashion as we did in the case of A_1 for the first cycle of replacement (see (2.9)). To do this we shall need the following lemma.

Lemma : Let $\{f_n\}$ and $\{g_n\}$ be two sequences of positive real numbers such that

$$
\begin{aligned}
f_{n+1} &= f_n + d_n \\
g_{n+1} &= g_n + e_n \qquad (2.12) \\
d_n &= k \cdot h_n \cdot e_n
\end{aligned}
$$

where h_n is non-increasing, k is a constant. Then there exists a subscript m such that

$$\frac{f_m}{g_m} \geq \frac{f_n}{g_n} \quad \text{for all } n.$$

Proof : Now it follows from (2.12) that

$$(2.13) \quad \frac{f_{n+1}}{g_{n+1}} \leq \frac{f_n}{g_n} \iff \frac{f_n + d_n}{g_n + e_n} \leq \frac{f_n}{g_n} \iff \frac{f_n}{g_n} \geq \frac{d_n}{e_n} = k.\ h_n$$

We shall now show that

$$\frac{f_n}{g_n} \geq kh_n \implies \frac{f_{n+1}}{g_{n+1}} \geq kh_n$$

Suppose

$$\frac{f_n}{g_n} \geq k.\ h_n$$

Then

$$f_{n+1} = f_n + d_n$$

$$= f_n + k.\ h_n .\ e_n$$

(2.14) $$\geq k.\ h_n .\ g_n + k.\ h_n .\ e_n$$

$$= k.\ h_n\ g_{n+1}$$

$$\geq k.\ h_{n+1} .\ g_{n+1}$$

Let

$$m = \text{Min} \{ n : k.\ h_n \leq \frac{f_n}{g_n} \}$$

We shall show

$$\frac{f_m}{g_m} \geq \frac{f_n}{g_n} \qquad \text{for all } n$$

From (2.13) we have

(2.15) $$k.\ h_m \leq \frac{f_m}{g_m} \Longrightarrow \frac{f_{m+1}}{g_{m+1}} \leq \frac{f_m}{g_m}$$

From (2.14) we have

$$k.\ h_m \leq \frac{f_m}{g_m}$$

(2.16) $$\Longrightarrow k.\ h_{m+1} \leq \frac{f_{m+1}}{g_{m+1}}$$

$$\Longrightarrow \frac{f_{m+2}}{g_{m+2}} \leq \frac{f_{m+1}}{g_{m+1}} \qquad \text{by (2.12)}$$

From (2.15), (2.16) it is clear that

$$\frac{f_m}{g_m} \geq \frac{f_n}{g_n} \qquad \text{for } n > m$$

Again by definition of m above we have

$$k.h_n > \frac{f_n}{g_n} \quad \text{for all } n < m$$

$$\Longrightarrow \frac{f_{n+1}}{g_{n+1}} > \frac{f_n}{g_n} \quad \text{for all } n < m, \text{ by } (2.12).$$

Hence the lemma.

2.4.3 We can now characterise A_2. Let us define the common replacement age $(1+A_2)$ for the second and subsequent cycles as –

$$(2.17) \quad A_2 = \text{Max}\,\{A : r(A+u) > d^{-u}(1-d)\,Z(1, A-1)/(1-d^{A-1}),\ a_e \leq A \leq L-u\}$$

$$\text{if } r(a_e+u) > d^{-u}(1-d)\,Z(1, a_e-1)/(1-d^{a_e-1})$$

$$= a_e \qquad \text{if } r(a_e+u) \leq d^{-u}(1-d)\,Z(1, a_e-1)/(1-d^{a_e-1})$$

<u>Theorem</u> : The common replacement age $(1+A_2)$ for the second and subsequent cycles given by (2.17), is optimal.

<u>Proof</u> : We have to show that A_2 maximises $Z(1,A)/(1-d^A)$, $a_e \leq A \leq L-u$ (see 2.10).

Let $f_n = Z(1, a_e + n - 1)$

$$g_n = 1 + d + d^2 + \ldots\ldots + d^{a_e + n - 2}$$

$$n = 1, 2, \ldots., L - u - a_e + 1.$$

It is easy to see that the sequence

$$Z(1, A)/(1 - d^A), \qquad a_e \leq A \leq L - u$$

is identical with the sequence

$$f_n/(1 - d)g_n, \qquad 1 \leq n \leq L-u-a_e+1$$

in the sense that the corresponding terms are the same.

Now,

$$f_{n+1} - f_n = d^u \ r(a_e + u + n) \ d^{a_e + n - 1}$$

Let us put

$$k = d^u$$

$$h_n = R(a_e + u + n)$$

Again,

$$g_{n+1} - g_n = d^{a_e + n - 1}$$

So, let us put

$$e_n = d^{a_e + n - 1}$$

Thus, we have

$$d_n = f_{n+1} - f_n = k.\ h_n.\ e_n$$

and h_n is non-increasing from (1.2) and 2.2.2.

Now by applying the lemma in 2.4.2 we can see that there exists m such that

$$\frac{f_m}{g_m} \geq \frac{f_n}{g_n} \quad \text{for all} \quad n = 1, 2, \ldots., L - u - a_e + 1.$$

where

$$m = \text{Min} \{ n : k.\ h_n \leq \frac{f_n}{g_n} \}$$

Considering now the equivalent sequence

$$Z(1,A)/(1-d^A), \quad a_e \leq A \leq L-u,$$

$$\text{of } f_n/(1-d)g_n, \qquad n = 1, 2, \ldots.., L - u - a_e + 1$$

it is easy to see that A_2 given in (2.16) maximises $Z(1,A)/(1-d^A)$.

Hence the theorem.

2.5 Existence of Constant Cycle of Replacement in the Infinite Horizon Case

2.5.1 Theorem : The optimal replacement age $(1+\Lambda_1)$ for the first cycle and the common optimal replacement age $(1+\Lambda_2)$ for the subsequent cycles, given by (2.9) and (2.17) respectively, are the same i.e., $\Lambda_1 = \Lambda_2$.

Proof : (i) Let

$$r(a_e+u) \leq d^{-u}(1-d) \quad Z(1, a_e-1)/(1-d^{a_e-1})$$

Now from (2.10) and (2.11)

$$d^{-u}(1-d)E \geq d^{-u}(1-d) \; Z(1, a_e-1)/(1-d^{a_e-1})$$

since E is maximum of $Z(1, \Lambda)/(1 - d^{\Lambda})$.

Hence, $r(a_e + u) \leq d^{-u}(1-d)E$

Comparing (2.9) and (2.17) we have $\Lambda_1 = \Lambda_2$:

(ii) Let us now suppose

$$r(a_e+u) > d^{-u}(1-d) \; Z(1, a_e - 1)/(1-d^{a_e-1})$$

Again, we can write from (2.17) that

$$(2.18) \qquad r(\Lambda+u) \leq d^{-u}(1-d)E \qquad \text{for } \Lambda = 1 + \Lambda_2, \ldots, L-u$$

since E is the maximum of $Z(1,\Lambda)/(1-d^{\Lambda})$.

Now, we shall show that

$$r(\Lambda+u) > d^{-u}(1-d)E \qquad \text{for } \Lambda = a_e, \ldots, \Lambda_2.$$

As in the proof of the theorem in 2.4.3 let us consider $f_n/(1-d)g_n$ the equivalent sequence of $Z(1, \Lambda)/(1 - d^{\Lambda})$.

Let

$$m = \text{Min}\{ n : k.\, h_n \leq \frac{f_n}{g_n} \}$$

Then,

$$k.\, h_{m-1} > \frac{f_{m-1}}{g_{m-1}} \quad \text{by definition of } m$$

$$\Longleftrightarrow k.\, h_{m-1}\, g_{m-1} > f_{m-1}$$

(2.19)
$$\Longleftrightarrow k.\, h_{m-1}\, g_{m-1} + d_{m-1} > f_{m-1} + d_{m-1}$$

$$\Longleftrightarrow k.\, h_{m-1}\, g_{m-1} + k.\, h_{m-1} \cdot e_{m-1} > f_m$$

$$\Longleftrightarrow k.\, h_{m-1}\, g_m > f_m$$

$$\Longleftrightarrow k.\, h_{m-1} > \frac{f_m}{g_m}$$

$$\Longleftrightarrow k.\, h_n > \frac{f_m}{g_m} \quad \text{for } n = 1, 2, \ldots, m-1$$

since h_n is non-increasing in n.

Coming back to sequence $Z(1, A)/(1-d^A)$, (2.19) means that

$$r(A_2+u) > d^{-u}\,(1-d)\;\; Z(1, A_2-1)/(1-d^{A_2-1})$$

(2.20)
$$\Longleftrightarrow r(A+u) > d^{-u}(1-d)\, Z(1, A_2)/(1-d^{A_2})$$

$$= d^{-u}(1-d)E \qquad \text{for } A = a_e, \ldots, A_2.$$

Comparing (2.18), (2.20) with (2.9) we have $A_1 = A_2$.

Hence the theorem.

2.5.2 Thus, in the infinite horizon case we have found that there exists a constant cycle of replacement which is optimal. We shall refer to A^* as the Optimal Economic Life of a palm where A^* is given by,

(2.21)
$$A^* = \text{Max}\,\{ A : r(A+u) > d^{-u}(1-d)\, Z(1, A-1)/(1-d^{A-1}),\ a_e \leq A \leq L-u \}$$
$$\text{if } r(a_e+u) > d^{-u}\,(1-d)\, Z(1, a_e-1)/(1-d^{a_e-1})$$
$$= a_e \quad \text{otherwise.}$$

Srinivasan (1967) had found similar optimal constant cycle policy in the context of expansion of industrial capacity in the face of growing demand.

2.6 Replacement Model with Finite Horison

2.6.1 In the earlier sections we dealt with the objective function which was concerned with the discounted net returns from the palms over an infinite horizon. Consideration of infinite horizon in similar studies has become a convention. It is often mathematically convenient and also obviates arbitrariness in fixing a finite length of horizon. However, we shall study the finite horizon model as a special case of interest. We shall confine ourselves only to the first cycle of replacement, particularly, we shall consider a horizon of length not more than the given life of a palm.

2.6.2 Let H be the length (in years) of the horizon, $H \leq L$. As earlier, let x be the current age of the palm. In the finite horizon case, it will be convenient to consider the number of years a given palm can be retained before replacement. Let y be the number of years a palm of age x is retained. This means that a seedling is underplanted at age (x+y) of the palm, and the palm is removed at age (x+y+u); in other words, the replacement age is (x+y-1). Now, the underplantation should be performed not later than a point so that a positive net return from the new palm can be realised within the period of the horizon. So, we shall consider $a_f < H$ and $y \leq H - a_f$ where a_f is the first bearing age of a palm.

Let us define for a given palm of age x,

$$(2.22) \quad W(y,H) = \sum_{n=1}^{y+u} r(x+n-1)d^{n-1} + \sum_{n=y+1}^{H} r(n-y)d^{n-1}$$

$$\text{for } y = 0, 1, 2, \ldots, H - a_f$$

$$a_f < H \leq L$$

$$\text{and } r(n) = 0 \quad \text{for } n > L$$

W(y,H) is the discounted net returns from the existing palm and the new palm. The first term accounts for the net returns from the existing palm and the second term accounts for the same from the new palm.

In general, optimal y for a given age x and horizon H can be found out by directly computing W(y, H) for y = 0, 1, 2, ..., $H - a_f$. However, it is not possible to study the properties of the optimal y i.e., it's relationship with x and H. It would be interesting to restrict ourselves to the palms which are in their declining yield phase and are going to reach age (L-u) within $(H-a_f)$ years. This means that we shall consider a palm of age $x \geq \text{Max}\ (L - H - u + a_f + 1,\ a_e)$. Let us put,

$$(2.22a) \qquad x_o = \text{Max}\ (L - H - u + a_f + 1,\ a_e).$$

The following theorems are concerned with the relationship between the optimal y on one hand and x, H on the other.

2.6.3 <u>Theorem</u> : If the horizon is such that $H \geq L-u - (a_e - a_s) + 1$, then the optimal number of years y, a palm can be retained is given by

$$(2.23) \quad y = 1 + \text{Max}\ \{ y' : r(x+u+y') > d^{-u}\ D(y', H);\ \ y' = 0, 1, \ldots, L-u-x \}$$

$$\text{if } r(x+u)\ d^{-u}\ D(0, H).$$

$$= 0 \text{ otherwise}$$

$$\text{where } D(y', H) = (1-d) \sum_{n=1}^{H-y'-1} r(n)d^{n-1} + r(H-y')d^{H-y'-1}$$

Proof : Optimal y is the one that maximises W(y, H) given in (2.22).

Now, from (2.22) we have

$$W(y+1,H) - W(y,H) = r(x+u+y)d^{u+y} + \sum_{n=y+2}^{H}\{r(n-y-1) - r(n-y)\}d^{n-1} - r(1)d^{y}$$

$$= r(x+u+y)d^{u+y} - d^{y}(1-d)\sum_{n=1}^{H-y-1} r(n)d^{n-1} - r(H-y)d^{H-1}$$

for $y \leq L - u - x$.

Therefore,

$$W(y+1, H) > W(y, H).$$

if and only if

(2.24) $$r(x+u+y) > d^{u}\{(1-d)\sum_{n=1}^{H-y-1} r(n)d^{n-1} + r(H-y)d^{H-y-1}\}$$

for $y < L - u - x + 1$

Let us define

(2.25) $$D(y,H) = (1-d)\sum_{n=1}^{H-y-1} r(n)d^{n-1} + r(H-y)d^{H-y-1}$$

for $y \leq L - u - x$

Then,

(2.26) $$D(y+1,H) - D(y,H) = d^{H-y-1}\{r(H-y-1) - r(H-y)\}, \quad y \leq L-u-x$$

Now,

(2.27)
$$H - y - 1 \geq H-L+u+x-1 \qquad \text{since } y \leq L - u - x$$

$$\geq H-L+u+a_e-1 \qquad \text{since } x \geq a_e$$

$$\geq L-u-(a_e - a_s) + 1 - L+u+a_e - 1$$

since $H \geq L-u-(a_e-a_s) + 1$

$$= a_s$$

Hence,

$$r(H-y-1) - r(H-y) \geq 0, \qquad y \leq L - u - x,$$

since $r(n)$ is non-increasing for $n \geq a_s$.

Therefore, from (2.26), $D(y, H)$ is non-decreasing in y, $y \leq L - u - x$. But $r(x + u + y)$ is non-increasing in y since $x \geq a_e$.

Now the theorem follows from (2.24).

2.6.4 We can summarise the above two sub-sections as follows :

For a given palm of age x such that $x_o \leq x \leq L-u$ (x_o given by (2.22a)), if the horizon H satisfies $H \geq L-u-(a_e - a_s) + 1$ then the optimal replacement age A is given by $A = x+y$, where y is given by (2.23).

Lemma : The optimal replacement age A is independent of the age x of the old palm if

$$r(x+u) > d^{-u} D(0, H).$$

Proof : Let $x_1 = \text{Max}\ (L - H - u + a_f + 1, a_e)$ and A_{x_1} be the corresponding optimal replacement age given by

$$A_{x_1} = x_1 + y_1$$

where

$$y_1 = 1 + \text{Max}\ \{ y' : r(x_1+u+y') > d^{-u} D(y', H) \} \tag{2.28}$$

when $D(y', H)$ is given by (2.25).

$$y' = 0, 1, 2, \ldots, L-u-x_1$$

Let us now consider a palm of age x_2 such that

$$x_1 < x_2 \leq L-u, \qquad r(x_2 + u) > d^{-u} (0, H).$$

Let A_{x_2} be the corresponding optimal replacement age given by

$$A_{x_2} = x_2 + y_2$$

Then,

$$y_2 = 1 + \text{Max}\,\{\, y' : r(x_2+u+y') > d^{-u}\, D(y', H) \}$$

(2.29) $$= 1 + \text{Max}\,\{\, y' : r(x_1+u+y' + c) > d^{-u}\, D(y', H) \}$$

where c is such that $x_2 = x_1 + c$.

From (2.28) and (2.29) it is obvious that

$$y_1 = y_2 + c$$

Therefore,

$$A_{x_2} = x_2 + y_2 \quad = x_1 + y_1 \quad = A_{x_1}$$

Hence the lemma.

2.6.5 Let us note from theorem 2.6.3 that the optimal number of years y a palm can be retained is dependent on the length of the horizon H.

Let us denote y by y_H for a given H. The following theorem shows the relation between H and y_H.

<u>Theorem</u> : Supoose $a_e - a_s \geq L - u - a_e + 1$. Then,

(i) $y_{H+1} = y_H$ for $L - u - (a_e - a_s) + 1 \leq H < a_e$

(ii) $y_{H+1} \geq y_H$ for $a_e \leq H \leq L$.

<u>Proof</u> : From (2.25) we have

(2.30) $$D(y, H+1) - D(y, H) = d^{H-y}\,\{r(H-y+1) - r(H-y)\}, \quad y \geq 0.$$

From (2.27) we have $(H-y) > a_s$, hence

$$r(H-y+1) \leq r(H-y) \quad \text{for } y \geq 0$$

Further, for $H < a_e$ we have $H - y \leq H < a_e$ since $y \geq 0$.

Therefore, we can write

$$r(H-y+1) = r(H-y) \quad \text{for } y \geq 0,\ L-u-(a_e-a_s)+1 \leq H < a_e \text{ since } H-y < a_e$$

(2.31) and

$$r(H-y+1) \leq r(H-y) \quad \text{for } y \geq 0,\ a_e \leq H \leq L.$$

So, from (2.30) and (2.31) we have

$$D(y, H+1) = D(y, H) \quad \text{for } L - u - (a_e - a_s) + 1 \leq H < a_e$$

(2.32) and

$$D(y, H+1) \leq D(y, H) \quad \text{for } a_e \leq H < L \; ; \quad \text{for } y \geq 0.$$

(i) Suppose $r(x+u) \leq d^{-u} D(0, H)$ where x is the age of the old palm. Then from (2.32) we have

$$r(x+u) \leq d^{-u} D(0, H) \leq d^{-u} D(0, H+1)$$

Hence,

$$y_H = y_{H+1} = 0 \quad \text{for } L-u-(a_e - a_s) + 1 \leq H < L.$$

(ii) Suppose $r(x+u) > d^{-u} D(0, H)$. Now, since y_H is optimal we can write from (2.23) that

(2.33)
$$r(x+u+y) > d^{-u} D(y, H) , \quad y = 0, 1, \ldots, y_H$$
$$r(x+u+y) \leq d^{-u} D(y, H) , \quad y = 1 + y_H, \ldots, L-u-x.$$

But since for $L-u-(a_e-a_s) + 1 \leq H < a_e$ we have from (2.32) that $D(y, H+1) = D(y, H)$, it is easy to see that y_{H+1} also will satisfy (2.33).

Hence $y_{H+1} = y_H$ for $L-u-(a_e - a_s) + 1 \leq H < a_e$.

Again, from (2.32) and (2.33) it can be seen that for $a_e \leq H < L$,

$$r(x+u+y) > d^{-u} D(y, H) \geq d^{-u} D(y, H+1) \quad \text{for } y = 0, 1, \ldots, y_H.$$

Hence $y_{H+1} \geq y_H$ for $a_e \leq H < L$, and hence the theorem.

2.7 Comparison of the Optimal Solutions Corresponding to Infinite Horizon and Finite Horizon

2.7.1 For the finite horizon case we considered a palm of current age x such that $x \geq x_o$ where x_o is given by (2.22a), $a_f \leq H \leq L$.

Let $x' = \text{Max}\,(L - H - u + a_f + 1, a_e)$.

From section 2.6 we have that, for $H \geq L - u - (a_e - a_s) + 1$, the optimal replacement age A^H of the palm is given by,

(2.34)
$$A^H = x_o + y^H$$
where
$$y^H = \text{Max}\,\{ y : r(x_o+u+y) > d^{-u} D(y, H); \quad y = 0, 1, 2, \ldots, L-u-x_o \},$$
if $r(x_o+u) > d^{-u} D(0, H)$
$$D(y, H) = (1-d) \sum_{n=1}^{H-y-1} r(n) d^{n-1} + r(H-y) d^{H-y-1}$$

If we consider an infinite horizon for the same palm then, from section 2.5, we have that the optimal replacement age A^∞ of the palm is given by,

(2.35)
$$A^\infty = x_o + y^\infty$$
where
$$y^\infty = \text{Max}\{ y : r(x_o+u+y) > d^{-u} (1-d) \frac{Z(1, x_o+y-1)}{1-d^{x_o-1}}; \quad y = 0, 1, 2, \ldots, L-u-x \}$$
if $r(x_o+u) > d^{-u} (1-d) \dfrac{Z(1, x_o-1)}{1-d^{x_o-1}}$.

The theorem in 2.6.5 provides some insight into the relationship between the length of the horizon and the corresponding optimal replacement age. It would be interesting to compare y^H and y^∞ (defined above). But it is difficult to investigate the order relation between y^H and y^∞ in general since it . will also depend on the specific functional form of the age-specific mean yield profile, rate of discount etc. However, one intuitively feels that when H is sufficiently small y^H and y^∞ should be different in general. The following theorem substantiates this intuition.

