The Hoffman–Wielandt inequality in infinite dimensions

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Abstract. The Hoffman–Wielandt inequality for the distance between the eigenvalues of two normal matrices is extended to Hilbert–Schmidt operators. Analogues for other norms are obtained in a special case.

Keywords. Hoffman–Wielandt inequality; infinite dimensions; Hilbert–Schmidt operators; Schatten p-norms.

1. Introduction

In 1953 Hoffman and Wielandt [13] proved what has now become one of the best-known matrix inequalities. The aim of this paper is to obtain an infinite-dimensional version of this inequality.

Let A be an $n \times n$ complex matrix. An n-tuple $\{\alpha_1, \ldots, \alpha_n\}$ is called an *enumeration* of the eigenvalues of A if its elements are the eigenvalues of A each counted as often as its multiplicity. The eigenvalues of $(A^*A)^{1/2}$ are called the singular values of A and are denoted as $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$. We will use the symbol $\|A\|_2$ to denote what is often called the *Frobenius norm* in the matrix theory literature and the *Hilbert–Schmidt norm* in the operator theory literature. This is defined as

$$\|A\|_2 = (\operatorname{tr} A^*A)^{1/2} = \left[\sum_{j=1}^{n} s_j^2(A)\right]^{1/2}. \tag{1}$$

The Hoffman–Wielandt inequality says that if A and B are $n \times n$ normal matrices and if $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ are enumerations of their eigenvalues, then there exists a permutation π on n symbols such that

$$\left[\sum_{i=1}^{n} |\alpha_i - \beta_{\pi(i)}|^2\right]^{1/2} \leqslant \|A - B\|_2. \tag{2}$$

Now let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space. If an operator A on $\mathscr{H}$ is compact then the spectrum of A is a countable set of complex numbers with 0 as the only limit point. All nonzero points in the spectrum are eigenvalues of A with finite multiplicity. The point 0 may or may not be an eigenvalue of A, and if it is its multiplicity may be finite or infinite. By an *enumeration* of the eigenvalues of A we shall mean a sequence $\{\alpha_1, \alpha_2, \ldots\}$ whose terms consist of all the eigenvalues of A each counted as often as its multiplicity. By an *extended enumeration* of the eigenvalues of A we shall mean a sequence $\{\alpha'_1, \alpha'_2, \ldots,\}$ whose terms consist of all the nonzero eigenvalues of A each counted as often as its multiplicity and the term 0 repeated infinitely often.

The singular values of A are defined as before. Now they are infinite in number. If

$$\|A\|_2 := \left[\sum_{j=1}^{\infty} s_j^2(A)\right]^{1/2} < \infty \tag{3}$$

the operator A is said to be in the *Hilbert–Schmidt Class* and the collection of all such operators is denoted as $\mathscr{I}_2$.

A bijection π of the set of natural numbers $\mathbb{N}$ onto itself will be called a *permutation of* $\mathbb{N}$.

The following two theorems are infinite-dimensional analogues of the Hoffman–Wielandt Theorem:

Theorem 1. *Let A and B be normal Hilbert–Schmidt operators and let $\{\alpha_1, \alpha_2, \ldots\}$ and $\{\beta_1, \beta_2, \ldots\}$ be enumerations of their eigenvalues. Then for each $\varepsilon > 0$ there exists a permutation π of $\mathbb{N}$ such that*

$$\left[\sum_{i=1}^{\infty} |\alpha_i - \beta_{\pi(i)}|^2\right]^{1/2} \leqslant \|A - B\|_2 + \varepsilon. \tag{4}$$

Theorem 2. *Let A and B be normal Hilbert–Schmidt operators and let $\{\alpha'_1, \alpha'_2, \ldots\}$ and $\{\beta'_1, \beta'_2, \ldots\}$ be extended enumerations of their eigenvalues. Then there exists a permutation π of $\mathbb{N}$ such that*

$$\left[\sum_{i=1}^{\infty} |\alpha'_i - \beta'_{\pi(i)}|^2\right]^{1/2} \leqslant \|A - B\|_2. \tag{5}$$

It seems essential to either add an ε as in Theorem 1 or to extend the enumerations as in Theorem 2. This point will be discussed in § 2 after the theorems have been proved.

In the special situation when A and B are Hermitian our Theorem 2 has already been proved by Markus [16], Friedland [12] and Kato [14], each of whom proved generalisations of this in different directions. Another rather special case was considered by Sakai [18].

The Hilbert–Schmidt norm is one of a family of norms called Schatten p-norms. These norms are defined as

$$\|A\|_p = \left[\sum_{j=1}^{\infty} s_j^p(A)\right]^{1/p}, \qquad 1 \leqslant p < \infty \tag{6}$$

$$\|A\|_\infty = s_1(A). \tag{7}$$

The class of operators for which $\|A\|_p$ is finite is an ideal $\mathscr{I}_p$ in the space of compact operators which itself is denoted as $\mathscr{I}_\infty$. Basic facts about these norms may be found in several standard texts such as [19].

A problem of much interest in perturbation theory has been that of obtaining analogues of the Hoffman–Wielandt inequality for all these p-norms (and for the larger class of symmetric norms). See [3] for a detailed discussion. In both the finite and the infinite dimensional cases this problem has been solved completely when A, B are both Hermitian (see [2], [14], [15], [16]) and when A is Hermitian and B is skew-Hermitian (see [1], [20]). When A and B are both unitary this problem has been solved only partially: *sharp* analogues of the inequality (2) are known only for the values $p=1$ and $p=\infty$ and good bounds are known for other values. (See [2], [5], [7], [8], [10]). But when A and B are arbitrary normal operators a sharp analogue of (2) for *any* value of p other than 2 has not been found *even* in the finite-dimensional case. See [6] and [7] for the known results when $p=\infty$.

In this direction we shall prove:

Theorem 3. *Let A be a Hermitian and B a normal operator, both lying in the Schatten class $\mathscr{I}_p$ for some $1 \leqslant p \leqslant \infty$. Let $\{\alpha'_1, \alpha'_2, \ldots\}$ and $\{\beta'_1, \beta'_2, \ldots\}$ be extended enumerations of the eigenvalues of A and B. Then there exists a permutation π of $\mathbb{N}$ such that*

$$\left[\sum_{i=1}^{\infty} |\alpha'_i - \beta'_{\pi(i)}|^p\right]^{1/p} \leqslant 2^{2/p-1}\|A-B\|_p \qquad \text{if } 1 \leqslant p \leqslant 2, \tag{8}$$

and

$$\left[\sum_{i=1}^{\infty} |\alpha'_i - \beta'_{\pi(i)}|^p\right]^{1/p} \leqslant 2^{1/2-1/p}\|A-B\|_p \qquad \text{if } 2 \leqslant p \leqslant \infty, \tag{9}$$

In the finite-dimensional case, the inequality (9) for the special case $p=\infty$ has been observed earlier. See, e.g., [3, p. 112]. For other p these results seem to be new even in this case.

2. Proofs and remarks

The proofs of Theorems 1 and 2 are both built upon the finite-dimensional case. In the first this involves a straightforward approximation argument, in the second some more intricacies.

Proof of Theorem 1. Label the eigenvalues of A and B as $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ in such a way that

$$|\alpha_1| \geqslant |\alpha_2| \geqslant \ldots; \quad |\beta_1| \geqslant |\beta_2| \geqslant \ldots. \tag{10}$$

Then choose orthonormal bases $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ for $\mathscr{H}$ so that

$$A = \sum_{i=1}^{\infty} \alpha_i u_i u_i^*, \qquad B = \sum_{i=1}^{\infty} \beta_i v_i v_i^*. \tag{11}$$

Since A and B are both Hilbert–Schmidt operators, given any $\delta > 0$ we can choose a positive integer r such that

$$\sum_{i=r+1}^{\infty} |\alpha_i|^2 \leqslant \delta^2, \qquad \sum_{i=r+1}^{\infty} |\beta_i|^2 \leqslant \delta^2. \tag{12}$$

So, if we define operators A_r and B_r as

$$A_r = \sum_{i=1}^{r} \alpha_i u_i u_i^*, \qquad B_r = \sum_{i=1}^{r} \beta_i v_i v_i^*, \tag{13}$$

then,

$$\|A - A_r\|_2 \leqslant \delta, \qquad \|B - B_r\|_2 \leqslant \delta. \tag{14}$$

Now consider the linear space spanned by the vectors $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ together. This is a space of dimension s where $r \leqslant s \leqslant 2r$. Call this space $\mathscr{H}_s$. The operators A_r and B_r both leave $\mathscr{H}_s$ invariant and vanish on its orthogonal complement. Let $w_1, \ldots, w_s$ be an orthonormal basis for $\mathscr{H}_s$ in which $w_j = u_j$ for $j = 1, 2, \ldots, r$. Then $A_r w_j = \alpha_j w_j$ for $1 \leqslant j \leqslant r$ and $A_r w_j = 0$ for $r + 1 \leqslant j \leqslant s$. Define a normal operator A_s on $\mathscr{H}_s$ by putting $A_s w_j = \alpha_j w_j$ for $1 \leqslant j \leqslant s$. Then note that

$$\|A_s - A_r\|_2^2 = \sum_{j=r+1}^{s} |\alpha_j|^2 \leqslant \delta^2. \tag{15}$$