2.7.2 Theorem : Let us consider a palm of age x such that $x \geq x_0$ where x_0 is defined above. Let A^{∞} and A^{H} be the optimal replacement ages in the infinite and finite horizon cases given by (2.35) and (2.34) respectively. Then, $A^{\infty} \geq A^{H}$ if $H \leq a_e + u$.

Proof : It is enough to prove $y^{\infty} \geq y^{H}$ when $H \leq a_e + u$.

We have found in sub-sections 2.4.1 and 2.5.1 that A^{∞} is the one that maximises $Z(1, A)/(1 - d^{A})$. In otherwords, y^{∞} is the one that maximises $Z(1, x_0 + y)/(1 - d^{x_0+y})$, $y = 0, 1, 2, \ldots, L - x_0 - u$.

It can be seen from (2.1) that,

$$\{Z(1, x_0+y+1)/(1-d^{x_0+y+1})\} - \{Z(1, x_0+y)/(1 - d^{x_0+1})\}$$

(2.36) $$= \{d^{x_0+y-1}/(1-d^{x_0+y})(1-d^{x_0+y-1})\} \times \{r(x_0+y+u)d^{u} - dD(y, x_0+2y+u)\}$$

Again, from (2.23) we have,

$$D(y, x_0+2y+u) - D(y+1, x_0+2(y+1)+u)$$

(2.37) $$= d^{x_0+y+u+1}\{r(x_0+y+u) - r(x_0+y+u+1)\} \geq 0$$

since $r(x_0+y+u) \geq r(x_0+y+u+1)$.

Hence, $D(y, x_0+2y+u)$ is decreasing in y.

Therefore, from (2.37) we can write

$$r(x_0+y+u)d^{u} - dD(y, x_0+2y+u)$$

(2.38) $$> r(x_0+y+u)d^{u} - D(y, x_0+2y+u)$$

$$\geq r(x_0+y+u)\,d^{u} - D(y, H)$$

since $x + 2y + u \geq x_0 + u$

$\geq a_e + u$ (as $x_0 \geq a_e$)

$\geq H$ (as $H \leq a_e + u$)

Now, from (2.34), (2.38) and (2.36) we have

$$Z(1, x_0 + y^H) > Z(1, x_0 + y) \quad \text{for } y = 0, 1, \ldots, y^H - 1.$$

Hence, $y^{\infty} \geq y^H$ for $H \leq a_e + u$.

Hence the theorem.

2.8 Empirical Exercise

2.8.1 Let us now work out a few numerical examples with the results obtained in the previous sections for the purpose of illustration.

In order to examine the sensitivity of the optimal replacement age to the different parameters, we have chosen three different values for a_e i.e., the age beyond which the mean yield starts declining. We have further chosen two different rates of decline (s_2) of the mean yield for each value of a_e. These values will provide us six different mean yield profiles of the form specified in (1.4). For the purpose of comparison we specified three different time-horizons in each case. They are given by -

$$H = \infty$$

$$= L - u - (a_e - a_s) + 1$$

$$= L - 5$$

The second value for H specified above corresponds to the lower bound of H for which the results are derived in the finite horizon case (see 2.5.3). The third value above has been taken as a large value for H.

Let us note that for different values of a_e and s_2 the life of a palm (i.e., L) will be different in each case. The rest of the parameters will remain the same as specified in 1.3.

We have worked out the optimal age for two different rates of discount in each case. The results are presented below.

2.8.2 Table: Optimal Replacement Age Corresponding to Different Values of the Parameters

a_e : Age (in years) beyond which the mean yield starts declining.

s_2 : Rate of decline (nuts per year) of the mean yield.

L : Life of a palm in years

d : Rate of discount in per cent

H : Time-horizon in years

A^H : Optimal replacement age for time-horizon H.

a_e	s_2	L	d	H	A^H
60	3	80	5	29	66
				75	66
				∞	66
60	3	80	10	29	70
				75	70
				∞	70
60	2	90	5	39	68
				85	69
				∞	69

a_e	s_2	L	d	H	A^H
60	2	90	10	39	75
				85	75
				∞	75
55	3	75	5	29	61
				70	62
				∞	61
55	3	75	10	29	65
				70	65
				∞	65
55	2	85	5	39	63
				80	65
				∞	64
55	2	85	10	39	70
				80	70
				∞	70
40	3	60	5	29	46
				55	47
				∞	47
40	3	60	10	29	50
				55	50
				∞	50
40	2	70	5	39	48
				65	51
				∞	50
40	2	70	10	39	55
				65	55
				∞	55

2.8.3 It seems from the above table that the sensitivity of the optimal replacement age to the time-horizon increases with lower life span, lower rate of decline of the mean yield and lower rate of discount. In the case of the values assumed for different parameters mentioned in 1.3, the optimal replacement age is insensitive to the length of the time-horizon. However, rate of discount influences the optimal age considerably.

Chapter III

Deterministic Replacement Policy Based On Trajectory Of Net Annual Returns

3.1 Introduction

3.1.1 Let us consider a collection of palms in a given area. The number and age distribution of the palms in a given year determines the total annual yield of the year (also the annual cost and the net return). Given the age distribution of the palms in the initial year, the manner in which palms are replaced every year will determine the age distribution in the subsequent years. Thus, the total annual yield trajectory over the years will depend on the manner in which the old palms are replaced every year. A perusal of the results on the optimal replacement age (infinite horison case) presented in section 2.2 will show that the deterministic replacement rule does not depend on the initial age distribution. The optimal rule specifies an age so that, all the palms that cross the specified age in a given year are to be selected for replacement in that year.

3.1.2 In this chapter we shall first examine how the age distribution changes over time under such an optimal rule (as discussed in the previous chapter) and then consider a model which relates a given replacement rule with the changes in the age distribution and consequently the total yield trajectory.

3.2 Impact of the Deterministic Replacement Rule on the Age Distribution Over Time

3.2.1 Let (1+A) be the optimal replacement age i.e., a palm is to be replaced at age (1+A). We have already discussed this optimal replacement age in the previous chapter, section 2.2. For convenience, we shall for the time being consider replacement by replantation (see 1.2.4), later we can incorporate underplantation easily. Thus, in the initial year we remove all the palms of age above A then in every subsequent year we remove all the palms that enter age (1+A) in the given year. Let n_x be the number of palms at age x in an arbitrary year t. We shall write the age distribution in a given year (after completing replacements of the year) as $(n_1, n_2, \ldots, n_A)$ where the ith component gives the number of palms at age i. It is easy to see that the age distribution will change as follows.

Year	Age Distribution
t	$(n_1, n_2, n_3, \ldots, n_{A-1}, n_A)$
t+1	$(n_A, n_1, n_2, \ldots, n_{A-2}, n_{A-1})$
t+2	$(n_{A-1}, n_A, n_1, \ldots, n_{A-3}, n_{A-2})$
...	... ...
t+A-1	$(n_2, n_3, n_4, \ldots, n_A, n_1)$
t+A	$(n_1, n_2, n_3, \ldots, n_{A-1}, n_A)$

It is clear from above that at every (A+1)th year the age distribution becomes the same as it was A years before. More precisely, the age distribution in (t+A)th year is same as that in the tth year

for $t \geq 1$. We can say that under the optimal replacement rule the age distribution rotates over the years and the period for one complete rotation is A years, where A is the optimal replacement age. The annual age distributions over the period of one complete rotation will be determined by the initial age distribution existing when the replacement scheme begins.

3.2.2 As noted already, the annual distributions over a period of rotation will determine the total annual yield trajectory over the period which will repeat itself in the successive periods. Although the discounted stream of net returns from this periodically repetitive yield trajectory is maximum, there may be two undesirable consequences. First, an initial age distribution which has a large proportion of palms within a narrow range of age, will lead to a trajectory with a upward swing followed by a similar downward swing. This is easily verifiable by considering a distribution which has all the palms at the same age. Secondly, the number of palms to be replaced in the initial year may be (actually very likely to be) very large both in proportion as well as in absolute number. This will mean a heavy initial expenditure for the replacement scheme.

It may be noted here that the two consequences of the optimal replacement rule outlined above are partly related to each other. Replacement of a large proportion of palms in the initial year will not only mean a heavy initial expenditure and low annual net returns in the immediately following years but also will generate a distribution which has a large proportion of palms at a single age that will contribute to the fluctuations in the yield trajectory. It is easy to see that more the distribution deviates from a uniform one more will be the year to year fluctuations in the total yield.

3.3 A Model for Considering the Yield Trajectory

3.3.1 We are interested in the formulation of a model depicting the relationship between a replacement rule and the consequent yield trajectory. We shall confine ourselves to the deterministic framework used in the previous chapter.

Since we are considering replacement of palms only in their declining yield phase, let us club all the palms above a certain age a ($a \geq a_e$) into a single age group and refer to the same as the 'old-age' group. We shall define the replacement rule here as follows.

First we specify a certain proportion, to be called the 'rate of underplantation'. Every year we select a proportion of palms, equal to the rate of underplantation, from the old-age group, and underplant an equal number of seedlings near them. Palms are selected in the discending order of their age starting with the oldest palm. Also, we remove all the palms that were selected for replacement u years before (where u is as defined in 2.2.1). Let us note that under the above replacement rule the number of palms (inclusive of the seedlings) existing in a given year will change from year to year. For convenience, throughout this chapter, by age distribution we shall mean number of palms in various age groups. Given a rate of underplantation we can now generate the annual age distribution of the palms from year to year. The annual yield trajectory can thus be computed from the generated age distributions.

It will be shown here that given a rate of underplantation the age distribution converges to a stationary one and, consequently, under certain conditions, the annual total yield becomes stable. We are particularly interested in the stable yield level as a function of the rate of underplantation.

3.3.2 Let us define the following matrix G for generating the age distribution from year to year for a given rate of underplantation. G generates the age distribution in the <u>nth</u> year given the age distribution of the (n-1)<u>th</u> year. For our purpose, we will be interested in the distribution of the existing palms, in a given year, into (a +u+1) exclusive and exhaustive age groups. The columns of G correspond to these age groups in year n and the rows correspond to the year (n-1). Thus, the (i,j)<u>th</u> entry in G accounts for the proportion of palms in age group i in year (n-1) moving into age group j in year n.

$$
(3.1)\quad G = \begin{array}{c} \\ 1 \\ 2 \\ \cdots \\ a \\ a^{+} \\ v_1 \\ \cdots \\ v_u \end{array}
\begin{array}{c}
\begin{array}{ccccc:ccccc} 1 & 2 & 3 & \cdots & a & a^{+} & v_1 & v_2 & \cdots & v_u \end{array} \\
\left[\begin{array}{ccccc:ccccc}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
q & 0 & 0 & \cdots & 0 & 1-q & q & 0 & \cdots & 0 \\
\hdashline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right]
\end{array}
$$

where $0 < q < 1$.

The collection of palms of age above 'a' consist of some which are not yet selected for replacement and the rest which have been selected for replacement in the year n or (n-1) ...... or (n-u-i). Column a^+ corresponds to the palms above age a excluding those already selected for replacement. v_1 corresponds to the palms selected in n th year for replacement and so on.

In a given year n, all palms which were in age group i in the (n-1)th year will move to the next age group, i = 1, 2, ..., a, $v_1, v_2, \ldots, v_{u-1}$. From the age group a^+ a proportion q of the palms are isolated for underplanting. Thus, a proportion q of the palms moves from age group a^+ to group v_1 and an equal proportion of palms is accounted in the age group 1 to take care of the underplanted seedlings. Palms which were in group v_u in year (n-1) will be removed in year n and hence the row v_u contains only zero entries. This explains the construction of G.

3.3.3 Let $\underset{\sim}{f}_i$ be the age distribution of palms in year i. Let us define,

$$\underset{\sim}{f}_i = (n_{1i} \quad n_{2i} \quad \ldots \quad n_{ai} \quad n_{a^+i} \quad n_{v_1} \quad \ldots \quad n_{v_u})'$$

(3.2) $$\underset{\sim}{f}_i = (\underset{\sim}{g}_i \vdots \underset{\sim}{h}_i) \quad \text{where}$$

$$\underset{\sim}{g}_i = (n_{1i} \quad n_{2i} \quad \ldots \quad n_{ai} \quad n_{a^+i})'$$

$$\underset{\sim}{h}_i = (n_{v_1 i} \quad n_{v_2 i} \quad \ldots \quad n_{v_u i})'$$

$$i = 0, 1, 2, \ldots\ldots$$

where n_{xi} is the number of palms in the age group x in year i.

Let us notice that if $q = 1$, we will have the similar kind of rotation of the annual age distribution discussed in section 3.2. We shall thus consider $q < 1$.

Theorem : Under the replacement rule defined in 3.3.1, the age distribution of palms $\underset{\sim}{f}_i$, as defined in (3.2), converges to a stable one i.e., there exists an age distribution f such that $\underset{\sim}{f}_i \to \underset{\sim}{f}$ as $i \to \infty$,

$$\underset{\sim}{f} = (n_1, n_2, \ldots, n_{a+}, n_a, n_{v_1}, \ldots, n_{v_u})$$

where $n_i = qN/(1 + qa)$ for $i = 1, 2, \ldots, a, v_1, v_2, \ldots, v_u$

$n_{a+} = N/(1 + qa)$ N = No. of palms in the initial year.

Proof : From (3.1) and (3.2) we have

(3.3) $$\underset{\sim}{f}_i' = \underset{\sim}{f}_{i-1}' G = \underset{\sim}{f}_0' G^i ; \quad i = 1, 2, \ldots\ldots$$

From (3.2) we have,

(3.4) $$\underset{\sim}{f}_i' = (\underset{\sim}{g}_i' \vdots \underset{\sim}{h}_i')$$

Let us now partition the matrix G as shown in (3.1) by dotted lines. Thus, let us define,

(3.5) $$\underset{(a+u+1,\ a+u+1)}{G} = \left[\begin{array}{c|c} \underset{(a+1,\ a+1)}{G_1} & \underset{(a+1,\ u)}{G_2} \\ \hline \underset{(u,\ a+1)}{G_3} & \underset{(u,\ u)}{G_4} \end{array}\right]$$

From (3.3), (3.4) and (3.5) we now have,

(3.6) $$\underset{\sim}{g}_i' = \underset{\sim}{g}_0' G_1^i \quad \text{since } G_3 \text{ is a null matrix.}$$

Now, G_1 is a stochastic matrix and it is a well known result that there exists a matrix G_1^* such that $G_1^i \longrightarrow G_1^*$ as $i \to \infty$.

Let $\underset{\sim}{g}' = \underset{\sim}{g}_c' G_1^*$, $\underset{\sim}{g}$ is the limit of $\underset{\sim}{g}_i$.

Let $\underset{\sim}{g} = (n_1, n_2, \ldots, n_{a^+})$

Again, from (3.3), (3.4) and (3.5) we have,

$$\underset{\sim}{h}_i' = \underset{\sim}{g}_{i-1}' G_2 + \underset{\sim}{h}_{i-1}' G_4$$

We can therefore write that

$$n_{v_1 i} = q \; n_{a^+(i-1)}$$

$$n_{v_2 i} = n_{v_1 (i-1)}$$

$$n_{v_3 i} = n_{v_2} (i-1)$$

$$\ldots \quad \ldots \quad \ldots$$

$$n_{v_u i} = n_{v_{u-1}} (i-1)$$

Since $\underset{\sim}{g}_i \to \underset{\sim}{g}$ as $i \to \infty$, we have $n_{a^+ i} \to n_{a^+}$ as $i \to \infty$.

Therefore, $n_{v_1 i} = q \cdot n_{a^+(i-1)} \longrightarrow q \cdot n_{a^+}$ as $i \to \infty$

Thus, we can write that

$\underset{\sim}{h}_i \to \underset{\sim}{h}$ as $i \to \infty$ where

(3.7) $\underset{\sim}{h} = (rn_{a^+} \; rn_{a^+} \; \ldots\ldots \; rn_{a^+})'$

Let us define,

$$\underset{\sim}{f}' = (\underset{\sim}{g}' \vdots \underset{\sim}{h}')$$

We have shown that $\underset{\sim}{f}_i \to \underset{\sim}{f}$ as $i \to \infty$ where

$$\underset{\sim}{g} = (n_1, \; n_2, \ldots\ldots, n_{a^+})'$$

$$\underset{\sim}{h} = (rn_{a^+}, rn_{a^+}, \ldots\ldots, \; rn_{a^+})'$$

Let $N = n_{1o} + n_{2o} + \ldots\ldots n_{ao} + n_{a^+o}$. N is the total number of palms existing in the initial year before the replacement rule is applied.

It is easy to see from (3.6) that,

$$n_{1i} + n_{2i} + \ldots\ldots + n_{ai} + n_{a^+i} = N \text{ for all } i,$$

since G_1 is a stochastic matrix.

Therefore, we have

(3.8) $$N = n_1 + n_2 + \ldots\ldots + n_a + n_{a^+}$$

Again, since $\underset{\sim}{g} = \underset{\sim}{g}\, G_1$, we can write,

(3.9)
$$\begin{aligned} n_1 &= qn_{a^+} \\ n_2 &= n_1 \\ n_3 &= n_2 \\ \ldots &\quad \ldots \\ n_a &= n_{a-1} \\ n_{a^+} &= n_a + (1 - q)\, n_{a^+} \end{aligned}$$

From (3.8) and (3.9) it can be seen that

(3.10) $$n_{a^+} = N/(1 + qa)$$

The theorem now follows from (3.7), (3.9) and (3.10).

3.3.4 It is not clear if the stable distribution given by the theorem above (in 3.3.3) will necessarily generate a stable yield level over time. The reason can be seen as follows.

The yield of the palms in each of the age groups 1 through 'a' can be easily computed since in each age group there are palms at a single age. But, yield of the palms in the age group a^+ will depend on the age composition of the palms in the group. So is the case with the yield of the palms in the age groups v_1 through v_u. Thus, the total yield of the palms in age groups 1 through a will remain stable for the stable age distribution but the total yield of the palms in age groups a^+, v_1, ..., v_u will be stable if the age composition of the palms in each of these groups remain stable. The following theorem shows that such is the case here.

Theorem : The total yield corresponding to the stable age distribution obtained in the theorem in 3.3.3 is stable over time.

Proof : To prove this we have to show that the age composition in each of the age groups a^+, v_1, ..., v_u is stable.

It can be seen from the stable distribution obtained in the theorem in 3.3.3 and the matrix G in (3.1) that every year $qN/(1 + qa)$ number of palms enter age group a^+ from age group a. These palms will be at age (a + 1) when they enter a^+. Since we are taking out palms from group a^+ in the descending order of age it is clear that depending on q, either group a^+ will have palms at all ages (a+1) through L or at the first few of the ages. Let us suppose there are palms in a^+ at ages (a+1), (a+2), ..., (a+b) where $1 \leq b \leq L-a$. It can be seen from the theorem in 3.3.3 that every year $qN/(1 + qa)$ number of palms enter group a^+, number of palms at age (a+1), (a+2) etc., will be $qN/(1+qa)$ except probably at age (a + b).

Number of palms in a^+ is $N/(1 + qa)$. Therefore, we have,

$$b = \text{Min}\,(L - a,\ c) \quad \text{if } \frac{1}{q} \text{ is an integer}$$

(3.11)

$$= \text{Min}\,(L - a,\ c + 1) \quad \text{if } \frac{1}{q} \text{ is not an integer.}$$

where c is the integral of $\frac{1}{q}$.

Since there are $N/(1+aq)$ number of palms in age group a^+ and q is the rate of underplantation every year $qN/(1+aq)$ number of palms enter in group v_1 from group a^+. Therefore, it can be seen that, if $b=c \leq L-a$ (i.e., whenever $\frac{1}{q}$ is an integer), every year there will be $qN/(1+aq)$ number of palms at each of the ages $(a + 1)$ through $(a + b)$. If $b = L - a < c$, every year there will be $qN/(1 + aq)$ number of palms at each of the ages $(a + 1)$ through $(a + b - 1)$ and $N(1 - (b - 1)q)/(1 + aq)$ number of palms at age $a + b$. Let us note that in this latter case

$$N(1 - (b - 1)q)/(1 + aq) \geq qN/(1 + aq).$$

Again, whenever $\frac{1}{q}$ is not an integar, every year there will be $qN/(1 + aq)$ number of palms at each of the ages $(a + 1)$ through $(a + b - 1)$ and $(1 - (b - 1)q)N/(1 + aq)$ number of palms at age $(a + b)$. Let us note here that in this case

$$N(1 - (b - 1)q)/(1 + aq) \lesseqgtr qN/(1 + aq)$$

depending on

$$c + 1 \lessgtr l - a.$$

Hence the age composition of palms in age group a^+ will remain the same over time in the stable distribution.

Similarly, if $\frac{1}{q}$ is an integer, group v_1 will contain $qN/(1+qa)$ palms at a single age $(a + b + 1)$ since $(1 -(b-1)q)N/(1+aq) \geq qN/(1+aq)$. If $\frac{1}{q}$ is not an integer and $c+ 1 < L-a$ then v_1 will contain $(1 - (b - 1)q)N/(1 + qa)$ palms at age $(a + b + 1)$ and $(b - 1)qN/(1+qa)$ palms at age $(a + b)$, every year. If $c + 1 > L - a$, then every year v_1 will contain $qN/(1 + aq)$ palms at age $(a + b + 1)$. Hence, age composition in v_1 is also stable.

Now, since v_2 has palms moving in from V_1 and so on, age composition in each of the groups v_2, v_3, ..., v_u will also be stable.