By a similar construction we can define a normal operator B_s on $\mathscr{H}_s$ which has eigenvalues $\beta_1, \ldots, \beta_s$ and is such that

$$\|B_s - B_r\|_2 \leqslant \delta. \tag{16}$$

Now apply the Hoffman–Wielandt Theorem to the operators A_s and B_s on the finite-dimensional space $\mathscr{H}_s$. This gives a permutation π of the set $\{1, 2, \ldots, s\}$ such that

$$\sum_{j=1}^{s} |\alpha_j - \beta_{\pi(j)}|^2 \leqslant \|A_s - B_s\|_2^2. \tag{17}$$

Now extend this permutation π to all of $\mathbb{N}$ by defining $\pi(j) = j$ if $j > s$. Then the inequalities (12), (14), (15), (16) and (17) together give

$$\sum_{j=1}^{\infty} |\alpha_j - \beta_{\pi(j)}|^2 \leqslant (\|A - B\|_2 + 4\delta)^2 + 4\delta^2.$$

Since δ was arbitrary this proves the theorem. ■

Proof of Theorem 2. Once again label the eigenvalues of A and B as in (10). Define extended enumerations $\{\alpha_i'\}$, $\{\beta_i'\}$ of eigenvalues of A and B as the two sequences whose terms are

$$\begin{aligned} &\alpha'_{2i-1} = \alpha_i, \quad \alpha'_{2i} = 0, \quad i = 1, 2, \ldots, \\ &\beta'_{2i-1} = \beta_i, \quad \beta'_{2i} = 0, \quad i = 1, 2, \ldots. \end{aligned} \tag{18}$$

By a slight modification of the argument used in proving Theorem 1 we can find a sequence ε_n of positive numbers and a sequence π_n of permutations of $\mathbb{N}$ such that

$$\sum_{i=1}^{\infty} |\alpha_i' - \beta'_{\pi_n(i)}|^2 \leqslant \|A - B\|_2^2 + \varepsilon_n^2, \tag{19}$$

and

$$\lim \varepsilon_n = 0. \tag{20}$$

To see this adopt the same notations as in the proof of Theorem 1 up to the inequality (14). Now let $\mathscr{H}_n$ be any subspace of dimension $n = 2r$ which contains all the vectors $u_1, \ldots, u_r, v_1, \ldots, v_r$. The operators A_r and B_r both leave $\mathscr{H}_n$ invariant and their restrictions to this space have eigenvalues α'_i, β'_i, $i = 1, 2, \ldots, n$. So, by the Hoffman–Wielandt Theorem there exists a permutation π_n of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i=1}^{n} |\alpha'_i - \beta'_{\pi_n(i)}|^2 \leqslant \|A_r - B_r\|^2 \leqslant (\|A - B\| + 2\delta)^2.$$

Extend the permutation π_n to all of $\mathbb{N}$ by putting $\pi_n(j) = j$ if $j > n$ and define ε_n via δ to get (19) and (20). Let

$$\nu_n = \pi_n^{-1}, \qquad n = 1, 2, \ldots. \tag{21}$$

We now construct a permutation π of $\mathbb{N}$ that will satisfy (5). To do this we will describe a procedure that defines π and its inverse ν by successively assigning values to $\pi(1), \nu(1), \pi(2), \nu(2), \ldots$. At the same time a subsequence of the sequence $\{\pi_n\}$ of permutations defined in the preceding paragraph is chosen. The procedure is described below in the form of an algorithm. This has two steps α and β to be run alternately and in each of these three mutually exclusive choices have to be made.

For $i = 1, 2, \ldots,$ do

α Look successively at the following three options, do as instructed, then go to β:

- (I) (void if $i = 1$). If $i = \nu(j)$ for some $j < i$ define $\pi(i) = j$.
- (II) If the set $\{\pi_n(i): n = 1, 2, \ldots\}$ is bounded let j be the minimal number which occurs infinitely often in this set. Define $\pi(i) = j$. Replace $\{\pi_n\}$ by a subsequence, denoted again by $\{\pi_n\}$, such that now $\pi_n(i) = j$ for all n.
- (III) If the set $\{\pi_n(i): n = 1, 2, \ldots\}$ is unbounded let j be the smallest even number which has not yet been called $\pi(k)$ for any $k < i$. Define $\pi(i) = j$. (Note that in this case $\lim_{n\to\infty} \beta'_{\pi_n(i)} = 0$ and we have defined π in such a way that $\beta'_{\pi(i)} = 0$.)