Hence the theorem.

3.3.5 We can now write down the expression for the stable yield level. Let Y(a, q) be the stable total annual yield when the replacement is performed from the collection of palms of age above a ($a \geq a_e$) at the rate of underplantation q. It follows from the proof of the theorem in 3.3.4 that,

$$Y(a, q) = \frac{qN}{1+aq} \sum_{n=1}^{a+b+u} \mu_n \quad \text{if } \frac{1}{q} \text{ is an integer and } b = c \leq L-a$$

(3.12)

$$= \frac{qN}{1+aq} \sum_{n=1}^{a+b+u-1} \mu_n + \frac{(1-(b-1)q)N}{1+aq} \mu_{a+b+u}$$

$$\text{if } \frac{1}{q} \text{ is not an integer and } b = c+1 \leq L-a$$

$$= \frac{qN}{1+aq} \sum_{n=1}^{a+b-1} \mu_n + \frac{(1-(b-1)q)N}{1+aq} \mu_L + \frac{qN}{1+aq} \sum_{n=a+b+1}^{a+b+u} 0$$

$$\text{if } \frac{1}{q} \text{ is an integer and } b = L-a < c$$

$$\text{or } \frac{1}{q} \text{ is not an integer and } b = L-a < c+1.$$

where b and c are given by (3.11).

Let us note that when $b = L - a$, we will have palms of age L and above in the age groups a^+, v_1, ..., v_u. But, since palms are dead by age L, we have taken 0 as the yield of such palms in the above expressions.

We shall consider the cases where $q > \frac{1}{L-a}$ since for $q \leq \frac{1}{L-a}$ we will have $L - a < c + 1$ which means that q will be too low leading to accumulation of dead plams in the old-age group.

The following set of lemma show the relation between $Y(a, q)$ and a, q.

Lemma 1 : Let q_1 and q_2 be two points in the range of q i.e., $(\frac{1}{1-a}, 1)$, such that, $\frac{1}{q_1}$ and $\frac{1}{q_2}$ are two successive integers i.e., $(\frac{1}{q_1}) = (\frac{1}{q_2}) + 1$. Let $c_1 = \frac{1}{q_1}$ and $c_2 = \frac{1}{q_2}$. Then, the stable yield level $Y(a, q)$ is increasing in q for q in the interval $(q_1, q_2]$ if and only, if,

$$(3.13) \qquad \frac{1}{(c_2+a)} \sum_{n=1}^{a+c_2+u} \mu_n > \mu_{a+c_2+u+1}$$

Proof : From (3.12) we can write,

$$Y(a, q) = \frac{q.N}{1+aq} \sum_{n=1}^{a+u+c_2} \mu_n + \frac{(1+c_2q)}{1+aq} \mu_{a+u+c_2+1}$$

for $q_1 < q \leq q_2$

$$= \frac{N}{1+aq} \left[q \{ \sum_{n=1}^{a+u+c_2} \mu_n - c_2 \mu_{a+u+c_2+1} \} + \mu_{a+u+c_2+1} \right]$$

Thus, $Y(a, q)$, $q_1 < q \leq q_2$, is of the form $(b_1 + b_2 q)/(1+b_3 q)$.

It is easy to see that a function of q of the above form is increasing in q if and only if $b_2 - b_1 b_3 > 0$.

In this case, the above condition is equivalent to (3.13). Hence the lemma.

Lemma 2 : Let q_1, q_2, c_1 and c_2 be as defined in Lemma 1. Then, $Y(a, q) > Y(a, q_1)$ for q in the interval $(q_1, q_2]$ if and only if the condition given by (3.13) is satisfied.

Proof : From (3.12) we can write,

$$Y(a,\ q) - Y(a,\ q_1) = \frac{qN}{1+aq} \sum_{n=1}^{a+u+c_2} \mu_n - \frac{q_1\ N}{1+aq_1} \sum_{n=1}^{a+u+c_1} \mu_n + \frac{(1-c_2q)N}{1+aq} \mu_{a+u+c_2+1}$$

$$= \frac{N}{(1+aq)(1+aq_1)} \{ q\ (1+aq_1) \sum_{n=1}^{a+u+c_2} \mu_n - q_1(1+aq) \sum_{n=1}^{a+u+c_2+1} \mu_n \} + \frac{(1-c_2q)N}{1+aq} \mu_{a+u+c_2+1}$$

since $c_1 = c_2+1$

$$= \frac{(q-q_1)N}{(1+aq)(1+aq_1)} \{ \sum_{n=1}^{a+u+c_2} \mu_n - (c_2+a)\ \mu_{a+u+c_2+1} \}$$

Hence the lemma.

Lemma 3 : Consider $Y(a,\ q)$ such that $\frac{1}{q}$ is an integer. Then $Y(a,\ q)$ is an decreasing function of a if and only if

$$\frac{1}{c+a} \sum_{n=1}^{a+u+c} \mu_n > \mu_{a+u+c+1} \quad \text{where} \quad c = \frac{1}{q},\quad a \geq a_e$$

Proof : From (3.12) we have,

$$Y(a+1,\ q) - Y(a,q) = \frac{q}{1+aq+q} \sum_{n=1}^{a+u+c+1} \mu_n - \frac{q}{1+aq} \sum_{n=1}^{a+u+c} \mu_n$$

$$= \frac{q}{(1+aq)(1+aq+q)} \{ (1+aq) \sum_{n=1}^{a+u+c+1} \mu_n - (1+aq+q) \sum_{n=1}^{a+u+c} \mu_n \}$$

$$= \frac{q}{(1+aq)(1+aq+q)} \{ (1+aq)\ \mu_{a+u+c+1} - q \sum_{n=1}^{a+u+c} \mu_n \}$$

Hence the lemma.

From Lemma 1 and Lemma 2 above the following theorem easily follows.

Theorem : If the stable yield level Y(a, q) has a maximum for q in ($\frac{1}{1-a}$, 1) for a given 'a', it will be at some point q such that $\frac{1}{q}$ is an integer and such a point q exists if and only if there exists q′ such that

$$\frac{1}{q'} = \frac{1}{q} + 1 = c$$

(3.14)
$$\frac{1}{c+a} \sum_{n=1}^{a+u+c} \mu_n \leq \mu_{a+u+c+1}$$

Proof : We have only to show that

$$\frac{1}{(c+a)} \sum_{n=1}^{a+u+c} \mu_n \leq \mu_{a+u+c+1}$$

$$\Longrightarrow \frac{1}{(c-1+a)} \sum_{n=1}^{a+u+c-1} \mu_n \leq \mu_{a+u+c}$$

This can be seen easily as follows.

$$\sum_{n=1}^{a+u+c} \mu_n \leq (c+a)\, \mu_{a+u+c+1}$$

$$\Longrightarrow \sum_{n=1}^{a+u+c-1} \mu_n + \mu_{a+u+c}$$

$$\leq (c-1+a)\, \mu_{a+u+c+1} + \mu_{a+u+c+1}$$

$$\leq (c-1+a)\, \mu_{a+u+c} + \mu_{a+u+c}$$

since $\mu_{x+1} \leq \mu_x$ for $x \geq a_0$

$$\Longrightarrow \sum_{n=1}^{a+u+c-1} \mu_n \leq (c-1+a)\, \mu_{a+u+c}$$

Now, the theorem follows from Lemma 1 and Lemma 2.

3.3.6 **Remarks :** Let us note that in our present model if $q = 1$ and $a = A^*$ where $(1 + A^*)$ is the optimal replacement age (infinite horizon) given by (2.21) then it becomes equivalent to the optimal replacement rule discussed in the previous chapter. The corresponding annual yield trajectory will have the highest discounted annual net returns although there will not be any stable age distribution in the limit. Thus, the replacement rule considered here is sub-optimal by the criterion of discounted net returns. The replacement rule in the present model is to be defined by specifying a and q — the cut-off age for the old-age group and the rate of underplantation respectively. The choice of a $(\geq a_e)$ and q $\left(\frac{1}{L-a} < q < 1 \right)$ is to be made by considering the stable yield level, the trajectory leading to the stable yield level and the level of sub-optimality.

3.4 Phased Replacement of the Old Palms

3.4.1 The model considered in the previous section 3.3 was chiefly concerned with the annual age distribution of the palms which will converge to a stable one over time. The stable age distribution was sought for in order to eliminate the fluctuations in the yield trajectory. Let us now consider the second problem noted in 3.2.2. This is the problem of initial heavy expenditure and low annual net returns due to the presence of large number of old palms to be replaced immediately under the optimal replacement rule.

One of the ways of easing the burden of initial heavy expenditure would be to replace the existing old stock of palms in a number of phases since in that case the initial expenditure will get distributed over a number of years. Suppose the initial age distribution is given by $(n_1, n_2, \ldots, n_A, n_{A+1}, \ldots, n_L)$ where n_x is the number of palms at age x and (1 + A) is the optimal replacement age. Under the optimal replacement rule, all the palms above age A should be chosen for removal by underplanting. Let us call the collection of all the palms of age above (A + 1) as the 'old stock'. These are the palms which have survived their optimal economic life and were due for replacement at least an year ago. Thus, under the optimal replacement rule, the old stock should be chosen for removal by underplanting along with the palms which are at age (A + 1) in the initial year. This, we shall refer to as the replacement of the old stock in a single phase. When the number of palms in the old stock is proportionately large we can think of replacing the old stock in, say, w number of phases. This means that in year i we shall select $(z_i + w_i)$ number of palms for underplanting where z_i is the number of palms at age (A + 1) in year i and w_i be the number of palms chosen (in descending order of age) from the old stock, i = 1, 2, ..., w,

$$\sum_{i=1}^{w} w_i = \sum_{j=A+2}^{L} n_j$$

Obviously, phased replacement of the old stock will lead to a fall in the discounted value of the annual net return stream from it's optimal level. The following lemma provides an upper bound to such a fall.

3.4.2 <u>Lemma</u> : The upper bound to the fall in the total discounted value of the annual net returns due to phased replacement (described above in 3.4.3) is given by,

$$(3.15)\quad \{ \sum_{x=A+2}^{L-u} n_x + \sum_{x=L-u+1}^{L} n_x (1-d^{L-x+1}) \} E - \sum_{x=A+2}^{L-u} \sum_{n=x+u}^{L} r(n)d^{n-x}$$

if replacement of a palm in the old stock is delayed upto it's age L at the latest. E is given by (2.11).

Proof : The fall in the total discounted value will be only due to the fall in the discounted value of the net return stream corresponding to the old stock.

The discounted value corresponding to the old stock if the optimal rule is applied (i.e., the palms are removed in a single phase) is given by,

$$(3.16)\quad \sum_{x=A+2}^{L} n_x \{ E + \sum_{n=x}^{Min(x+u-1,L)} r(n)d^{n-x} \}$$

The same, when all the palms in the old stock are retained till age L before being replaced, is given by,

$$(3.17)\quad \sum_{x=A+2}^{L} n_x \{ d^{L-x+1} E + \sum_{n=x}^{L} r(n)d^{n-x} \}$$

Obviously, difference between (3.16) and (3.17) gives us the required upper bound.

Therefore, the required upper bound

$$= \sum_{x=A+2}^{L-u} n_x \{(1 - d^{L-x+1})E + \sum_{n=x}^{Min(x+u-1,L)} r(n)d^{n-a} - \sum_{n=x}^{L} r(n)d^{n-x} \}$$

$$= \sum_{x=A+2}^{L-u} n_x \{ (1-d^{L-x+1})E + \sum_{n=x}^{x+u-1} r(n)d^{n-x} - \sum_{n=x}^{L} r(n)d^{n-k} \}$$

$$+ \sum_{x=L-u+1}^{L} n_x \{ (1-d^{L-x+1})E + \sum_{n=x}^{L} r(n)d^{n-x} - \sum_{n=x}^{L} r(n)d^{n-x} \}$$

$$= \sum_{x=A+2}^{L-u} n_x \{(1-d^{L-x+1})E - \sum_{n=x+u}^{L} r(n)d^{n-x}\} + \sum_{x=L-u+1}^{L} n_k(1-d^{L-x+1})E$$

$$= \{\sum_{n=A+2}^{L-u} n_x + \sum_{x=L-u+1}^{L} n_x(1-d^{L-x+1})\} E - \sum_{x=A+2}^{L-u} \sum_{n=x+u}^{L} r(n)d^{n-x}.$$

Hence the lemma.

3.4.3 Choice of a suitable phasing scheme i.e., a suitable combination of $(w, w_1, w_2, \ldots, w_w)$ will depend chiefly on the constraints to the initial expenditure, the minimum desirable annual net returns during the immediate years etc. In the appendix we present a linear programming solution to the choice of $(w, w_1, w_2, \ldots, w_w)$ which is applicable in a variety of situations.

Appendix

A3.1 Suppose that we do not want to delay the replacement of the palms in the old stock beyond (L-A-1) years, by which time the youngest palm (at age A+2) among them reaches age L. Again, we have already seen in 3.2.1 that the age distribution of the palms rotate over a period of A years. Suppose we want to complete the phased replacement within the first period of rotation so that the rotations repeat themselves in the successive periods. Thus, let us choose $w = \text{Min}(A, L - A - 1)$. We shall first derive the expressions for the annual net returns during the first period of rotation i.e., the first A years.

A3.2 For convenience, let us for the time being consider replacement by replantation. Let $(n_1, n_2, \ldots, n_L)$ be the initial age distribution before the replacement programme starts. All the palms at age (A + 1) will be removed now and an equal number of seedlings will be planted. Thus, let us put $n_1 = n_{A+1}$ and $n_{A+1} = 0$. Let P_j be the total net return in year j corresponding to the palms of age between 1 and A in the initial year. The expression for P_j can be written as,

$$(3.19) \qquad P_j = \sum_{x=1}^{A-j+1} n_x\, r(j+x-1) + \sum_{x=A-j+2}^{A} n_x\, r(x+j-A-1); \qquad j = 2, 3, \ldots, A.$$

$$= \sum_{x=1}^{A} n_x\, r(j + x - 1); \qquad j = 1$$

where r(x) is the expected net return from a palm at age x.

We obtained (3.19) in the following way. Let us refer to the table (3.2.1).

Palms at age 1 in the initial year will be at age j in the <u>jth</u> year.

Palms at age 2 in the initial year will be at age (j+1) in the <u>jth</u> year.

In general, palms at age x in the initial year will be at age (j+x−1) in the jth year, x = 1, 2, ..., A−j+1. Again, palms at age A in the initial year will be at age (j−1) in the jth year (not necessarily the same palms but it may be the palms that replaced the initially existing ones). In general, palms at age x initially will be at age (x+j−A−1) in the jth year, x = A−j+2, ..., A.

Let us define w_x^j as the number of palms to be replaced in year j from the ones which were at age x in the initial year $x \geq A + 2$, j = 1, 2, ..., A. Let Q_j be the annual net return in year j corresponding to the old stock. The expression for Q_j can be written as,

$$(3.20) \qquad Q_j = \sum_{k=1}^{j} \left\{ \sum_{x=A+2}^{L} w_x^k \right\} r(j-k+1) + \sum_{x=A+2}^{L} \left\{ n_x - \sum_{k=1}^{j} w_x^k \right\} r(x+j-1);$$

$$j = 1, 2, \ldots, A.$$

We obtained (3.20) in the following way. In year k, the number of palms removed from the old stock is given by $\sum_{x=A+2}^{L} w_x^k$. The palms which are planted in year k will be at age (j − k + 1) in year j. Thus we get the first term in the expression for Q_j. The palms which are at age x in the initial year and are yet to be replaced will be at age (x + j − 1) in year j. Thus we get the second term. Therefore, $(P_j + Q_j)$ is the net return in the year j from all the palms. Now, underplantation can be easily incorporated by rewriting P_j and Q_j from (3.19) and (3.20) as follows.

$$(3.21)\quad P_j = \sum_{x=1}^{A-j+1} n_x\, r'(j+x-1) + \sum_{x=A-j+2}^{A} n_x\, r(x+j-A-1); \quad j = 2, 3, \ldots, A.$$

$$= \sum_{x=1}^{L} n_x\, r'(j + x - 1); \qquad j = 1$$

where $r'(x) = r(x) + r(A+x), \quad x = 1, 2, \ldots, u$

$= r(x), \quad x > u$

$$Q_j = \sum_{k=1}^{j} \sum_{x=A+2}^{L} w_x^k \{r(j - k + 1) + r(x + j - 1)\}$$

$$+ \sum_{x=A+2}^{L} (n_x - \sum_{k=1}^{j} w_x^k)\, r(x + j - 1); \quad j = 1, 2, \ldots, u$$

$$(3.22)\quad = \sum_{k=1}^{j-u} \sum_{x=A+2}^{L} w_x^k\, r(j - k + 1) + \sum_{k=j-u+1}^{j} \sum_{x=A+2}^{L} w_x^k\, r''(x, j-k+1)$$

$$+ \sum_{x=A+2}^{L} (n_x - \sum_{k=1}^{j} w_x^k)\, r(x + j - 1); \quad j = u+1, \ldots, A$$

where $r''(x, y) = r(y) + r(x+j-1); \quad y = 1, 2, \ldots, u$

$= r(y); \quad y > u$

The above expressions in (3.21) and (3.22) are derived from the fact that, in the case of replacement by underplantation net return from every palm that has replaced an old palm is accompanied by the net return from the old palm for the first u years.

A3.3 Now we are in a position to formulate the linear programming problem

Since we are considering the phased replacement within the first period of rotation we may consider the annual net return stream only for the first A years for the objective function. The LP problem can thus be formulated as follows.

(1) Objective function : $\sum_{j=1}^{A} (P_j + Q_j)d^{j-1}$ where P_j and Q_j are given by (3.21) and (3.22).

(2) Optimising variables : w_x^j; $x = A+2, A+3, \ldots, L$;

$j = 1, 2, \ldots, A$

Let us note that if we choose $w < A$ then $w_x^j = 0$ for $j = w+1, \ldots, A$.

(3) Constraints : We may choose various kinds of constraints. Following area few examples.

(i) $P_j + Q_j \geq bP_j$; $0 < b < 1$; $j = 1, 2, \ldots, A$.

$$\sum_{j=1}^{A} w_x^j = n_x \; ; \quad x = A+2, \ldots, L.$$

This specifies that the phasing should be such that the annual net returns do not fall below a certain proportion of what we could get in absence of the old stock.

(ii) $P_j + Q_j \geq P$; $j = 1, 2, \ldots, A$

$$\sum_{j=1}^{A} w_k^j = n_x; \quad x = A + 2, \ldots, L.$$

This specifies that the phasing should be such that the annual net returns do not fall below a certain bound. The bound P may be chosen to be Min (P_j; $j = 1, 2, \ldots, A$) or some other level adjudged to be satisfactory.

(iii) $P_j + Q_j \geq b\,Q$; $1 < b$; $j = 1, 2, \ldots, A$

$$\sum_{j=1}^{A} w_x^j = n_x \; ; \quad x = A + 2, \ldots, L$$

This specified that the annual net return should be raise by a certain proportion of the level Q where Q is the

Min $(P_j + Q_j;\ j = 1, 2, \ldots, A)$ when replacement is performed in a single phase i.e., $w = 1$.

A3.4 Remarks : (i) It may require a few trials of the LP formulation with different number of phases and different bounds for the trajectory to arrive at a suitable phasing scheme.

(ii) Let us note that the phasing scheme will determine the annual age distributions in the successive periods of rotation. It can be seen that phasing will reduce the lumping of a large number of palms in a single age group since palms that replace the old stock will spread over different age groups.

(iii) Easing of the initial heavy expenditure can be achieved also in the model for the trajectory considered in section 3.3. This can be done by simply chosing suitable rates of underplantation q_1, q_2, ..., etc., during the early years before settling to a constant rate of underplantation q.

Chapter IV

Stochastic Replacement Rule : An Improvement Over Deterministic Rule

4.1 Introduction

4.1.1 The deterministic modelling exercises in the previous chapters do not use the dependence of the future yield stream of a palm on it's past yield performance. Thus, an obvious limitation of the deterministic models lies in the risk of retaining a consistently low yielding palm or removing a palm which has a potential for yielding 'satisfactorily' beyond the replacement age. Here we shall first try to modify the deterministic replacement rule discussed in Chapter II and examine the extent of improvement, in terms of reduction in the risks just mentioned, that can be achieved. Later, we shall try to formulate further rules in the light of the results we obtain from this exercise.

4.1.2 Let us recall the deterministic replacement rule (infinite horizon case) we discussed in Chapter II. There we were considering replacement of a palm in it's declining yield phase i.e., beyond age a_e.

Let A be the Optimal Economic Life (OEL) of an existing palm. Then, A is given by

$$r(x + u) > (1 - d)d^{-u} E, \qquad x = a_e + 1, \ldots, A$$

$$r(x + u) \leq (1 - d)d^{-u} E, \qquad x = A + 1, \ldots, L - u.$$

where r(.), d, E, u are as given in 2.2.2, (2.11), 2.2.1 respectively.

We shall refer to this replacement rule as the Rule 1 or in short R 1.

4.1.3 Let $Y(x)$ be the annual yield of a palm at age x, $x = 1, 2, \dots, L$, where L is the life of a palm. As mentioned in the introductory chapter we shall assume that $(Y(1), Y(2), \dots, Y(L))$ has a Multivariate Normal Distribution with specification as stated in 1.2.2. Given the past yield record of a palm over a certain period we can easily obtain the conditional expectation of the future annual yield profile.

Let

$$X(x) = E(Y(x) \mid Y(x-1) = y_{x-1}, \dots, Y(x-n) = y_{x-n})$$

$$= b_{ox} + b_{1x}\, y_{x-1} + \dots + b_{nx}\, y_{x-n}; \quad n < x$$

$X(x)$ is the conditional mean of the yield of a palm at age x given it's past n years' yield, $b_{ox}, b_{1x}, \dots, b_{nx}$ are the corresponding regression coefficients.

In general

(4.1) $$X(x') = b_{ox'} + b_{1x'}\, y_{x-1} + \dots + b_{nx'}\, y_{x-n}$$

where x is the present age of a palm and $x' = x, x+1, \dots, L$, and $b_{ox'}, b_{1x'}, \dots, b_{nx'}$ are the regression coefficients corresponding to the regression of $Y(x')$ on $Y(x-1), Y(x-2), \dots, Y(x-n)$.

4.2 Modification of Rule 1

4.2.1 The deterministic rule R 1 was based on the expected annual yield stream of a palm. Let us instead consider the conditional expected yield of the palm in the future given it's yield record over past n years. Let x be the present age of the palm. A stochastic version of Rule 1 is obtained by replacing the expected yield stream $\mu_x, \mu_{x+1}, \ldots, \mu_L$ by the conditional expected yield stream $X(x), X(x+1), \ldots, X(L)$ in the deterministic version (see 2.3.2). It is easy to see that we should replace the palm at the current age, in order to maximise the discounted value of the future net return stream, if

$$r'(x + u) \leq (1 - d)d^{-u} E$$

$$\text{where } r'(x + u) = p.\ X(x + u) - s(x + u)$$

p and s are as defined in (2.2.2), $r'(x + u)$ is the conditional expected net return given the past yield record. X(.) is given by (4.1).

Thus, we can define the modified rule as follows.

<u>Rule 2</u> : Given x , the present age of a palm, and given $Y(x - 1), Y(x - 2), \ldots\ldots, Y(x - n)$ — the yield record of the palm duing the past n years, replace at current age 'x' only if

$$r'(x + u) \leq (1 - d)d^{-u} E$$

$$\text{or, } X(x + u) \leq \frac{1}{p} \left[(1 - d)d^{-u} E + s(x + u)\right]$$

$$\text{or, } \sum_{k=1}^{1} b_{k,\,x+u}\, y_{x-k} \leq \frac{1}{p} \left[(1-d)d^{-u} E + s(x+u)\right] - b_{o,x+u}$$

by (4.1)

Thus, we can write : replace only if,

$$\sum_{k=1}^{n} b_{k,x+u}\, y_{x-k} \leq k_1 \tag{4.2}$$

$$\text{where } k_1 = \frac{1}{p} \left[(1-d)d^{-u} E + s(x + u)\right] - b_{o,\, x+u}$$

Let us call this rule R 2 in short.

4.2.2 Now the question is — how good is the improvement of R 2 over R 1 ?

One of the measures of the degree of improvement can be given by the probability that R 2 generates a net return stream with higher discounted value than R 1. Let 'x' be the present age of a palm. We shall consider two situations separately $x \leq A$ and $x > A$ where A is the optimal replacement age according to the deterministic version (R 1). In the first case i.e., when $x \leq A$, R 1 recommends no replacement at current age. In the second case i.e., when $x > A$, R 1 recommends replacement at the current age.

<u>Case (i)</u> : $x \leq A$

Consider

P (discounted value corresponding to R 2
$\geq$ discounted value corresponding to R 1
| R 2 recommends replacement)

If this probability is greater than or equal to half then we shall consider R 2 a better rule than R 1.

<u>Remark</u> : The choice of the number $\frac{1}{2}$ in this context requires some explanation. What really matters is R 2 performs better than R 1 in an absolute (unconditional) sense. For this purpose we require:

P (discounted value corresponding to R 2
$\geq$ discounted value corresponding to R 1,
R 2 recommends replacement)

which is equal to

P (discounted value corresponding to R 2
$\geq$ discounted value corresponding to R 1
| R 2 recommends replacement)
X p (R 2 recommends replacement)

It can thus be seen that even if one replaces $\frac{1}{2}$ by an arbitrarily large number, say 0.95, the problem would still remain since P(R 2 recommends replacement) can be small. Moreover, the replacement of $\frac{1}{2}$ by an arbitrary number α leads to analytical complexities involving in particular non-normal distributions. It is for this reason that we have not persued this line further.

In any case, what is more important is that the probability that R 2 recommends replacement whenever the past yields are low, is quite high. A modified way of looking at the problem in this manner is attempted in the next chapter.

Case (ii) : $x > A$

In similar fashion we shall consider R 2 a better rule than R 1 if

P (discounted value corresponding to R 2

$\geq$ discounted value corresponding to R 1

| R 2 recommends no replacement)

$\geq$ 0.5

We shall derive a sufficient condition for R 2 to be better (in the sense explained above) than R 1 in each of the two situations.

4.2.3 Consider $x \leq A$

Let DV(R I) be the discounted value corresponding to the rule R I = R 1, R 2. Let R(x) be the annual net return from a palm at age x i.e.,

$$R(x) = p\,Y(x) - s(x), \quad x = 1, 2, \ldots, L.$$

Given that R 2 recommends replacement at the current age x,

$$DV(R\ 2) = \sum_{n=x}^{x+u-1} R(n)d^{n-x} + E$$

and

$$DV(R\ 1) = \sum_{n=x}^{A+u} R(n)d^{n-x} + d^{A-x+1}\ E$$

Therefore, given that R 2 recommends replacement at current age x,

$$DV(R\ 2) \geq DV(R\ 1)$$

$$\iff \sum_{k=x+u}^{A+u} R(k)d^{k-x} \leq (1 - d^{A-x+1})\ E$$

$$\iff \sum_{k=x+u}^{A+u} Y(k)d^{k-x} \leq \frac{1}{p}\left[\sum_{k=x+u}^{A+u} s(k)d^{k-x} + (1-d^{A-x+1})E\right] = k_2, \text{say.}$$

Thus, given (4.2) i.e., R 2 recommends replacement,

$$DV(R\ 2) \geq DV(R\ 1) \iff$$

(4.3) $$\sum_{k=x+u}^{A+u} Y(k)p^{k-x} \leq k_2$$

$$\text{where } k_2 = \frac{1}{p}\left[\sum_{k=x+u}^{A+u} s(k)d^{k-x} + (1-d^{A-x+1})\ E\right]$$

Now, using (4.2) and (4.3) we have

$$P(\ DV(R\ 2) \geq DV(R\ 1)\ \big|\ R\ 2 \text{ recommends replacement at age } x\ )$$

(4.4)

$$= P(\sum_{k=x+u}^{A+u} Y(k)d^{k-x} \leq k_2\ \Big|\ \sum_{k=1}^{n} b_{k,x+u}\ Y(x-k) \leq k_1\ )$$

$$= P\ (Z_2 \leq k_2\ \big|\ Z_1 \leq k_1\ )$$

$$\text{where } Z_2 = \sum_{k=x+u}^{A+u} Y(k)d^{k-x}$$

$$Z_1 = \sum_{k=1}^{n} b_{k,\ x+u}\ Y(x-k)$$

k_1 and k_2 are defined already in (4.2) and (4.3).

Let us first consider

$P(Z_2 \leq k_2 \mid Z_1 = k_1')$; k_1' is an arbitrary number so that $k_1' < k_1$

(4.5)
$$= \Phi\left[\frac{k_2 - E(Z_2) - \frac{Cov.(Z_1, Z_2)}{V(Z_1)}\{k_1' - E(Z_1)\}}{\sqrt{V(Z_2)}\{1 - r^2(Z_1, Z_2)\}^{\frac{1}{2}}}\right]$$

where $r(Z_1, Z_2) = Corr.(Z_1, Z_2)$

$\Phi(.)$ = Standard Normal Distribution Function.

Let
$$b = \frac{Cov(Z_1, Z_2)}{V(Z_1)}$$

4.2.4 Lemma : $P(Z_2 \leq k_2 \mid Z_1 = k_1')$ increases with decrease of $k_1' < k_1$ where Z_1, Z_2 are defined in (4.4) and k_1, k_2 are defined in (4.2) and (4.3) respectively.

Proof : Note that

$$k_1' - E(Z_1)$$
$$\leq k_1 - E(Z_1)$$
$$= \frac{1}{p}\left[(1 - d)d^{-u} E + s(x + u)\right] - b_{o,\, x+u} - E(Z_1) \quad \text{by (4.2)}$$

But $b_{o,\, x+u} = E(Y(x+u)) - E(Z_1)$

$$\therefore\ k_1 - E(Z_1)$$
$$\leq \frac{1}{p}\left[(1 - d)d^{-u} E + s(x + u)\right] - \mu_{x+u}$$
$$= \frac{1}{p}\left[(1 - d)d^{-u} E - r(x + u)\right]$$
$$\leq 0 \quad \text{from 4.1.2 since } x \leq A$$

Also $b > 0$ since Corr. $(Y(i), Y(j)) > 0$ for all $i, j = 1, 2, \ldots, L$.

Hence the lemma.

4.2.5 Thus, a condition which is sufficient for

$$P(Z_2 \leq k_2 \mid Z_1 = k_1) \geq 0.5$$

is also sufficient for R 2 to be better than R 1 in the sense explained in 4.2.2(i).

Now $P(Z_2 \leq k_1 \mid Z_1 = k_1) \geq 0.5 \iff$

(4.6) $$k_2 - E(Z_2) - b(k_1 - E(Z_1)) \geq 0$$

We have already found in 4.2.4 that

$$k_1 - E(Z_1) = \frac{1}{p}\left[(1-d)d^{-u} E - r(x+u)\right]$$

Again $k_2 - E(Z_2) = \frac{1}{p}\left[\sum_{k=k+u}^{A+u} s(k)d^{k-x} + (1-d^{A-x+1})E\right] - E(Z_2)$ by (4.3)

Also $E(Z_2) = E\{\sum_{k=x+u}^{A+u} Y(k)d^{A-x}\}$

$$= \sum_{k=x+u}^{A+u} \mu_k d^{k-u}$$

$$\therefore k_2 - E(Z_2) = \frac{1}{p}\left[(1-d^{A-x+1})E - \sum_{k=x+u}^{A+u} R(k)d^{k-x}\right]$$

And finally, from (2.11)

$$(1 - d^A)E = \sum_{k=1}^{A+u} R(k)d^{k-1}$$

Thus, with necessary rearrangement of terms in (4.6) we have proved the following.

<u>Theorem</u>: Let $x(\leq A)$ be the present age of a palm whose past n years' yield performance is known. Thus, a sufficient condition that R 2 is a better rule than R 1, in the sense explained in 4. 2. 2 is given by,

$$\sum_{k=1}^{L} l_k R(k) \geq 0$$

where $l_k = l \frac{d^{k-1}}{(1 - d^{A})}$ for $k = 1, 2, \ldots, a+u-1$

$= (ld^{x+u-1} - d^{u} + b)$ for $k = a+u$

$= d^{k-x} (d^{x-1} - 1)$ for $k = a+u+1, \ldots, A+u$

$= ld^{k-1}$ for $k = A+u+1, \ldots, L$

$$l = 1 - d^{A-x+1} - b(1 - d)d^{-u}$$

4.2.6 Let us now consider $x > A$.

Note that at every age x of the palm the rule R 2 decides, on the basis of past yield record, whether to replace in the current year or to postpone replacement by another year.

Given $Y(x-1), Y(x-2), \ldots, Y(x-n)$, let us suppose R 2 recommends postponement of replacement by another year i.e.,

(4.7)

$$R'(x+u) > (1 - d)d^{-u} E$$

$$\text{or} \quad \sum_{k=1}^{n} b_{k,x+u} Y(x-k) > k_1 \qquad \text{(see 4.2)}$$

Then, $DV(R\ 2) = \sum_{x=a}^{x+u} R(k)d^{k-x} + dE$

Again, since $x > A$, R 1 recommends replacement in the current year.

Thus,

$$DV(R\ 1) = \sum_{k=x}^{x+u-1} R(k)d^{k-x} + E$$

Thus, we can write, given (4.6)

$$DV\ (R\ 2) \geq DV\ (R\ 1)$$

$$R'(x+u) \geq (1 - d)d^{-u} E$$

(4.8) $$Y(x+u) \geq \frac{1}{p} \left[(1-d)d^{-u} E + s(x+u)\right] = k_3, \text{ say.}$$

Let us define

(4.9, $Z_3 = Y(x + u)$

Using (4.7) and (4.8) we can write,

P (DV(R 2) $\geq$ DV(R 1) | R 2 recommends no replacement at current age x)

$$= P\,(Z_3 \geq k_3 \mid Z_1 > k_1)$$

Again as in the case $x \leq A$ (see 4.2.3), let us first consider

(4.10) $P(Z_3 \geq k_3 \mid Z_1 = k_1')$, where k_1' is an arbitrary number so that $k_1' \geq k_1$

$$= 1 - \phi\left[\frac{k_3 - E(Z_3)\ \frac{Cov(Z_1, Z_3)}{V(Z_1)}\{k_1' - E(Z_1)\}}{\sqrt{V(Z_3)}\{1 - r^2(Z_1, Z_3)\}^{\frac{1}{2}}}\right]$$

4.2.7 <u>Lemma</u> : $P(Z_3 \geq k_3 \mid Z_1 = k_1')$ increases with increase in $k_1' \geq k_1$ where Z_1, k_1 are defined in (4.2) and Z_3, k_3 are defined in (4.9) and (4.10) respectively.

<u>Proof</u> : $k_1' - E(Z_1)$

$$\geq k_1 - E(Z_1)$$

$$= \frac{1}{p}\left[(1-d)d^{-u}E + s(x+u)\right] - b_{o,x+u} - E(Z_1)$$

$$= \frac{1}{p}\left[(1 - d)d^{-u}E - r(x+u)\right]$$ (see proof of the lemma in 4.2.4)

≥ 0 from 4.1.2 since $x > A$

Also $\frac{Cov(Z_1, Z_3)}{V(Z_1)} > 0$ since Corr. $(Y(i), Y(j)) \geq 0$ for all $i,j = 1,2,\ldots,L$.

The lemma now follows from (4.10).

4.2.8 **Lemma :** Cov $(Z_1, Z_3) = V(Z_1)$ where Z_1, Z_3 are defined in (4.2) and (4.9).

Proof : Let us use the following notations.

$$\underset{\sim}{g} = (b_{1,x+u}, b_{2,x+u}, \ldots, b_{n,x+u})'$$

$$Y = (Y(x-1), Y(x-2), \ldots, Y(x-n))'$$

$$D = V(Y)$$

$$\underset{\sim}{f} = \text{Cov.}\ (Y(x+u), Y)$$

Now,

$$\underset{\sim}{g} = D^{-1}\underset{\sim}{f}$$

$$Z_1 = \underset{\sim}{g}'Y$$

$$V(Z_1) = \underset{\sim}{g}'D\underset{\sim}{g} = \underset{\sim}{f}'D^{-1}\underset{\sim}{f}$$

$$\text{Cov.}(Z_3, Z_1) = \text{Cov.}\ (Y(x+u), \underset{\sim}{g}'Y)$$

$$= \underset{\sim}{g}'\underset{\sim}{f}$$

$$= \underset{\sim}{f}'D^{-1}\underset{\sim}{f}$$

Hence the lemma.

4.2.9 **Lemma :** $k_3 - E(Z_3) = k_1 - E(Z_1)$

Proof : $k_3 - E(Z_3)$

$$= \frac{1}{p}\left[(1-d)d^{-u}E + s(x+u)\right] - \mu_{x+u}$$

$$\frac{1}{p}\left[(1-d)d^{-u}E - r(x+u)\right]$$

$$= k_1 - E(Z_1) \qquad \text{(see proof of the lemma 4.2.7)}.$$

4.2.10 Using the lemmas in 4.2.8 and 4.2.9 it is easy to see that

$$P(Z_3 > k_3 \mid Z_1 = k_1) = 0.5$$

where Z_1, Z_3 are defined in (4.2) and (4.9) respectively.

Now, using the lemma in 4.2.7 we can prove the following.

Theorem : Let $x(>\Lambda)$ be the present age of a palm whose past yield record is known. Then, R 2 is always a better rule than R 1 in the sense explained in 4.2.2.

Chapter V

Best Linear Stochastic Replacement Rule

5.1 Introduction

5.1.1 In the previous chapter (section 4.2) we modified the deterministic replacement rule R 1 (discussed in chapter 2 and 4) to incorporate the stochastic nature of the yield profile of palms. The modified replacement rule R 2 (see 4.2.1) is based on the association between the future and the past yield performance.

Let us notice that R 2 given by (4.2) is based on a linear function of the past yields of the palm. We shall refer to this function of the past performance as the 'decision function'. The objective function in this case is the discounted value of the future net return stream which also is a linear function of the future yield performance (see 2.2.2). Given the objective function, R 2 is formulated by choosing a particular decision function which is derived from the regressions of the future yields on the past yield record.

In this chapter we shall consider the problem of choosing a decision function for a given objective function. In general, both decision and objective functions can be arbitrary functions of the yield performance. We shall, however, restrict ourselves to linear functions only.

5.1.2 Let x be the present age of a palm, g be a linear function of $Y(x)$, $Y(x+1)$, ... $Y(L)$ and f be a linear function of $Y(x-1)$, $Y(x-2)$, ..., $Y(x-n)$, $n < x$. g is the objective function and f is the decision function on the basis of which replacement rule can be constructed. In general, a replacement rule should be such that whenever replacement is recommended the future yields of the palm should be unsatisfactory in some sense. Let us assume that g is an increasing function of the future yield performance i.e., the better the yield in future higher the value of g. We shall define the unsatisfactory future yield performance by a pair (g, kg) where kg is a specified constant so that $[g \leq kg]$ is an unsatisfactory event.

5.1.3 Let us consider the discounted value of the future net return stream as the objective function for example (see Chapter I). We can redefine the objective function and thereby the unsatisfactory event as follows.

The replacement rule is to provide the decision whether to replace the palm (at age x) in the current year or to postpone it by another year. Each year such a decision is to be taken. The following table gives the discounted values corresponding to each decision.

Decision	Value of the objective function
1. Postpone replacement by an year	$\sum_{n=x}^{x+u} R(n)d^{n-x} + dE$
2. Replace in the current year	$\sum_{n=x}^{x+u-1} R(n)d^{n-x} + E$

where R(n) is the annual net return at age n and E is the discounted value of the net returns from the subsequent sequence of palms (see (2.11)).

The difference between the discounted values corresponding to decisions 1 and 2 is given by,

$$(5.1) \qquad R(x + u) - (1 - d)d^{-u} E$$

Since replacement should be recommended whenever it will be gainful in terms of the discounted value, the unsatisfactory event in this case can be defined from (5.1) as $[R(x + u) \leq (1 - d)d^{-u} E]$. Thus, in the present example, $g = R(x + u)$ and $kg = (1 - d)d^{-u} E$.

Let us note that the objective function in the above example is concerned with the net return stream over an infinite future. This kind of objective function can be referred to as a 'long-sighted' one. In contrast, there can be 'short-sighted' objective functions concerned with the immediate future. For example, (i) average net return over the next few years, (ii) ratio of the value of the total yield to the total cost over a certain finite period, and so on. The finite horizon case considered in 2.6 is also an example of a 'short-sighted' objective function.

Given an objective function g and the unsatisfactory event $[g \leq kg]$ i.e., given the pair (g, kg) we shall define a replacement rule through a pair (f, k_f), where f is a decision function and k_f is a chosen constant; as follows.

Replacement Rule : replace in the current year if and only if $f \leq k_f$.

The above formulation is similar to the screening problem considered by Marshall and Olkin (1969). But, instead of minimising the expected loss (after specifying loss functions) we are chiefly interested in reducing the probabilities of the risks (explained in the next section) involved in replacement decisions.

Now let us consider the class of all (f, k_f) such that f is a linear function and k_f is a chosen constant. We shall refer to this as the class of 'linear replacement rules (LRR)'.

For a given objective (g, kg) we are interested in the 'best' rule (f, k_f) in the class of LRR.

5.2 Criteria for Comparing (f, k_f)

5.2.1 For decision (to replace or to postpone replacement by another year) at a given age of the palm, there are two types of risks involved. The risks are as follows:

(i) replacing a palm when retaining would have been satisfactory,

(ii) retaining a palm when replacement would have been satisfactory.

For a given objective (g, kg) let us consider a replacement rule (f, k_f). The extent of protection offered by (f, k_f) against the above mentioned risks can be assessed by the following two probabilities.

1. $P(g \leq kg \mid f \leq k_f)$
2. $P(f \leq k_f \mid g \leq kg)$

The first probability is the chance that the future performance of the palm will be unsatisfactory given that the rule recommends replacement. We shall refer to this probability as the Reliability of a rule (f, k_f). The second probability is the chance that the rule recommends replacement given that the future performance will be unsatisfactory. We shall refer to this probability as the Efficiency of a rule (f, k_f).

Now we can compare the goodness of any two given rules in terms of Reliability and Efficiency.