β Look successively at the following three options, do as instructed, then go back to α with $i + 1$ in place of i:

- (IV) If $i = \pi(j)$ for some $j \leqslant i$ define $\nu(i) = j$.
- (V) If the set $\{\nu_n(i): n = 1, 2, \ldots\}$ is bounded let j be the minimal number which occurs infinitely often in this set. Define $\nu(i) = j$. Replace $\{\nu_n\}$ by a subsequence, denoted again by $\{\nu_n\}$, such that now $\nu_n(i) = j$ for all n. This also gives a new subsequence of $\{\pi_n\}$ if we put $\pi_n = \nu_n^{-1}$.
- (IV) If the set $\{\nu_n(i): n = 1, 2, \ldots\}$ is unbounded let j be the smallest even number that has not yet been called $\nu(k)$ for any $k < i$. Define $\nu(i) = j$. (Note in this case we had $\lim_{n\to\infty} \alpha'_{\nu_n(i)} = 0$ and we have defined ν in such a way that $\alpha'_{\nu(i)} = 0$).

We claim that the permutation π defined above satisfies the inequality (5). For this it is enough to show that for every positive integer N we have

$$\sum_{i=1}^{N} |\alpha'_i - \beta'_{\pi(i)}|^2 \leqslant \|A - B\|_2^2. \tag{22}$$

Let $\Phi_N = \{\pi_1, \pi_2, \ldots\}$ be the subsequence of the original sequence $\{\pi_n\}$ obtained after running N steps of α and β in the above procedure. We will split the set $\{1, 2, \ldots, N\}$ into three disjoint subsets S_1, S_2 and S_3 by separating indices according to what happened to them in the above algorithm. These sets are defined as

$$S_1 = \{i : 1 \leqslant i \leqslant N, \exists \pi_m \in \Phi_N \text{ such that } \pi(i) = \pi_m(i)\}.$$

Note that if $i \in S_1$ then by (II) and (V) in the above construction $\pi(i) = \pi_m(i)$. for all $\pi_m \in \Phi_N$.

$$S_2 = \{i : 1 \leqslant i \leqslant N, \pi(i) \text{ was defined by (III) above}\}.$$

Note that

$$\beta'_{\pi(i)} = \lim_{n \to \infty} \beta'_{\pi_n(i)} = 0 \qquad \text{if } i \in S_2. \tag{23}$$

$S_3 = \{i : 1 \leqslant i \leqslant N, i \text{ was defined as } i = \nu(j) \text{ for some } j \leqslant i \text{ by (VI) above}\}$. Note that

$$\alpha'_i = \lim_{n \to \infty} \alpha'_{\nu_n(\pi(i))} = 0 \qquad \text{if } i \in S_3. \tag{24}$$

Now for any element π_n of Φ_N we can use the above splitting to write

$$\begin{aligned}\sum_{i=1}^{N} |\alpha'_i - \beta'_{\pi(i)}|^2 &= \sum_{i \in S_1} |\alpha'_i - \beta'_{\pi_n(i)}|^2 + \sum_{i \in S_2} |\alpha'_i|^2 + \sum_{i \in S_3} |\beta'_{\pi(i)}|^2 \\ &= \left[\sum_{i \in S_1} |\alpha'_i - \beta'_{\pi_n(i)}|^2 + \sum_{i \in S_2} |\alpha'_i - \beta'_{\pi_n(i)}|^2 + \sum_{i \in S_3} |\alpha'_{\nu_n(\pi(i))} - \beta'_{\pi(i)}|^2 \right] \\ &\quad + \sum_{i \in S_2} \{|\alpha'_i|^2 - |\alpha'_i - \beta'_{\pi_n(i)}|^2\} + \sum_{i \in S_3} \{|\beta'_{\pi(i)}|^2 - |\alpha'_{\nu_n(\pi(i))} - \beta'_{\pi(i)}|^2\}.\end{aligned} \tag{25}$$

As $n \to \infty$ the last two sums in (25) go to zero, since both are finite sums of terms going to zero. The limit of the three sums inside the square brackets can be written as

$$\lim_{n \to \infty} \sum_{i=1}^{N} |\alpha'_i - \beta'_{\pi_n(i)}|^2.$$

This is bounded above by

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} |\alpha'_i - \beta'_{\pi_n(i)}|^2.$$

Hence, the inequality (22) follows from (19) and (20). ∎

The difference between the finite-dimensional and the infinite-dimensional case arises because of the fact that the *unitary orbit* of an operator A defined as the set $\{UAU^* : U \text{ unitary}\}$ is closed in the former case but not always in the latter. The following simple example illustrating this phenomenon was provided to us by Peter Rosenthal.