5.2.2 Let us note that a simultaneous maximisation of both the probabilities may not be possible since

$$P(g \leq kg \mid f \leq k_f) = P(f \leq k_f \mid g \leq kg) \frac{P(g \leq kg)}{P(f \leq k_f)}$$

It can be seen from above that with increase in k_f, $P(f \leq k_f \mid g \leq kg)$ increases and so also $P(f \leq k_f)$.

In order to find the 'best' LRR we shall proceed as follows.

We shall again assume that the yield sequence$(Y(1), Y(2), \ldots, Y(L))$ of a palm has a Multivariable Normal Distribution with specification as stated in 1.2.2.

For a given objective (g, kg), let us define F as the class of all LRR (f, k_f) such that

$$\text{Corr.}\ (f, g) > 0$$

$$P(g \leq kg \mid f = k_f) = 0.5$$

i.e., F is the class of all (f, k_f) such that f is linear, positively correlated with g and reliability of (f, k_f) is greater than half. We have already discussed the choice of the number $\frac{1}{2}$ in 4.2.2.

We shall define the Best Linear Replacement Rule (BLRR) as the (f^*, k_{f^*}) in F which has the highest efficiency i.e.,

$$P(f^* \leq k_{f^*} \mid g \leq kg) \geq P(f \leq k_f \mid g \leq kg) \quad \text{for all } (f, k_f) \varepsilon\ F.$$

We shall refer to f^* as the Best Linear Decision Function (BLDF).

5.3 <u>The Best Linear Replacement Rule (BLRR)</u>

5.3.1 **Theorem :** The BLRR (f^*, k_{f^*}) corresponds to the regression of g on $Y(x-1), Y(x-2), \ldots, Y(x-n)$; $n < x$, if $kg < E(g)$ where (g, kg) is the given objective.

Proof : For $(f, k_f) \in F$ we have,

$$P(g \leq kg \mid f = k_f) = 0.5$$

$$\iff kg - E(g) - \frac{Cov(f, g)}{v(f)} (k_f - E(f)) = 0$$

(5.2) $$\iff (kg - E(g))/\sqrt{v(g)} = r_f(k_f - E(f))/\sqrt{v(f)}$$

where $r_f = Corr(f, g) > 0$

Let us define

(5.3) $$k'g = (kg - E(g))/\sqrt{v(g)} \text{ and}$$

$$k'_f = (k_f - E(f))/\sqrt{v(f)}$$

Thus, from (5.2) and (5.3) we have,

(5.4) $$k'g = r_f \, k'_f$$

Let e_f be the efficiency of the rule (f, k_f)

$$e_f = P(f \leq k_f \mid g \leq kg)$$

(5.5) $$= P(f' \leq k'_f \mid g' \leq k'g)$$

where $f' = (f - E(f)) / \sqrt{v(f)}$

$$g' = (g - E(g)) / \sqrt{v(g)}$$

From (5.4) and (5.5) we now have -

(5.6) $$e_f = P\left(f' \leq \frac{k'g}{r_f} \mid g' \leq k'g\right)$$

Let us note that $P(f' \leq \theta \mid g' \leq k'g)$, $-\infty < \theta < \infty$, , is a distribution function conditional on $g' \leq k'g$ and f', g' are Standard Normal variates. Hence, it is evident from (5.6) that e_f is an increasing function of r_f if $kg < E(g)$ since $r_f > 0$ (see 5.2.2) and $k'g < 0$.

Therefore, BLRR corresponds to the decision function f for which r_f is maximum or, in other words, (f^*, k_{f^*}) corresponds to the decision functions of the form a + bf where a, b are finite constants, $b \neq 0$, and f is the regression of g on $Y(x-1), Y(x-2), \ldots, Y(x-n)$.

5.3.2 It is easy to see that the above theorem is not valid for (g, kg) where $kg > E(g)$. In this case, one can see from (5.6) the BLRR would correspond to f which has the minimum r_f. The case is a curious one, but we do not persue this here. The question of replacement arises only in the case of palms which are already yielding low. We are interested in finding out whether a minimum level of yield performance (somewhat below the expectation) can be realised in future from a low yielding palm.

5.4 Dependence of Efficiency of a Replacement Rule on the Variance of the Age-Specific Yield and the Current Age

5.4.1 We shall use the following lemma due to Lehmann (1966) in this section.

Lemma : If F and G are two random variables such that $P(F \leq \theta \mid G = \lambda)$ is non-increasing in λ then,

$$P(F \leq \theta \mid G < \lambda) \geq P(F \leq \theta \mid G \leq \lambda')$$

for all $\lambda' < \lambda$ and for all θ

The following theorem establishes the relationship between the efficiency of a rule and the common variance σ^2 of the yield stream

$$\{Y(x), \quad x = 1, 2, \ldots, L\}$$

5.4.2 Theorem : The efficiency e_f of a replacement rule $(f, k_f) \varepsilon F$ for a given objective (g, kg), $kg < E(g)$, is an increasing function of the common variance σ^2 of the age-specific yield

Proof : Let us first note that

$$e_f = P(f \leq k_f \mid g = k)$$
$$= P(f' \leq k_f' \mid g' = k')$$
$$= \not{D}\left[\frac{k_f' - r_f k'}{\sqrt{1 - r_f^2}}\right]$$

where $k' = (k - E(g))/\sqrt{v(g)}$,

r_f, f', g', k_f' are as defined in (5.2), (5.5) and (5.3),

$\not{D}(.)$ is the Standard Normal Distribution function.

Thus, e_f is non-increasing in k' since $r_f > 0$. Applying the lemma in 5.4.1 we have,

(5.7) $P(f' \leq k_f' \mid g' \leq k_g') \geq P(f' \leq k_f' \mid g' \leq k')$

for all $k' < kg'$ and for all kg'

where kg is defined in (5.3).

Now, g is a linear function of Y(x), Y(x+1), ..., Y(L). Let us define,

(5.8) $$g = \sum_{i=x}^{L} l_i \, Y(i), \qquad l_i \text{ are constants, not all zero.}$$

Then, $v(g) = \sigma^2 \, l' D l$

where $l = (l_x, l_{x+1}, \ldots, l_L)'$

D = Correlation matrix of (Y(x), ..., Y(L))

$\sigma^2 = v(Y(i))$, $i = x, x + 1, \ldots, L$.

$v(g)$ is therefore an increasing function of σ^2 since D is positive definite i.e., $l' Dl > 0$ for all non-null l (see (1.3)).

Let us consider $\sigma_1^2 \leq \sigma_2^2$. Correspondingly we have e_{f_1}, e_{f_2}, k'_{g_1}, k'_{g_2} for a given f and g.

$$e_{f_1} = P(f' \leq \frac{k'_{g_1}}{r_f} \mid g' \leq k'_{g_1}) \qquad \text{from (5.6)}$$

$$\leq P(f' \leq \frac{k'_{g_2}}{r_f} \mid g' \leq k'_{g_1}) \qquad \text{since } k'_{g_1} < k'_{g_2} \text{ a. } k_g < E(g)$$

$$\leq P(f' \leq \frac{k'_{g_2}}{r_f} \mid g' \leq k'_{g_2}) \qquad \text{by (5.7)}$$

$$= e_{f_2}$$

Hence the theorem.

5.4.3 In the proof of the above theorem we have also proved that

$$e_f = P(f' \leq k'_f \mid g' \leq k'_g)$$

in an increasing function of k'_g since k'_g increases with an increase in σ^2 where e_f is the efficiency of the rule $(f, k_f) \varepsilon F$. Since g is an increasing function of the yield performance, it is reasonable to assume that E(g) is decreasing with the age of the palm x,(see 5.8).

Let us now suppose that $k_g = kE(g)$ where k is a constant, $0 < k < 1$, $k_g < E(g)$

$$V(g) = \sigma^2 \; l' Dl \qquad \text{(see the proof of the theorem in 5.4.2).}$$

It can be seen from (1.3) that the structure of the correlation matrix D is such that V(g) is independent of the current age x. Hence

$$k'_g = (E(g)(k-1))/\sqrt{V(g)}$$

in an increasing function of age x. Therefore, the efficiency e_f of a replacement rule $(f, k_f) \in F$ is an increasing function of age x of the palm.

5.5 Condition under which the BLDF for a given Objective Function Provides as Efficient a Rule as the BLDF for the Objective Function Extended to a Longer Future

5.5.1 Let us consider an objective function

(5.9) $$g = \sum_{i=1}^{m} l_i \, Y(x + i - 1) \; ; \qquad m \leq L - x + 1$$

where x is the age of the palm.

Let the BLDF for the above objective function be given by

(5.10) $$f = \sum_{i=1}^{n} w_i \, Y(x - i) \; ; \qquad n < x$$

By the theorem in 5.3.1, w_i are the regression coefficients corresponding to the regression of g on Y(x-1), Y(x-2), ..., Y(x-n).

Let us write,

$$\underset{\sim}{w} = D_p^{-1} \, C \, l \qquad \text{where}$$

$$\underset{\sim}{w} = (w_1, w_2, \ldots, w_n)'$$

$$\underset{\sim}{l} = (l_1, l_2, \ldots, l_m)'$$

(5.11) $$D_p = V(Y_p)$$

$$Y_p = (Y(x-1), \; Y(x-2), \ldots, Y(x-n))'$$

$$C = \text{Cov}\,(Y_p, k_g)$$

$$Y_g = (Y(x), \; Y(x+1), \ldots, Y(x + m - 1))'$$

Now let us consider the following objective function.

(5.12) $$h = \sum_{i=1}^{t} l_i \, Y(x + i - 1); \quad t < m, \quad m \leq L - x + 1$$

The objective function g given by (5.9) can be looked upon as the objective function h given by (5.12) extended to a longer future.

Let the BLDF for h be given by,

(5.13) $$s = \sum_{i=1}^{n} q_i \, Y(x - i) \; ; \quad n < x$$

Again, the theorem in 5.3.1, q_i are given by the regression of s on Y(x − 1), Y(x − 2), ..., Y(x − n). Let us write,

(5.14)
$$\underset{\sim}{q} = D_p^{-1} \, C_1 \, \underset{\sim}{k} \quad \text{where}$$

$$\underset{\sim}{q} = (q_1, q_2, \ldots, q_n)'$$

$$\underset{\sim}{k} = (l_1, l_2, \ldots, l_t)'$$

$$C_1 = \text{Cov.}\,(Y_p, Y_s)$$

$$Y_s = (Y(x), Y(x+1), \ldots, Y(x+t-1))'$$

D_p as defined in (5.11)

Let us further define the following,

(5.15)
$$\bar{\underset{\sim}{k}} = (l_{t+1}, l_{t+2}, \ldots, l_{x+m-1})'$$

$$C_2 = \text{Cov}\,(Y_p, \bar{Y}_s)$$

$$\bar{Y}_s = (Y(x+t), Y(x+t+1), \ldots, Y(x+m-1))$$

so that we have,

$$\underset{\sim}{l} = (\underset{\sim}{k}' \vdots \bar{\underset{\sim}{k}}')' \qquad \text{see (5.11) and (5.14)}$$

$$\bar{C} = (C_1 \vdots C_2) \qquad \text{see (5.11) and (5.14)}$$

The following theorem shows the condition under which s, the BLDF for h, will be a BLDF for g also. It will be shown in the next chapter that this condition is satisfied in the Markov case.

5.5.2 <u>Theorem</u> : The BLDF s (given by (5.13 and (5.14)) for h (given by (5.12)) will provide a rule for g (given by (5.9)) as efficient as the corresponding BLDF f (given by (5.10) and (5.11)) if $C_1\underset{\sim}{k}$ and $C_2\bar{\underset{\sim}{k}}$ are linearly dependent where C_1, $\underset{\sim}{k}$, C_2, $\bar{\underset{\sim}{k}}$ are given by (5.14) and (5.15).

<u>Proof</u> : $C_1\underset{\sim}{k}$ and $C_2\bar{\underset{\sim}{k}}$ are linearly dependent implies that there exists constants α, β not both zero, such that

$$\alpha C_1\underset{\sim}{k} + \beta C_2\bar{\underset{\sim}{k}} = \underset{\sim}{0}, \quad \underset{\sim}{0} \text{ is the null vector}$$

$$\alpha C_1\underset{\sim}{k} + \beta (C\underset{\sim}{l} - C_1\underset{\sim}{k}) = \underset{\sim}{0} \quad \text{since}$$

$$C\underset{\sim}{l} = (C_1 \vdots C_2) \begin{pmatrix} \underset{\sim}{k} \\ \cdots \\ \bar{\underset{\sim}{k}} \end{pmatrix} = C_1\underset{\sim}{k} + C_2\bar{\underset{\sim}{k}}$$

$$C\underset{\sim}{l} = \upsilon C_1\underset{\sim}{k} \quad \text{where } \upsilon = \beta/(\beta - \alpha)$$

Now, $C\underset{\sim}{l} = \upsilon C_1\underset{\sim}{k}$ implies that -

$$(5.16) \qquad C_1'\underset{\sim}{k}' D_p^{-1} C\underset{\sim}{l} = (C_1'\underset{\sim}{k}' D_p^{-1} C_1\underset{\sim}{k} \cdot C'\underset{\sim}{l}' D_p^{-1} C\underset{\sim}{l})^{\frac{1}{2}}$$

by Cauchy-Schwartz inequality.

A perusal of the proof of the theorem in 5.3.1 shows that two decision functions with the same correlation coefficient with a given objective function will provide equally efficient rules. Thus, we will have to show here that -

$$r_{sg} = r_{fg} \quad \text{where}$$

$$r_{sg} = \text{Corr.} (s, g)$$

$$r_{fg} = \text{Corr.} (f, g)$$

Now,

$$r_{fg} = \frac{\underset{\sim}{w}'C\underset{\sim}{1}}{(\underset{\sim}{w}'D_p w \;\; \underset{\sim}{1}'D_g\underset{\sim}{1})^{\frac{1}{2}}}$$

where $D_g = V(Y_g)$, Y_g given by (5.11)

$$= \left(\frac{\underset{\sim}{1}'C'D_p C\underset{\sim}{1}}{\underset{\sim}{1}'D_g\underset{\sim}{1}} \right)^{\frac{1}{2}}$$

since $w = D_p^{-1}C\underset{\sim}{1}$ by (5.11)

Again,

$$r_{sg} = \frac{\underset{\sim}{q}'C\underset{\sim}{1}}{(\underset{\sim}{q}'D_p\underset{\sim}{q}. \;\; \underset{\sim}{1}'D_g\underset{\sim}{1})^{\frac{1}{2}}}$$

$$= \frac{C_1'k'D_p^{-1}\, C\underset{\sim}{1}}{(C_1'\underset{\sim}{k}'D_p^{-1}\, C_1\underset{\sim}{k} \;\; \underset{\sim}{1}'D_g\underset{\sim}{1})^{\frac{1}{2}}}$$

since $\underset{\sim}{q} = D_p^{-1}\, C_1\underset{\sim}{k}$ by (5.14),

Now the theorem follows from (5.16).

5.5.3 Remark : If s is the BLDF for a given objective function h and g is an objective function obtained by extending h (see 5.5.1) then the above theorem provides the condition under which s will be a BLDF for g also. Now, if s is merely a linear combination of the BLDF f for g given by (5.10) then the above theorem is not very useful. Let us note from the previous section that -

$$\underset{\sim}{w} = D_p^{-1}C\underset{\sim}{1} = D_p^{-1}\,(C_1 \vdots C_2)\,\begin{pmatrix} k \\ \cdots\cdots \\ \bar{k} \end{pmatrix}$$

$$= D_p^{-1}\, C_1\underset{\sim}{k} + D_p^{-1}\, C_2\underset{\sim}{\bar{k}}$$

$$= \underset{\sim}{q} + D_p^{-1}\, C_2\underset{\sim}{\bar{k}}$$

Hence, $$f = \underset{\sim}{w}'Y_p = \underset{\sim}{q}'Y_p + \underset{\sim}{\bar{k}}'\, C_2'\, D_p^{-1}\, Y_p$$

$$= s + \underset{\sim}{\bar{k}}'\, C_2'\, D_p^{-1}\, Y_p$$

Thus, f is not a linear combination of s. This means that s is computationally more convenient decision function than f for the given objective function g.

Chapter VI

A Markov Replacement Rule

6.1 Introduction

6.1.1 In this Chapter we shall consider a special case where the Normal density of the yield vector of the palm $\underset{\sim}{Y} = (Y(1), Y(2)\ldots,Y(L))'$ is Markovian.

Let us recall the assumptions on the vector $\underset{\sim}{Y}$ outlined in (1.1) through (1.7) in Chapter I.

$\underset{\sim}{Y}$ has a Multivariate Normal density such that

$$
(6.1)\qquad
\begin{aligned}
V(Y(x)) &= \sigma^2 &&\text{for } x = a_f, a_f + 1, \ldots, L\\
&= 0 &&\text{for } x = 1, 2, \ldots, a_f - 1.\\
\text{Cov.}(Y(x), Y(x')) &= \sigma^2 \rho_{xx'} &&\text{for } x, x' \geq a_f\\
\text{where } \rho_{xx'} &= \pi^{|x-x'|-1}\rho, \quad x \neq x', && 0 < \rho < 1, \; 0 < \pi < 1
\end{aligned}
$$

Let us define,

$$
(6.2)\qquad \pi = \rho
$$

This would imply that $\underset{\sim}{Y}$ has a Markovian density. In fact the above condition is both necessary and sufficient for $\underset{\sim}{Y}$ to have a Markovian density (see Feller (1966)).

6.1.2 Replacement rules in this case obviously need be based only on $Y(x-1)$ where x is the current age of the palm. In other words, a replacement decision will only depend on whether the yield of the palm in the previous year was below a certain level. Rules discussed in the last two chapters (IV and V) can be applied in this case simply by putting $n = 1$ and the results on those rules will be still valid. Moreover, the Markovian case has the following property for the decision rules.

Let $g = \sum_{i=1}^{m} l_i \, Y(x + i - 1)$, $m \leq L - x + 1$, l_i are constants not all zero, x is the current age of the palm. The BLDF (see Chapter V) for the above objective function g will be given by the regression of g on Y(x- 1).

Now, from (6.1) and (6.2) we have

$$\text{Cov.}(Y(x-1), Y(x+j)) = \rho^{j+1} \; ; \qquad j = 0, 1, 2, \ldots, m-1$$

Let us put

$$\underset{\sim}{k} = (l_1)$$
$$\bar{\underset{\sim}{k}} = (l_2, l_3, \ldots, l_m)$$
$$C_1 = (\rho)$$
$$C_2 = (\rho^2, \rho^3, \ldots, \rho^m)$$

Obviously, there exists a constant c such that $C_1 \underset{\sim}{k} = cC_2\bar{\underset{\sim}{k}}$. By applying the theorem in 5.5.2 it can be seen that the regression of $l_1Y(x)$ on Y(x-1) will provide a replacement rule corresponding to the objective function g as efficient as the BLDF corresponding to g.

6.2 Formulation of a Markov Replacement Model

6.2.1 We shall formulate now a Markov Replacement Rule (MRR) with the discounted value of net return in future (over infinite horizon) as the objective function.

For a given palm the strategy is to replace it in the current year depending on it's age and yield in the previous year. Obviously, it is not meaningful to consider replacement decision before a palm starts yielding. We shall consider the replacement decision only if the palm is above age a_o, $a_o \geq a_f$. Let J(t) be the age of a palm in year t, t = 1, 2, ...

Let us define,

$$J(1) = k \quad \text{where k is arbitrary age,} \quad 1 \leq k \leq L.$$

(6.3)
$$J(t) = J(t-1)+1 \quad \text{if } J(t-1) < a_o \quad \text{or } a_o \leq J(t-1) < L \text{ and } Y(J(t-1)) > Z(J(t-1))$$

$$= 1 \quad \text{if } J(t-1) = L \quad \text{or} \quad a_o < J(t-1) < L \text{ and } Y(J(t-1)) \leq Z(J(t-1))$$

where Z(.) are given constants.

We shall refer to Z(x) as the 'threshold' value at age x. Thus, the replacement rule is — replace a palm at age x only if $a_o < x < L$ and $Y(x-1) \leq Z(x-1)$, or $x \geq L$. It is easy to see that J(t) is a Markov Chain.

Markov replacement rules have been considered by a few authors in the case of ageing assets. See, for example, Kao (1973) who formulated replacement rules when the asset deteriorates over time and the changes of state are Semi-Markovian. Ward and Faris (1968) considered Palm trees whose yield performance is Markovian.

Now, the problem is to determine $(Z(x), x = a_o, \ldots\ldots, L-1))$. A given $(Z(x), x = a_o, \ldots\ldots, L-1))$ will be referred to as a given replacement rule. We shall attempt a formulation here which is similar to the Markov Reward Process Model due to Haward (1960). But we will not be concerned with an optimal rule $(Z(x), x = a_o, \ldots\ldots, L-1))$ which will maximise the objective function. We merely propose to develop a computational procedure by which a Markov Replacement Rule can be derived to improve upon the deterministic replacement rule given in Chapter II. By improvement we mean an increase in the value of the objective function i.e., the discounted value of future net returns.

6.2.2 The transition matrix $((P_{ij}))$ corresponding to the Markov Chain J(t), given by (6.3), can be constructed as follows.

It can be seen that J(t) is a Markov Chain with L states. From any state i, $a_o \leq i < L$, there can be transition either to state (i+1) or to state 1. From any state i, $1 \leq i < a_o$, there can be transition only to state (i+1). From state L, there can be transition only to state 1. Thus, we have,

$$
\begin{aligned}
&P_{i1} = 0 && \text{for } i = 1, 2, \ldots, a_o - 1 \\
&P_{i1} = P(Y(i) \leq Z(i)) && \text{for } i = a_o \\
(6.4)\qquad &P_{i1} = P(Y(i) \leq Z(i) \mid Y(i-1) > Z(i-1)) && \text{for } i = 1 + a_o, \ldots, L-1. \\
&P_{i1} = 1 && \text{for } i = 1 \\
&P_{i\,i+1} = 1 - P_{i1} && \text{for } i = 1, 2, \ldots, L \\
&P_{ij} = 0 && \text{for } j \neq i + 1,\ j \neq 1
\end{aligned}
$$

Corresponding to every transition from state i to state j we define a reward R_{ij}. Let us notice that we need to define R_{ij} only for those transitions (i, j) for which $P_{ij} \neq 0$. Also, these R_{ij} will depend on the given rule $(Z(x), x = a_o, \ldots\ldots, L-1))$. For a given rule, we define R_{ij} in the case of replacement by replantation as follows.

$$R_{ij} = E(R(i)) = r(i) \qquad \text{for } i = 1, 2, \ldots, a_o \text{ and } j = i+1$$

(6.5)

$$= E(R(i) \mid Y(i) > Z(i)) \qquad \text{for } i = 1 + a_o \text{ and } j = i+1$$

$$= E(R(i) \mid Y(i) \leq Z(i)) \qquad \text{for } i = 1 + a_o \text{ and } j = 1$$

$$= E(R(i) \mid Y(i) > Z(i), Y(i-1) > Z(i-1))$$
$$\text{for } i = 2+a_o, \ldots, L-1 \text{ and } j = i+1$$

$$= E(R(i) \mid Y(i) \leq Z(i), Y(i-1) > Z(i-1))$$
$$\text{for } i = 2+a_o, \ldots, L \text{ and } j = 1$$

where $R(i) = p. Y(i) - s(i)$, the actual net return from a palm at age i,

$r(i) = E(R(i))$, see 2.2.2.

In the case of replacement by underplantation we need to modify R_{ij} for the transitions from state i to state 1 since the palm to be replaced will be retained for u years more before it is finally removed. Thus, in this case we define,

(6.6)

$$R_{i1} = \sum_{n=i}^{i+u} E(R(u) \mid Y(i) \leq Z(i))d^{n-i-1} \qquad \text{for } i = a_o$$

$$= \sum_{n=1}^{i+u} E(R(n) \mid Y(i) \leq Z(i), Y(i-1) > Z(i-1))$$
$$\text{for } i = 1 + a_o, \ldots, L$$

R_{ij} defined above in (6.5) and (6.6) are consistent with the deterministic replacement model (infinite horizon case) developed in Chapter II. In the present model, the deterministic replacement rule corresponds to the rule given below.

(6.7)

$$a_o = A$$

$Z(x) = +\infty$ for x = [redacted] $a_o, \ldots, L-1$

where (A+1) is the replacement age in the deterministic version.

It can be seen that the above rule specifies that replace a palm only if it's age is above A. This is exactly the deterministic rule discussed in Chapter II. It will be shown in 6.3.4 that the above rule leads to the same expression for the objective function in the present model as that in the deterministic version given in 2.2.

6.3 Procedure to Derive a Markov Replacement Rule

6.3.1 We shall consider the case of replacement by replantation here. The necessary modification for replacement by underplantation can be easily achieved by replacing R_{i1} given in (6.5) by the expressions given in (6.6).

Let V_i be the discounted value of future net returns for a given rule $(Z(x), x = a_o, \ldots\ldots, L-1))$ corresponding to a palm at age i in the initial year. It is easy to see that the following recursive relation holds true.

$$V_i = \sum_{j=1}^{L} P_{ij} R_{ij} + d \sum_{j=1}^{L} P_{ij} V_j, \quad i = 1, 2, \ldots, L \tag{6.8}$$

where d is the discount factor.

Let us define,

$$Q_i = \sum_{j=1}^{L} P_{ij} R_{ij}, \qquad i = 1, 2, \ldots, L \tag{6.9}$$

From (6.8), (6.9) and (6.4) we can write,

$$\begin{aligned} V_i &= Q_i + dV_{i+1} && \text{for } i = 1, 2, \ldots, a_o-1 \\ &= Q_i + d(P_{i1} V_1 + P_{ii+1} V_{i+1}) && \text{for } i = a_o, \ldots\ldots, L-1 \\ &= Q_L + dV_1 && \text{for } i = L \end{aligned} \tag{6.10}$$

$$\begin{aligned} \text{where } Q_i &= R_{ii+1} && \text{for } i = 1, 2, \ldots, a_o - 1 \\ &= P_{i1} R_{i1} + P_{ii+1} R_{ii+1} && \text{for } i = a_o, \ldots\ldots, L-1. \\ &= R_{L1} && \text{for } i = L \end{aligned}$$

It [redacted] is easy to see from (6.5) and (6.10) that

$$Q_i = r(i) \qquad \text{for } i = 1, 2, \ldots, a_o$$

(6.11) $$= E(R(i) \mid Y(i-1), Z(i-1)) \quad \text{for } i = 1 + a_o, \ldots, L.$$

6.3.2 For a given rule $(Z(x), x = a_o, \ldots\ldots, L-1)$ the discounted values V_i can obtained by solving the equations given in (6.10). From (6.10) we have,

$$V_1 = Q_1 + dV_2$$
$$V_2 = Q_2 + dV_3$$
$$\ldots \quad \ldots \quad \ldots$$
$$V_{a_o-1} = Q_{a_o-1} + dV_{a_o}$$

So, $$V_1 = Q_1 + d\{Q_2 + dV_3\}$$
$$= Q_1 + dQ_2 + d^2V_3$$

and so on.

Thus, we can write

(6.12) $$V_1 = \sum_{i=1}^{a_o-1} Q_i\, d^{i-1} + d^{a_o-1} V_{a_o}$$

Again, from (6.10) we have

$$V_{a_o} = Q_{a_o} + d\{ P_{a_o 1} V_1 + P_{a_o(1+a_o)} V_{1+a_o} \}$$

$$= Q_{a_o} + d P_{a_o 1} V_1 + dP_{a_o(1+a_o)} \Big[Q_{1+a_o}$$

$$+ d\{ P_{(1+a_o)1} V_1 + P_{(1+a_o)(2+a_o)} V_{2+a_o} \} \Big]$$

$$= Q_{a_o} + dP_{a_o(1+a_o)} Q_{1+a_o} + dP_{a_o 1} V_1$$

$$+ d^2 P_{a_o(1+a_o)} P_{(1+a_o)1} V_1$$

$$+ d^2 P_{a_o(1+a_o)} P_{(1+a_o)(2+a_o)} V_{2+a_o}$$

Thus, we can write

$$(6.13) \quad V_{a_o} = Q_{a_o} + \sum_{i=a_o}^{L-2} d^{\,i-a_o+1} \{ \prod_{j=a_o}^{i} P_{jj+1} \} Q_{i+1}$$

$$+ d \Big[P_{a_o 1} + \sum_{i=a_o}^{L-2} d^{\,i-a_o+1} \{ \prod_{j=a_o}^{i} P_{jj+1} \} P_{(i+1)1} \Big] V_1$$

$$+ d^{\,L-a_o} \{ \prod_{i=a_o}^{L-1} P_{ii+1} \} V_L$$

So, we have

$$V_1 = C_1 + d^{\,a_o - 1} V_{a_o} \qquad \text{from (6.12)}$$

$$V_{a_o} = C_2 + C_3 V_1 + C_4 V_L \qquad \text{from (6.13)}$$

$$V_L = C_5 + dV_1 \qquad \text{from (6.10)}$$

which imply

$$(6.14) \quad V_1 = \{ C_1 + d^{a_o - 1} (C_2 + C_4 C_5) \} / \{ 1 - d^{a_o - 1} (C_3 + dC_4) \}$$

where $C_1 = \sum_{i=1}^{a_o - 1} Q_i d^{i-1}$

$$C_2 = Q_{a_o} + \sum_{i=a_o}^{L-2} d^{i-a_o+1} \{ \prod_{j=a_o}^{i} P_{jj+1} \} Q_{i+1}$$

$$C_3 = d \left[P_{a_o 1} + \sum_{i=a_o}^{L-2} d^{i-a_o+1} \{ \prod_{j=a_o}^{i} P_{jj+1} \} P_{(i+1)1} \right.$$

$$C_4 = \{ \prod_{i=a_o}^{L-1} P_{i(i+1)} \} d^{L-a_o}$$

$$C_5 = Q_L$$

The rest of $V_2, V_3, \ldots, V_L$ can be obtained as follows.

$$V_L = C_5 + dV_1 \qquad \text{where } C_5 = Q_L$$

$$(6.15) \quad V_i = Q_i + d \{ P_{i1} V_1 + P_{ii+1} V_{i+1} \}, \quad i = a_o, \ldots, L-1$$

$$V_i = Q_i + dV_{i+1}, \qquad i = 2, \ldots, a_o - 1$$

where Q_i are given by (6.11).

6.3.3 Ideally, one should derivethe optimal rule $(Z(x), x = a_o, \ldots, L-1)$ over the set of $(L - a_o)$ - tuples of positive real numbers. This can be obtained by maximising V_1 expressed in (6.14) using one of the standard mathods available for the purpose.

Now, the Markov Replacement Rule (MRR) is considered here as an improvement over the Deterministic rule. We have already obtained a replacement rule given by the deterministic version in Chapter II. In the present formulation the deterministic optimal rule will be given by,

$$a_o \leq A$$

$$Z(x) = +\infty \qquad \text{for } x = A, \ldots, L-1$$

$$= -\infty \qquad \text{for } x = a_o, \ldots, A-1 \quad \text{if } a_o \leq A-1$$

where 1 + A is the deterministic optimal replacement age.

As mentioned earlier, it will be shown in the next sub-section that V_1 corresponding to the above rule is the same as the objective function specified in the deterministic model. Here we propose a simple procedure for deriving a MRR by improving upon the deterministic rule mentioned above. The procedure maximises V_1 given a small set of alternative values for each $Z(x)$, $x = a_o, \ldots, L-1$, by direct enumeration. The steps in the computation are as follows.

Let V_1^o be the value of V_1 corresponding to the deterministic rule to be denoted by $\{Z^o(x)\}$.

For $x = a_o$, find out the alternative value $Z_k(a_o)$ that maximises V_1 over the set of alternative values (pre-specified) and $Z^o(a_o)$. This is to be done by direct enumeration of the expression of V_1 given by (6.14). Now, consider the new rule obtained by replacing $Z^o(a_o)$ by the $Z_k(a_o)$. Starting with this new rule, for $x = 1 + a_o$, again find out the alternative value $Z_k(1 + a_o)$ that maximises V_1 over the set of alternative values (pre-specified for $Z(1+a_o)$) and $Z^o(1+a_o)$. Thus, we obtain the next improved rule by replacing $Z^o(1+a_o)$ by $Z_k(1+a_o)$. Repeat the same procedure for

$x = 3 + a_o$, ..., L-1. This will complete the first iteration. Let $\{Z^1(x)\}$ be the new rule derived in the first iteration and V_1^1 be the corresponding value of V_1.

Now, for the second iteration, start with $\{Z^1(x)\}$ and repeat the same procedure as in the first iteration with the same set of pre-specified alternative values for each $Z(x)$. Thus, we would obtain $\{Z^2(x)\}$ and V_1^2 at the end of the second iteration.

Carry on the iterations till the value of V_1 converges or one has obtained a satisfactory improvement over V_1^o — the value of V_1 corresponding to the deterministic optimal rule.

The above procedure is illustrated in 6.5 with nummerical examples.

Let us note that it may require to try a number of sets of alternative values for each $Z(x)$ before one obtains a satisfactory improvement in the value of V_1.

6.3.4 Now, we shall show that with the matrices $((P_{ij}))$ and $((R_{ij}))$ i.e., the transition matrix and the reward matrix specified in 6.2.2, the present model is consistent with the deterministic model (infinite horizon case) discussed in Chapter II. To do this we derive the expression for the objective function in the present model corresponding to the rule given in (6.7) which is equivalent to the deterministic replacement rule. Only the case of replacement by replantation is considered here. The case of underplantation can be similarly derived.

It can be seen from (6.4) that corresponding to $(Z(x), x = a_o, \ldots, L-1)$ we have

$$(6.16) \qquad P_{ij} = 1 \quad \text{if } i = 1, 2, \ldots, A-1 \text{ and } j = i+1. \quad \text{or } i = A, \ldots, L \text{ and } j = 1. \qquad = 0 \text{ otherwise}$$

Therefore, from (6.14) we have

$$C_1 = \sum_{i=1}^{A-1} Q_i \, d^{i-1}$$

$$(6.17) \qquad C_2 = Q_A \qquad \text{since from (6.16) } P_{jj+1} = 0 \text{ for } j \geq A$$

$$C_3 = d \qquad \text{'' '' '' '' '' ''}$$

$$C_4 = 0$$

$$C_5 = Q_L$$

Now, from (6.14) and (6.17) we have

$$V_1 = \sum_{i=1}^{A} Q_i \, d^{i-1} / (1 - d^A)$$

Again, from (6.11), we have

$$Q_i = r(i), \qquad i = 1, 2, \ldots, A$$

Therefore, we get

$$V_1 = \sum_{i=1}^{A} r(i) \, d^{i-1} / (1 - d^A)$$

which is nothing but the objective function in the deterministic version in the case of replacement by replantation discussed in Chapter II. (See (2.11) and put $u = 0$).

6.4 Simplified Formulae for Computation

6.4.1 For computing the transition matrix $((P_{ij}))$ given by (6.4), bivariate normal probabilities have to be computed. We have used the method suggested by Owen (1956) in our empirical exercise. For computing the univariate normal probabilities we have used the IBM subroutine available.

6.4.2 In order to compute the Q_i values given by (6.11) we have used the following result.

Let $E(Y(j) \mid Y(x-1) = \lambda) = \alpha + \beta\lambda, \quad j \geq x.$

Let $f(s,t)$ be the joint density of $(Y(j), Y(x-1))$ and $f_1(t)$ be the density of $Y(x-1)$.

Then,

$$E(Y(j) \mid Y(x-1) > Z(x-1)) = \frac{\int_{-\infty}^{\infty} \int_{Z(x-1)}^{\infty} s\, f(s,t)\, ds\,dt}{\int_{Z(x-1)}^{\infty} f_1(t)\, dt} \tag{6.18}$$

$$= \frac{\int_{Z(x-1)}^{\infty} (\alpha + \beta t)\, f_1(t)\,dt}{\int_{Z(x-1)}^{\infty} f_1(t)\, dt}$$

$$= \alpha + \beta \frac{\int_{Z(x-1)}^{\infty} t\, f_1(t)\, dt}{\int_{Z(x-1)}^{\infty} f_1(t)\,dt}$$

$$= \alpha + \beta\, E(Y(x-1) \mid Y(x-1) > Z(x-1))$$

In order to compute $E(Y(x) \mid Y(x) > Z(x))$ we have used the following formula given by Johnson and Kotz (1970).

Let G be a normal variate with mean μ and variance σ^2. Then,

$$E(G \mid A < G \le B)$$

$$(6.19) \qquad = \mu + \left[\left\{ H\left(\frac{A-\mu}{\sigma}\right) - H\left(\frac{B-\mu}{\sigma}\right) \right\} \div \left\{ \Phi\left(\frac{B-\mu}{\sigma}\right) - \Phi\left(\frac{A-\mu}{\sigma}\right) \right\} \right] \sigma$$

where $H(\cdot)$ and $\Phi(\cdot)$ are the ordinate and the distribution function of the standard normal density.

6.5 Empirical Exercise

6.5.1 Let us now work out a few numerical examples by applying the iterative procedure described in the previous section. We shall assume the same mean yield profile as specified in 1.3. In order to illustrate the advantage of a Markov Replacement Rule (MRR) over the Deterministic one we shall examine the sensitivity of MRR to two relevant parameters viz., ρ and σ^2. These parameters are discussed in 6.1.1. We have chosen two different values for each of the two parameters mentioned above. Let us specify that $a_o = 55$ i.e., decision to replace is considered only age 55 years onwards. We choose (10, 30, 50, 70) as the set of alternative threshold values to be considered for each age x, $x = a_o, \ldots \ldots, L-1$.

In all the four cases the iterative procedure converged by not more than four steps. The results are presented in the next two sub-sections.

6.5.2 **TABLE** : Optimal Threshold Values in Different Cases

Age x	Threshold values Z(x)				
	Deterministic case	$\rho = 0.62$, $\sigma = 30$	$\rho = 0.62$, $\sigma = 15$	$\rho = 0.95$, $\sigma = 30$	$\rho = 0.95$, $\sigma = 15$
55	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
56	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
57	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
58	$-\infty$	$-\infty$	$-\infty$	10	$-\infty$
59	$-\infty$	$-\infty$	$-\infty$	10	$-\infty$
60	$-\infty$	$-\infty$	$-\infty$	30	10
61	$-\infty$	$-\infty$	$-\infty$	30	30
62	$-\infty$	10	$-\infty$	30	30
63	$-\infty$	10	10	30	30
64	$-\infty$	10	30	30	30
65	$-\infty$	30	30	30	30
66	$-\infty$	30	30	30	30
67	$+\infty$	30	50	30	50
68	$+\infty$	30	50	30	50
69	$+\infty$	50	50	30	50
70	$+\infty$	50	50	50	50
71	$+\infty$	50	50	50	50
72	$+\infty$	50	50	50	50
73	$+\infty$	50	50	50	50
74	$+\infty$	50	$+\infty$	50	50
75	$+\infty$	50	$+\infty$	50	50
76	$+\infty$	70	$+\infty$	50	50
77	$+\infty$	$+\infty$	$+\infty$	50	50
78	$+\infty$	$+\infty$	$+\infty$	50	50
79	$+\infty$	$+\infty$	$+\infty$	50	50

6.5.3 **TABLE :** Optimal Values of a Few Selected V_x in Different Cases

Age x	Deterministic case $V_x^{(o)}$	Optimal discounted values: $\rho=0.62$, $\sigma=30$, $V_x^{(1)}$	$\rho=0.62$, $\sigma=15$, $V_x^{(2)}$	$\rho=0.95$, $\sigma=30$, $V_x^{(3)}$	$\rho=0.95$, $\sigma=15$, $V_x^{(4)}$	$V_x^{(1)} \div V_x^{(o)}$	$V_x^{(2)} \div V_x^{(o)}$	$V_x^{(3)} \div V_x^{(o)}$	$V_x^{(4)} \div V_x^{(o)}$
1	470.63	471.44	470.90	473.38	471.38	1.0017	1.0006	1.0053	1.0016
10	823.94	825.20	824.36	828.21	825.10	1.0015	1.0005	1.0052	1.0014
20	844.83	846.87	845.50	851.78	846.72	1.0024	1.0008	1.0082	1.0022
30	813.92	817.24	815.01	825.25	817.00	1.0041	1.0013	1.0139	1.0038
40	763.57	768.99	765.36	782.02	768.59	1.0071	1.0023	1.0242	1.0066
50	681.57	690.39	684.47	711.62	659.74	1.0129	1.0043	1.0441	1.0120
60	547.99	562.35	552.71	603.03	561.30	1.0262	1.0086	1.1005	1.0243
70	463.79	494.30	480.01	517.87	502.85	1.0658	1.0350	1.1166	1.0342
75	450.29	479.98	450.54	527.49	492.04	1.0659	1.0006	1.1714	1.0927

6.5.4 Let us recall that V_x is the discounted value of net returns over an infinite horizon starting with a palm at age x for a given set of threshold values which defines a replacement rule. Since the life span of a palm is very long (80 years) and replacement decision is considered only at a late age ($a_o = 55$) of the palm the improvement in V_1 corresponding to a MRR over that corresponding to the Deterministic one is not vary pronounced. So, we have presented V_x for a few selected x instead of V_1 only.

It is clear from the table in 6.5.3 that a Markov Replacement Rule performs better when the variance of yield is larger and more so when the coefficient of correlation (ρ) between yields of successive ages is higher. However, it must be noted that the extent of improvement achieved by a Markov rule over the Deterministic rule is not very impressive.

Chapter VII

Estimation of the Future Mean Yield Profile When the Current Age in Unknown

7.1 Introduction

7.1.1 So far we had been discussing construction of replacement rules assuming all the while that the current age of the given palm is known. Let us suppose now that the current age is not known. In this case, we do not have any knowledge of the future mean yield sequence of the palm. But suppose we have the past performance of the palm known over a certain period of time.

Let $Z(1), Z(2), \ldots, Z(n)$ be the actual annual yields of the palm observed during the past n years, $Z(i)$ being the yield in the ith year counted backward.

Let $Z(i) = Y(x-i)$, $i = 1(1)n$, where x is the present age of the palm, x is unknown. Let us suppose that $x - n > a_s$ i.e., the palm is in it's full bearing or declining yield phase (see 1.2).

Let $Z = (Z(1), Z(2), \ldots, Z(n))'$

$V(Z) = D$

where D is the dispersion matrix of Z (see (1.3)).

$E(Z) = N$ which is not known since the present age is not known, a (n x 1) column vector.

We shall assume D to be known in this Chapter. Let us note from (1.3) that D is independent of the unknown age x. We shall consider the following two cases.

(i) $a_s < x-n < a_e$ i.e., declining yield phase has set in sometime during the past n years, if at all. m_2 and s_2 i.e., the mean of stable yield during the full bearing phase and the annual rate of decline in the mean during declining yield phase (see (1.4)), are known and so also σ^2 — the common variance of Z(i).

(ii) $a_s < x-n$ i.e., declining yield phase might or might not have set in even before the period for which the yield performance is known i.e., before n years. Here both m_2, s_2 are unknown. For convenience, we shall drop the suffixes and simply refer to m and s. Since we are considering palms beyond age a_s this should cause no confusion.

7.1.2 We are interested in constructing replacement rules when the current age x of the palm is not known. For this purpose, we need an estimate of the expected yield stream from the palm in future. With this objective in mind, in case (i), we shall construct a test of significance to detect whether the declining yield phase has set in sometime in the past n years. We shall obtain a maximum likelihood estimator of the year at which the decline in the mean begins.

Note that an estimation of the current age x is not possible in general in case (ii). For example, if we find that the mean is declining since the beginning of the period of observation of Z then all we can say is $a_e \leq x-n$. What we can do in this case is to find out the current mean and the rate of decline.

7.2 Maximum Likelihood Estimation and Test of Significance in Case (i) : $a_s < x - n \leq a_e$, m, s and σ^2 known

7.2.1 Let us consider $Z(1), Z(2), \ldots, Z(n)$ as a sample of n units ordered (descending) in time. In this case the declining yield phase has set in (if at all) sometime during the period of observation i.e., $a_s < x - n < a_e$. If we look at the problem as one of detecting a change in the mean over time it becomes similar to the well-known 'slippage problem' with two important differences. Firstly, here the sample units are dependent random variables and secondly, there is more than one shift in the mean.

There is a considerable literature on the slippage problems. For the material in this section we particularly refer to Hinkley (1970), [redacted] Sen and Srivastava (1975) and Hawkins (1977). We also refer to Barnard (1959) where dependent sample units for constructing control charts were considered.

7.2.2 We consider the following hypotheses.

(7.1)

$$H_0 : Z \sim N\left(M^{(n)}, D\right)$$

$$H_1 : Z \sim N\left(M^{(k)}, D\right), \qquad k = 1, 2, \ldots, n-1.$$

$$\text{where } \underset{(n \times 1)}{M^{(n)}} = (m, m, \ldots, m)'$$

$$\underset{(n \times 1)}{M^{(k)}} = (\underbrace{m, m, \ldots, m}_{k \text{ times}}\ m-s, \ldots, m-(n-k)s)'$$

with m, s and D known but k unknown, $k = 1, 2, \ldots, n-1$, $m > 0$, $s > 0$

H_0 specifies that there was no decline in the mean over the past n years.

H_1 specifies that a decline in the mean has started from the (k+1)th year, $k = 1, 2, \ldots, n-1$.

Let $L_i(Z)$ be the likelihood of Z under hypothesis H_i, i= 0, 1.

Consider

$$\frac{L_1(Z)}{L_o(Z)} = \frac{\exp - \frac{1}{2} \{(Z - M^{(k)})' D^{-1} (Z - M^{(k)})\}}{\exp - \frac{1}{2} \{(Z - M^{(n)})' D^{-1} (Z - M^{(n)})\}}$$

Thus,

$$\log_e \frac{L_1(Z)}{L_o(Z)} = -\frac{1}{2} \{ (Z - M^{(k)})' D^{-1} (Z - M^{(k)}) - (Z - M^{(n)})' D^{-1} (Z - M^{(n)}) \}$$

With some rearrangement we obtain

$$\log_e \frac{L_1(Z)}{L_o(Z)} = Z' D^{-1} (M^{(k)} - M^{(n)}) - \frac{1}{2}(M^{(k)}+M^{(n)})' D^{-1} (M^{(k)}-M^{(n)})$$

$$= w(k), \quad \text{say}, \quad k = 1, 2, \ldots, n-1.$$

Let us define

(7.1a)
$$\underset{(n \times 1)}{\underset{\sim}{e}} = (1, 1, \ldots, 1)'$$

$$\underset{\sim}{h}_k = (\underbrace{0, 0, \ldots, 0}_{k \text{ times}}, -1, -2, \ldots, -(n-k))'$$

Note that

$$M^{(k)} - M^{(n)} = s \underset{\sim}{h}_k$$

$$M^{(k)} + M^{(n)} = 2m \underset{\sim}{e} + s \underset{\sim}{h}_k$$

It is easy to see that

$$w(k) = Z' D^{-1} \underset{\sim}{h}_k s - \frac{1}{2} (2m \underset{\sim}{e} + s \underset{\sim}{h}_k)' D^{-1} \underset{\sim}{h}_k s$$

$$= s Z' D^{-1} \underset{\sim}{h}_k - s m \underset{\sim}{e}' D^{-1} \underset{\sim}{h}_k - \frac{s^2}{2} \underset{\sim}{h}_k' D^{-1} \underset{\sim}{h}_k$$

The maximum likelihood estimate of k is given by -

$$\hat{k} = \text{such that } w(\hat{k}) = \max_i w(i)$$

Thus, the likelihood ratio rule for testing H_o against H_1 is given by $w(\hat{k})$ — reject H_o if $w(\hat{k}) > c$, where c is a suitable constant.

$$\left.\begin{aligned} E(w(\hat{k})) &= \frac{s^2}{2}\, \underset{\sim}{h}'_{\hat{k}}\, D^{-1}\, \underset{\sim}{h}_{\hat{k}} \\ V(w(\hat{k})) &= s^2\, \underset{\sim}{h}'_{\hat{k}}\, D^{-1}\, \underset{\sim}{h}_{\hat{k}} \end{aligned}\right\} \quad \text{under } H_o$$

So, we obtain the test statistic as -

$$(7.2) \qquad W^* = \frac{1}{\sqrt{\underset{\sim}{h}'_{\hat{k}}\, D^{-1}\, \underset{\sim}{h}_{\hat{k}}}} (Z - m\,\underset{\sim}{e})'\, D^{-1}\, \underset{\sim}{h}_{\hat{k}} \sim N(0, 1)$$

It is interesting to note that the test statistic above does not involve s — the rate of decline.

7.3 An Alternative Approach in Case (i)

7.3.1 For the purpose of constructing replacement rules we are chiefly interested in an estimate of the future mean yield profile of the palm. For this reason our main attempt in this Chapter is to find out whether a decline in the mean of the annual yield has set in already and if so since when ?

We shall adopt here a discrimination or classification approach which was first suggested by Page (1957) for slippage problems.

Let us consider the following hypotheses

$$H_k : \; N = M^{(k)} = (m, \; m, \; \dots, \; m, \; m-s, \; \dots, \; m-(n-k)s)'$$

where $k = 1, 2, \dots, n$

$$N = E(Z) = M^{(k)} \quad \text{under } H_k$$

In order to detect if a decline in the mean of the annual yield of the palm has set in sometime in thepast n years we are to discriminate between hypotheses H_k, $k = 1(1)n$. A method of discrimination or classification is specified by the definition of a division of the whole sample space into mutually exclusive regions I_k, $k = 1(1)n$, so that if the sample point falls within region I_k the hypothesis H_k is accepted.

Let q_k be the a priori probability that H_k is true,

$$\sum_{k=1}^{n} q_k = 1$$

By Rao (1973) the probability of misclassification is minimised when I_k are defined by -

$$Z \varepsilon I_k \quad \text{if } q_k L_k(Z) \geq q_i L_i(Z); \quad i \neq k$$

where $L_i(Z)$ is the likelihood of Z under H_i, $i = 1(1)n$. But in this particular case we do not have any way of allocating the a priori probabilities. We shall assume $q_k = \frac{1}{n}$, $k = (1)n$. Such a priori probabilities has been suggested in various papers, see for example — Randles and et. al (1978).

Thus, by the present method hypothesis H_k will be preferred if

$$Z \varepsilon I_k \quad \text{i.e., } L_k(Z) \geq L_i(Z), \quad i \neq k$$

This method of discrimination is well-known and has been studied in different papers. For Normal case, this method has been well summarised by Anderson (1958).

Here we propose another method of discrimination whose simplicity in this specific case is appealing for practical applications.

7.3.2 Suppose we consider the likelihood of $\bar{Z}$ instead of Z where

$$\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z(i)$$

We propose to accept hypothesis H_k if

$$\bar{Z} \varepsilon J_k \quad \text{i.e.,} \; L_k(\bar{Z}) \geq L_i(\bar{Z}), \qquad i \neq k$$

where $L_k(\bar{Z})$ is the likelihood of $\bar{Z}$ under H_k.

Now,

$$E(\bar{Z} \mid H_k) = \left[\sum_{i=1}^{k} m + \sum_{i=k+1}^{n} \{ m - (i-k)s \} \right] \frac{1}{n}$$

(7.2a)

$$= m - \frac{s}{2n} (n-k)(n-k+1) \quad \text{for } k = 1, 2, \ldots, n-1$$

$$= m \quad \text{for } k = n$$

$$V(\bar{Z} \mid H_k) = \frac{1}{n^2} e' D e \quad \text{for all } k$$

where $\underset{(1 \times n)}{e'} = (1, 1, \ldots, 1)$ so that $\bar{Z} = \frac{1}{n} e' Z$

So, $\bar{Z} \varepsilon J_k$ implies

$$\frac{L_k'(\bar{Z})}{L_i'(\bar{Z})} \geq 1, \qquad i \neq k$$

$$\text{i.e. } \log_e \frac{L_k'(\bar{Z})}{L_i'(\bar{Z})} \geq 0, \qquad i \neq k$$

$$\text{i.e. } -\frac{n^2}{2\underset{\sim}{e}' D \underset{\sim}{e}} \left\{ \left(\bar{Z} - \frac{1}{n} \underset{\sim}{e}' M^{(k)}\right)^2 - \left(\bar{Z} - \frac{1}{n} \underset{\sim}{e}' M^{(i)}\right)^2 \right\} \geq 0, \qquad i \neq k$$

$$\text{i.e. } \left\{ \left(\bar{Z} - \frac{1}{n} \underset{\sim}{e}' M^{(k)}\right)^2 - \left(\bar{Z} - \frac{1}{n} \underset{\sim}{e}' M^{(i)}\right)^2 \right\} \leq 0, \qquad i \neq k$$

$$\text{i.e. } \frac{1}{n} \left(\underset{\sim}{e}' M^{(k)} - \underset{\sim}{e}' M^{(i)}\right) \left(\frac{1}{n} \underset{\sim}{e}' M^{(k)} + \frac{1}{n} \underset{\sim}{e}' M^{(i)} - 2\bar{Z} \right) \leq 0, \qquad i \neq k$$

where $\underset{\sim}{e} = (1, 1, \ldots, 1)'$, $(n \times 1)$ column vector.

But $\underset{\sim}{e}'M^{(k)} - \underset{\sim}{e}'M^{(i)} = \frac{s_1}{2}\{(n-i)(n-i+1) - (n-k)(n-k+1)\}$ from (7.2a)

$$\geq 0 \quad \text{for } i = 1, 2, \ldots, k$$

$$\leq 0 \quad \text{for } i = k+1, \ldots, n$$

$\therefore \bar{Z} \varepsilon J_k$ implies for $k < n$

$$2\bar{Z} \geq \frac{1}{n}(\underset{\sim}{e}M^{(k)} + \underset{\sim}{e}M^{(i)}) \quad \text{for } i = 1, 2, \ldots, k-1.$$

$$2\bar{Z} \leq \frac{1}{n}(\underset{\sim}{e}M^{(k)} + \underset{\sim}{e}M^{(i)}) \quad \text{for } i = k+1, \ldots, n$$

and for $k = n$

$$2\bar{Z} \geq \frac{1}{n}(\underset{\sim}{e}'M^{(k)} + \underset{\sim}{e}'M^{(i)}) \quad \text{for } i = 1, 2, \ldots, k-1$$

But $\underset{\sim}{e}'M^{(k)} + \underset{\sim}{e}'M^{(i)} = 2nm - \frac{s_1}{2}\{(n-k)(n-k+1) + (n-i)(n-i+1)\}$

Thus, it is easy to see that

(7.3) $\quad \bar{Z} \varepsilon J_k$ implies

$$a_{k-1} < \bar{Z} \leq a_k, \qquad k = 1, 2, \ldots, n-1$$

$$a_{n-1} < \bar{Z} \qquad k = n$$

where $a_o = -\infty$

$$a_k = m - \frac{s_1}{2n}(n-k+1)^2, \qquad k = 1, 2, \ldots, n-1$$

7.3.3 Thus, we have two methods of discrimination between hypotheses H_k, $k = 1(1)n$. One is the standard method found in the literature which is based on the originally observed Z and the other is based on $\bar{Z}$ — the sample mean.

Method based on Z is given by the partition of sample space of Z—

$$\{ I_k : L_k(Z) \geq L_i(Z), \quad i \neq k, \quad k = 1(1)n \}$$

Method based on $\bar{Z}$ is given by the partition of sample space of $\bar{Z}$ —

$$\{ J_k : L'_k(\bar{Z}) \geq L'_i(\bar{Z}), \quad i \neq k, \quad k = 1(1)n \}$$

As mentioned earlier, the simplicity of the classification method (computationally) based on $\bar{Z}$ as demonstrated in (7.3) makes it attractive. Let us now compare the above two methods in the following lemma.

<u>Lemma</u> : The classification based on $\bar{Z}$ is as efficient as the classification based on Z in terms of the probability of correct classification.

<u>Proof</u>: Probability of correct classification for a given method of classification is given by -

$$\frac{1}{n} \sum_{k=1}^{n} P(k)$$

where P(k) is the probability that hypothesis H_k is preferred when it is true.

Let $\quad I'_k = \{ Z : \bar{Z} \varepsilon J_k \}, \quad k = 1(1)n$

Notice that $\{I_k, k = 1(1)n \}$ maximises the probability of correct classification over all partitions of the sample space of Z. So we have

$$(7.4) \qquad \sum_{k=1}^{n} P(Z \varepsilon I_k \mid H_k) \geq \sum_{k=1}^{n} P(Z \varepsilon I'_k \mid H_k)$$

Let $\quad J'_k = \{ \bar{Z} : Z \varepsilon I_k \}, \quad k = 1(1)n$

Again, notice that $\{ J_k, k = 1(1)n \}$ maximises the probability of correct classification over all partitions of the sample space of $\bar{Z}$. So, we have

$$(7.5) \qquad \sum_{k=1}^{n} P(\bar{Z} \varepsilon J_k \mid H_k) \geq \sum_{k=1}^{n} P(\bar{Z} \varepsilon J'_k \mid H_k)$$

But $P(\bar{Z} \varepsilon J_k \mid H_k) = P(Z \varepsilon I'_k \mid H_k)$

$P(\bar{Z} \varepsilon J'_k \mid H_k) = P(Z \varepsilon I_k \mid H_k)$

for $k = 1(1)n$.

Therefore, from (7.5) we have

(7.6) $$\sum_{k=1}^{n} P(Z \varepsilon I'_k \mid H_k) \geq \sum_{k=1}^{n} P(Z \varepsilon I_k \mid H_k)$$

Hence the lemma from (7.4) and (7.6).

7.4 Maximum Likelihood Estimation and Test of Significance in Case (ii) : $a_s < x - n$, and m, s are unknown

7.4.1 In this case we do not know whether the declining yield phase has set in sometime during the past n years (the period of observation) or even before that point i.e., we only know that $a_s < x - n$. Let us recall the hypotheses formulated in Case (i) as given in (7.1) :

$$H_0 : Z \sim N(M^{(n)}, D)$$

$$H_1 : Z \sim N(M^{(k)}, D), \quad k = 1, 2, \ldots, n-1.$$

where $\underset{(n \times 1)}{M^{(n)}} = (m, m, \ldots, m)$

$$\underset{(n \times 1)}{M^{(k)}} = (\underbrace{m, m, \ldots, m}_{k \text{ times}}, m-s, m-2s, \ldots, m-(n-k)s)'$$

with m, s, k unknown, $k = 1, 2, \ldots, n-1$, $m > 0$, $s > 0$.

Note that in this case we do not know x – the age of the palm and $a_s < x - n$. So it is not necessarily true that $m = m_2$ (see (1.4)) since there is the possibility of $x - n > a_e$.

We shall proceed in the same fashion as in the Case (i) in 7.2.2 to derive the likelihood ratio rule in this case, after replacing the unknown parameters in the likelihood $L_k(Z)$ by their conditional (on $k = j$, $j = 1(1)n$) maximum likelihood estimates.

7.4.2 The likelihood corresponding to H_1 is given by

$$L_k(Z) = (2\pi)^{-\frac{n}{2}} |D|^{-\frac{1}{2}} \exp. -\frac{1}{2} \{(Z-M^{(k)})' D^{-1} (Z-M^{(k)})\}$$

$k = 1, 2, \ldots, n-1.$

Let us denote the elements of the matrix D^{-1} by t_{ij} i.e.,

$$D^{-1} = ((t_{ij})) \quad i, j = 1(1)n$$

Now

$$(Z-M^{(k)})' D^{-1} (Z-M^{(k)}) = Z' D^{-1} Z - 2Z' D^{-1} M^{(k)} + M^{(k)'} D^{-1} M^{(k)}$$

and $M^{(k)} = me+sh_k$, $\quad k = 1(1)n-1$

So we can write

$$(Z-M^{(k)})' D^{-1} (Z - M^{(k)})$$

$$= Z' D^{-1} Z - 2m Z' D^{-1} \underset{\sim}{e} - 2s Z' D^{-1} \underset{\sim}{h}_k + m^2 \underset{\sim}{e}' D^{-1} \underset{\sim}{e}$$

$$+ s^2 \underset{\sim}{h}_k' D^{-1} \underset{\sim}{h}_k + 2ms\underset{\sim}{e}' D^{-1} \underset{\sim}{h}_k \qquad \text{for } k = 1, \ldots\ldots, n-1.$$

$$= Z' D^{-1} Z - 2mZ' D^{-1} \underset{\sim}{e} + m^2 \underset{\sim}{e}' D^{-1} \underset{\sim}{e} \qquad \text{for } k = n$$

since $\underset{\sim}{h}_n$ is a null vector.

where $\underset{\sim}{e}$ and $\underset{\sim}{h}_k$ are given by (7.1a).

Thus, we obtain the maximum likelihood estimators (MLE) of m, s conditional on k as follows.

$$\hat{m}(k) = \frac{Z' B h_k}{e' B h_k} \qquad \text{for } k = 1, 2, \ldots, n-1.$$

(7.7)
$$= \frac{Z' D^{-1} e}{e' D^{-1} e} \qquad \text{for } k = n$$

$$\hat{s}(k) = \frac{Z' B' e}{e' B h_k} \qquad \text{for } k = 1, 2, \ldots, n-1$$

where $\underset{(n \times n)}{B} = D^{-1} (e h_k' - h_k e') D^{-1}$

7.4.3 Lemma : The Maximum likelihood estimators of m and s given by (7.7) are unbiased.

Proof : Let us first note that -

$$h_k' B h_k = h_k' D^{-1} (e h_k' - h_k e') D^{-1} h_k = 0, \quad k = 1, 2, \ldots, n-1.$$

$$e' B' e = e' D^{-1} (h_k e' - e h_k') D^{-1} e = 0$$

Therefore,

$$E(\hat{m}(k) \mid k) = E\{ \frac{Z' B h_k}{e' B h_k} \mid k \}$$

$$= \frac{M^{(k)'} B h_k}{e' B h_k} = \frac{(me + sh_k)' B h_k}{e' B h_k}$$

$$= \frac{m e' B h_k + s h_k' B h_k}{e' B h_k} = m \qquad \text{for } k = 1, 2, \ldots, n-1$$

$$E(\hat{m}(k) \mid k) = E\left(\frac{Z' D^{-1} e}{e' D^{-1} e} \mid k \right) = m \qquad \text{for } k = n$$

$$E(s(k) \mid k) = E\left\{ \frac{Z'B'e}{\underset{\sim}{e}'B\,\underset{\sim}{h}_k} \Big| k \right\}$$

$$= \frac{(m\underset{\sim}{e} + s\underset{\sim}{h}_k)'B'e}{\underset{\sim}{e}'B\,\underset{\sim}{h}_k} = \frac{me'B'e + sh_k'B'e}{e'Bh_k}$$

$$= \frac{se'B\,\underset{\sim}{h}_k}{\underset{\sim}{e}'B\underset{\sim}{h}_k} = s, \quad \text{for } k = 1, 2, \ldots, n-1$$

Hence the lemma.

Remark : Let us note that $E(Z - M^{(k)} \mid k) = \underset{\sim}{0}$ where $\hat{M}^{(k)} = \hat{m}(k)e + \hat{s}(k)\,h_k$ for $k = 1, 2, \ldots, n-1$ and $M^{(n)} = \hat{m}(n)\,\underset{\sim}{e}$, $\underset{\sim}{0}$ is the null vector. Also, it can be seen from (7.7) that $\hat{m}(k)$ and $\hat{s}(k)$ are simply linear combinations of $Z(1), Z(2), \ldots, Z(n)$.

7.4.4 Let $\hat{L}_k(Z)$ be obtained from $L_k(Z)$ in 7.2.2 by replacing the parameters by their respective m. l. e.s given in (7.7.).

$$\text{Let} \quad \hat{w}(k) = \frac{\hat{L}_k(Z)}{\hat{L}_n(Z)}$$

The m. l. e of k is given by

$$\hat{k} \text{ such that } \hat{w}(\hat{k}) = \max_i \hat{w}(i)$$

Asymptotic properties of $\hat{k}$ in the slippage problem with independent sample units has been studied by Hinkley (1970). We do not intend to persue the exercise here. However, we shall make a simulation study of the distribution of $\hat{k}$ in 7.6. Our likelihood ratio rule in this case will be given by -

reject H_0 if $\hat{w}(\hat{k}) > c$ where c is a suitably chosen constant.

Now,

$$\hat{w}(\hat{k}) = \left\{ \frac{\hat{Q}(n)}{\hat{Q}(k)} \right\}^{\frac{n}{2}}$$

Hence, our test statistic turns out to be

$$(7.8) \qquad \frac{\hat{Q}(n)}{\hat{Q}(k)} = \frac{(Z - \hat{M}^{(n)})' D^{-1} (Z - \hat{M}^{(n)})}{(Z - \hat{M}^{(k)})' D^{-1} (Z - \hat{M}^{(k)})}$$

$$= \frac{Z' E_1 Z}{Z' E_2 Z}$$

where

$$E_1 = F_1' D^{-1} F_1$$

$$E_2 = F_2' D^{-1} F_2$$

$$F_1 = I + \frac{1}{e' B \underset{\sim}{h}_k} DBDB$$

$$F_2 = I - \frac{1}{\underset{\sim}{e}' D^{-1} \underset{\sim}{e}} \underset{\sim}{e}\, \underset{\sim}{e}' D^{-1}$$

$$B = D^{-1} (\underset{\sim}{e}\, \underset{\sim}{h}_k' - \underset{\sim}{h}_k\, \underset{\sim}{e}') D^{-1}$$

$\underset{\sim}{e}$ and $\underset{\sim}{h}_k$ are given in (7.1a).

The distribution of this statistic can be dirived as follows.

$$P\left(\frac{Z' E_1 Z}{Z' E_2 Z} \leq z \right)$$

$$= P(Z'(E_1 - zE_2)Z \leq 0)$$

Let $E_z = E_1 - zE_2$. E_z is a symmetric matrix. Consider the transformation

$$Z_1 = F_2 Z \qquad \text{where } F_2 \text{ as given in (7.8)}$$

$$E(Z_1) = 0$$

$$V(Z_1) = F_2 D F_2' \qquad \text{since } V(Z) = D, \quad \text{see } 7.1.1.$$

F_2 is a non-singular matrix. Hence $F_2 D F_2'$ is also non-singular and symmetric. There exists a non-singular lower triangular matrix U such that

$$F_2 D F_2' = U U'$$

Consider

$$Z_2 = U^{-1} Z_1$$

$$Z = F_2^{-1} Z_1 = F_2^{-1} U Z_2$$

Thus,

$$P(Z' E_z Z \leq 0)$$

$$= P(Z_2' F Z_2 \leq 0) \qquad \text{where } F = U' F_2^{-1'} E_z F_2^{-1} U$$

Again, since F is symmetric, there exists the orthogonal matrix H such that $H_1' F H_1 = H$ where H is the diagonal matrix of eigen values of F. Thus, considering $W = H_1' Z_2$ we have

$$P\left(\frac{Z' E_1 Z}{Z' E_2 Z} \leq z\right)$$

$$= P(Z_2' F Z_2 \leq 0)$$

$$= P(W' H W \leq 0)$$

$$= P\left(\sum_{i=1}^{n} \lambda_i(z) \; W_i^2 \leq 0\right)$$

where W_i are independent standard normal variates and $\lambda_i(z)$ are the eigen values of $F = U' F_2^{-1'} E_z F_2^{-1} U$.

There are now several ways of deriving the distribution of $\sum_{i=1}^{n} \lambda_i(z) W_i^2$ available in the literature. For an exhaustive account please see Johnson and Kotz (1970). See also Gurland (1953) where the distribution of ratio of two quadratic forms in normal variates (with parameters known) has been derived for various cases.

But the likelihood ratio rule, as it can be seen, leads to a complicated procedure. Let us now explore the discrimination approach considered in Case (i) in 7.3.

7.5 <u>Discrimination Approach in Case (ii)</u>.

7.5.1 Let us recall the hypotheses formulated in 7.3.1. We shall use m and s instead of m_2 and s_2 considered earlier.

$$H_k : N = M^{(k)} = (m \quad m \quad \ldots\ldots \quad m \quad m-s \quad \ldots\ldots \quad m-(n-k)s)$$

$$k = 1, 2, \ldots\ldots, n$$

$$E(Z) = N = M^{(k)} \quad \text{under } H_k$$

When m and s are known, and the hypotheses H_k are a priori equally likely, the classification rule is given by —

Prefer H_k if $L_k(Z) = \max_i L_i(Z)$ where $L_i(Z)$ is the likelihood of Z under H_i, $i = 1(1)n$.

Now, $\underset{i}{\text{Max}} \; L_i(Z)$

$$= \underset{i}{\text{Max}} \left[(2\pi)^{-\frac{n}{2}} |D|^{-\frac{1}{2}} \exp. -\frac{1}{2} \{ (Z-M^{(i)})' D^{-1} (Z-M^{(i)}) \} \right]$$

$$= \underset{i}{\text{Max}} \left[\exp. -\frac{1}{2} \{ (Z-M^{(i)})' D^{-1} (Z-M^{(i)}) \} \right]$$

$$= \underset{i}{\text{Max}} \left[-\frac{1}{2} \{ (Z - M^{(i)})' D^{-1} (Z - M^{(i)}) \} \right]$$

$$= \underset{i}{\text{Min}} \; \{ (Z - M^{(i)})' D^{-1} (Z - M^{(i)}) \}$$

When the parameters are not known, they can be replaced by their respective suitable estimates conditional on the hypothasis H_i, $i = 1, 2, \ldots, n$. The resulting classification rules can be called 'plug-in classification rules' (PCR). The notion of PCR was first suggested by Fisher (1936). The estimates were meant to be on the basis of what are known as training samples, see Anderson (1958). Here of course our estimates of m and s are based on a single vector observation.

From 7.4.2, where the m.l.es of m and s have been derived, we can write

$$Z - M^{(i)} = U_i Z, \qquad i = 1, 2, \ldots, n$$

where $U_i = I + \frac{1}{e' B h_i} DBDB, \qquad i = 1, 2, \ldots, n-1$

$$U_n = I + \frac{1}{\underset{\sim}{e}' D^{-1} \underset{\sim}{e}} \underset{\sim}{e} \underset{\sim}{e}' D^{-1}$$

$$B = D^{-1} (\underset{\sim}{e} \underset{\sim}{h}_i' - \underset{\sim}{h}_i \underset{\sim}{e}') D^{-1}$$

$\underset{\sim}{e}$ and $\underset{\sim}{h}_k$ as defined in (7.1a)

Let $\bar{U}_i = U_i' D^{-1} U_i$, $i = 1, 2, \ldots, n$

Therefore, a plug-in classification rule in this case is given by -

$$\text{prefer } H_k \text{ if } L_k(Z) = \underset{i}{\text{Max}} \; L_i(Z)$$

$$\text{i.e. } \underset{i}{\text{Min}} \; (Z - M^{(i)})' D^{-1} (Z - M^{(i)})$$

$$= \underset{i}{\text{Min}} \; Z' \bar{U}_i Z.$$

7.5.2 The rule derived above is not difficult for application. But the efficiency of this rule in terms of the probability of correct classification is not easy to study theoretically. We note that, in exactly similar fashion as above, we could consider plug-in likelihood ratio (PLR) rules in section 7.4. But the PLR rule with the m.l.es turns out to be as complicated as the exact likelihood ratio rule. So, we didn't pursue the same in the earlier section.

7.6 Empirical Exercise

7.6.1 We have considered two methods in case (i) based on the classification approach in order to estimate k — the time-point beyond which the mean yield declines (see 7.3.1 and 7.3.2). In case (ii), we derived the maximum likelihood estimators of m and s (the stable mean yield and the rate of decline) and suggested a plug-in classification rule in order to estimate k (see 7.4.2 and 7.5.1)

Here we propose to get an idea about the sampling distributions of the estimators mentioned above. We shall also examine the sensitivity of these distributions to some of the parameters that specify the stochastic nature of the annual yield sequence of a palm.

Let us recall the following

$$V(Y(x)) = \sigma^2 \quad \text{where } x \text{ is the age of a palm.}$$

$$\text{Cov}\,(Y(x),\, Y(x')) = \pi^{|x - x'| - 1}\, \rho.\, \sigma^2, \quad x \neq x',$$

$$0 < \pi < 1,\ 0 < \rho < 1.$$

The study of the sampling distributions is performed by simulation.

We have chosen the period of observation of the actual yields as 10 years i.e., n = 10 (see 7.1.1). We have fixed $\pi = \rho$ for the purpose of our study and selected the following alternative values of ρ and σ.

$$\rho = 0.62,\ 0.95$$

$$\sigma = 15,\ 30$$

The first value for ρ is the coefficient estimated for Kerala (see 1.3). The second value hasbeen taken as a high correlation coefficient. Again, the second value chosen for σ is the estimate of variance of the annual yield in Kerala (see 1.3). The first value has been taken as a small variance.

The rest of the parameters are taken as specified in 1.3.

100 samples of the yield vector are generated in each case by generating Multivariate Normal samples with the specified parameters (m, s, π, ρ, σ and k). The results of the simulations are presented in the following tables.

7.6.2 **TABLE** : Estimation of the Sampling Distributions of k in Case (i) i.e., m and s are known.

Sample size = 100 in each case.

$n = 10$, $\pi = \rho$, $m = 60$, $s = 3$

$\hat{k}_1$: Estimator of k given by the method in 7.3.1.

$\hat{k}_2$: Estimator of k given by the method in 7.3.2.

Variance and Correlation	Specified value of k	Mean of the estimators		Standard Error of the estimators	
		$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_1$	$\hat{k}_2$
$\sigma = 30$	3	4.6	4.7	3.55	3.73
$\rho = 0.62$	7	6.0	5.9	3.39	3.86
$\sigma = 30$	3	5.2	5.1	3.16	4.08
$\rho = 0.95$	7	5.6	6.0	3.02	4.07
$\sigma = 15$	3	3.8	4.4	2.64	3.20
$\rho = 0.62$	7	6.4	6.8	3.17	3.60

7.6.3. TABLE : Estimation of the Sampling Distribution of k in Case (ii) i.e., m and s Are Unknown.

Sample Size = 100 in each case.

$n = 10$, $\pi = \rho$, $m = 60$, $s = 3$.

$\hat{k}$: Estimator of k given by the method suggested in 7.5.1.

$\hat{m}$: Estimator of m given by (7.7)

$\hat{s}$: Estimator of s given by (7.7).

Variance and Correlation	Specified value of			Mean of the estimators			Standard Error of the estimators		
	k	m	s	$\hat{k}$	$\hat{m}$	$\hat{s}$	$\hat{k}$	$\hat{m}$	$\hat{s}$
$\sigma = 30$	3	60	3	6.0	58.7	7.4	2.63	23.99	16.88
$\rho = 0.62$	7	60	3	6.6	60.7	5.6	2.43	20.90	18.90
$\sigma = 30$	3	60	3	5.0	62.9	4.9	2.64	28.83	6.20
$\rho = 0.95$	7	60	3	6.0	61.9	4.2	2.69	27.97	8.13
$\sigma = 15$	3	60	3	5.0	60.2	6.4	2.64	11.81	7.79
$\rho = 0.62$	7	60	3	5.7	60.6	4.2	2.83	8.71	8.57

7.6.4 In case (i), we considered the situation where the decline in the mean has set in sometime during the period of observation of actual yields i.e., $a_s < x-n < a_e$ where x is the current age (unknown) of the palm and n is the number of observations of annual yield (see 7.1.1). Both m (the stable mean yield) and s (the rate of decline of the mean yield) are assumed to be known. The change-point for the mean yield is to be estimated.

Let us recall that we considered two different classification rules in this case. The first one provides the estimator k_1 (say) based on the observed yield vector (see 7.3.1) and second one provides the estimator k_2 (say) based on the average of the observed yields (see 7.3.2).

It can be seen from the table in 7.6.2 that changes in σ or ρ do not make any significant difference to the standard errors of the estimates given by k_1 and k_2. But a reduction in σ definitely improves the estimates although an increase in ρ does not lead to any improvement. On the whole the estimates given by both the methods seem to be satisfactory and more so when $k = 7$ than when $k = 3$.

In case (ii), we considered the situation where the decline in the mean yield might or might not have set in during the period of observation i.e., $a_s < x-n$ (see 7.1.1). Here in addition to the change-point in the means both of m and s are assumed to be unknown and are to be estimated. We have suggested an estimator $\hat{k}$ (say) for the change-point based on a plug-in classification rule (see 7.5). The maximum likelihood estimators $\hat{m}$ and $\hat{s}$ of m and s are given by (7.7).

It can be seen from the table in 7.6.3 that the changes in σ or ρ again do not make any significant difference to the standard error of the estimate given by $\hat{k}$. It is interesting to note that standard error of $\hat{k}$ is roughly of the same order as that of k_1 and k_2 if not slightly better. However, $\hat{k}$ seems to be satisfactory only when $k=7$ and not when $k=3$.

We have already shown in 7.4.3 that $\hat{m}$ and $\hat{s}$ are unbiased estimators. The standard error of $\hat{s}$ improves with an increase in ρ or a decrease in σ but that of $\hat{m}$ seems to improve only with a decrease in σ (see 7.6.3). The reason for this is not clear. It should be noted that the standard error of $\hat{s}$ is rather large specially when $\sigma = 30$, $\rho = 0.62$ (the estimated values for Kerala).

Chapter VIII

Summary and Conclusion

8.1 We have considered in the foregoing chapters various replacement models (deterministic and stochastic) for Coconut palms. Palms were considered as a special case of ageing assets. The parameters defined in (1.1) through (1.2) in section 1.2 to characterise the yield profile of a Coconut palm can, in general, describe the performance of any ageing asset. For example, the performance of various machines can be described by putting $a_f = a_s = 1$. This would mean that the mean performance of the machines remain stable for a certain period of time before it begins to decline. Even the assumption of a Multivariate Normal distribution of the age-specific yield sequence (see (1.3)) of a palm is a reasonable one for a large class of assets. Let us note here that the assumption of a constant variance of the age-specific yield is not crucial for the analytical results derived in the previous chapters. It can be shown with little modification (whenever necessary), that all the results are valid even when the variance is not a constant provided the correlation structure specified in (1.3) remains the same. Thus, although the analytical results derived here are specific to the case of Coconut palms, they can be extended to a large class of ageing assets. We further note that we have been concerned with a static situation i.e., a piece of asset is replaced by an identical one.

8.2 In the economic literature, replacement (of capital asset) problems are studied with discounted value of the net income stream as the objective function. Usually, the net income stream is considered over an infinite time-horizon. The argument in favour of considering infinite horizon is that if one is not interested in the income generated in the distant future, a high discount rate can be used for necessary adjustment. We studied the replacement problem in the deterministic version in Chapter II. In this version we take the expected yields (μ_x, x = 1, 2, ..., L) as the actual ones. We investigated both finite and infinite horizon cases. Our focus was to analytically characterise the optimal replacement age. This reflects the relationship between the net return stream, or equivalently, the performance curve of μ_x of an asset, the rate of discount, length of time horizon etc., on one hand and the replacement age on the other.

In the infinite horizon case, if one starts with a brand new piece of an asset, the existence of an optimal constant cycle of replacement has a strong intuitive background. But what we found further is that the same constant cycle holds true even if one starts with an asset of an arbitrary vintage (see 2.5).

In the finite horizon case, we considered a single cycle replacement model. This means that the length of the horizon is not too long to warrant replacement more than once. We found that the optimal replacement age, in this case, is a non-decreasing function of the length where the length varies within a certain finite range (see 2.6.5). Also, we found that for sufficiently short horizon the corresponding optimal replacement age is less than or equal to that corresponding to the infinite horizon case, irrespective of the rate of discount (see 2.7).

This result is not inttuitively obvious although one could anticipate an order relationship between the solutions corresponding to a very short time-horizon and an infinite one. But,, what is more interesting is that the above two theorems leave one possibility open. Let A^H and A^∞ be the optimal replacement ages corresponding to a finite time-horizon (of length H) and infinite horizon case respectively. For sufficiently small H, let us suppose $A^H < A^\infty$. Now, as H increases it is possible that A^H increases beyond A^∞ i.e., there can exist H large enough such that $A^H > A^\infty$ since A^H is a non-decreasing function of H. We have found this to be true in certain empirical cases (see 2.8). Let us note that all these results depend on the nature of the mean yield profile of a Coconut palm (or, in general, the performance curve of an asset).

8.3 Consideration of the discounted value of the net income stream as an objective function ignores the changes in the age distribution of the palms over time. Consequently, the time-path of the annual net income and the annual total yield are not taken care of in such a formulation (see 3.1). Also, the optimal replacement age depends on the choice of the discount rate and the length of the time-horizon. In this context, we considered a replacement model based on the total yield (or annual net income) trajectory which depends on the changes in the age distribution of palms over time (Chapter III). The relationship between a given rate of replacement andthe consequent stable age distribution was established. The rate of replacement was defind as the proportion of old palms (above a given age) replaced every year. It was found that the stable age distribution is a uniform one. The maximum stable yield level corresponds to a particular rate of replacement and a cut-off age which defines the old palms. Both of these optimal solutions depend on the yield profile of a palm (see 3.3 for the above results).

8.4 An obvious limitation of the deterministic models summarised above is that they do not utilise the fact that future yield performance of a palm is dependent on the past performance. This leads to a simple modification of the earlier model presented in Chapter II. The future mean yield stream of the existing palm was replaced by the mean yield stream conditioned by the past yield performance. The resulting replacement rule was compared with the deterministic rule (Chapter IV). It was found that if the present age of a given palm is above the optimal replacement age given by the deterministic version, then the modified rule is better than the deterministic one on the basis of the following probability of the criterion.

$$P\,(\text{Discounted value corresponding to R 2} \geq \text{Discounted value corresponding to R 1} \mid \text{R 2 recommends no replacement}) \geq 0.5$$

where R 1 : Deterministic rule

R 2 : Modified rule

The choice of the number 0.5 is discussed in subsection 4.2.2.

8.5 Next we considered (Chapter V) a general set up for the stochastic replacement rules on the basis of the above results. Given any objective function g, an unsatisfactory performance of a palm in the future was defined by the event $[g \leq k_g]$ where k_g is a given constant, g is a function of the future yields of a palm. A replacement rule was defined by specifying a decision function f and a constant k_f as follows.

Replace only if $f \leq k_f$

f is a function of the past yields of a palm. For a given pair of (g, k_g) the choice of a decision function f was proposed to be based on the following two probabilities as the criteria.

1. $P(g \leq k_g \mid f \leq k_f)$: Reliability
2. $P(f \leq k_f \mid g \leq k_g)$: Efficiency.

We confined ourselves to only linear forms of g and f. A class of replacement rules (f, k_f) was defined by choosing the constant k_f for each f so that the corresponding Reliability is ≥ 0.5. It was found that the replacement rule with the highest Efficiency in the above class corresponds to the decision function f which is the regression of g on the past yields $Y(x-1), Y(x-2), \ldots, Y(x-n)$, $n < x$, where x is the present age of the palm and $Y(i)$ is the yield of the palm at age i. It was further found that the Efficiency of a replacement rule (for a given objective (g, k_g)) is an increasing function of the common variance of $Y(i)$. This is of course a desirable result since the question of stochastic replacement rules arises only when there is high variability in the yield.

8.6 As a special case we considered Markov Replacement Rules in Chapter VI. A replacement rule in this case was defined by specifying a threshold value $Z(x)$ at each age x. A palm was to be replaced at the current age x only if the yield $Y(x-1)$ was below the specified threshold value $Z(x-1)$. A computational procedure was developed to choose a suitable set of threshold values. The choice of the threshold values was made by improving the discounted value of net returns over the optimal level obtained in the deterministic version. In consistence with the findings of Chapter V, it was found that higher improvement over the value of the objective function could be achieved whenever the variance σ^2 of the yield $Y(x)$ was higher.

8.7 Finally (in Chapter VII) we considered the problem of estimating the future mean yield profile of a palm (μ_i, $i = x, x+1, \ldots, L$) given it's past yeild record $Y(x-1), Y(x-2), \ldots, Y(x-n)$, $n < x$, where x is the present age of a palm which is unknown. The common variance σ^2 of $Y(i)$ and the correlation matrix of the yields $Y(i)$ were assumed to be known. Since the replacement rules are particularly dependent on the mean yield stream of a palm this estimation becomes necesssary when the current age of a palm cannot be ascertained. A classification approach was considered along with the likelihood ratio rule for the purpose of this estimation. A simulation study showed that the classification procedure provides satisfactory estimates and this procedure was found to be operationally convenient.

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