Example. Let A be the normal operator given by

$$A = \operatorname{diag}\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right).$$

For $n = 1, 2, \ldots$, let

$$A_n = \operatorname{diag}\left(\frac{1}{n}, 1, \frac{1}{2}, \ldots, \frac{1}{n-1}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right)$$

Then each A_n is in the unitary orbit of A. However, A_n converges (in the Hilbert–Schmidt norm topology) to B where

$$B = \operatorname{diag}\left(0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$$

and B is not in the unitary orbit of A. By the same argument we can find a sequence of operators in the unitary orbit of A which converges to a diagonal operator having arbitrarily many zeroes on the diagonal.

One way to interpret the inequality (2) is that it gives a lower bound for the distance between the unitary orbits of two diagonal matrices. In the infinite-dimensional case such orbits are not closed. So, we should seek a lower bound for the distance between their *closures*. Such a bound is provided by Theorem 2.

The other, more standard, interpretation of (2) is that it gives an upper bound for the distance between the eigenvalues of two normal matrices. This distance is a metric on the space of unordered n-tuples of complex numbers. More precisely, consider the space $\mathbb{C}^n$ with the Euclidean norm $\|\cdot\|_2$. Let Π_n be the group of permutations on n indices. For $x \in \mathbb{C}^n$ let $x(\pi)$ be the vector whose coordinates are obtained by applying the permutation π to the coordinates of x. Calling two such vectors equivalent let $\tilde{x}$ be the equivalence class of x. Let $\tilde{\mathbb{C}}^n = \mathbb{C}^n/\Pi_n$ be the space of such equivalence classes. Then this is a metric space with the metric

$$d(\tilde{x}, \tilde{y}) = \min_{\pi} \|x - y(\pi)\|_2.$$

Since the eigenvalues of an $n \times n$ matrix are known only up to a permutation it is natural to identify them with a point in the space $\tilde{\mathbb{C}}^n$. The inequality (2) then gives a bound for the distance between the eigenvalues of two normal matrices A and B in terms of the distance between A and B. Now when A is a Hilbert–Schmidt operator we have to replace the space $\mathbb{C}^n$ in the above discussion by the space l_2. Let Π denote the set of all bijections of the set of natural numbers onto itself.

Consider the space $\tilde{l}_2 = l_2/\Pi$. The eigenvalues of A can be identified with a point in this space. We can now define for $\tilde{x}, \tilde{y}$ in this space

$$d(\tilde{x}, \tilde{y}) = \inf_{\pi} \|x - y(\pi)\|_2.$$

However, the example given above also shows that this does not give a metric on $\tilde{l}_2$. It only gives a *pseudometric*. Indeed, given any x in l_2 we can find a y which has the same nonzero entries as x but arbitrarily many additional zero entries, and for which $d(\tilde{x}, \tilde{y}) = 0$. The quotient space $\tilde{l}_2/d$ with respect to this pseudometric is a metric space. To identify this space let l'_2 be the subset of l_2 consisting of vectors with infinitely many zero entries. For each $x \in l_2$ let $x' = (x_1, 0, x_2, 0, \ldots)$. Then $x' \in l'_2$. Let $\tilde{x}'$ be the image of this point in $\tilde{l}'_2 = l'_2/\Pi$. Now define

$$d(\tilde{x}, \tilde{y}') = \inf_{\pi} \|x' - y'(\pi)\|_2.$$

It can be seen that this defines a metric on the space $\tilde{l}'_2$. It would be most natural to use this metric to measure the distance between the eigenvalues of two Hilbert–Schmidt operators. Theorem 2 is then seen to be the natural extension of the finite-dimensional Hoffman–Weilandt Theorem.

Since 0 is *always* an accumulation point of the eigenvalues of a compact operator, in any case there is good reason to include it with infinite multiplicity in a count of the eigenvalues.

Now we recall briefly some of the known results for the special class of Hermitian operators. Let A and B be $n \times n$ Hermitian matrices with eigenvalues enumerated as $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ and $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n$ respectively. Then we have

$$\left[\sum_{i=1}^{n} |\alpha_i - \beta_i|^p\right]^{1/p} \leqslant \|A - B\|_p \qquad \text{for } 1 \leqslant p \leqslant \infty. \tag{26}$$

This is a consequence of a theorem of Lidskii and Wielandt. See [3, Chapter 3]. This theorem was extended to infinite dimensions by Markus [16]. If A is a compact Hermitian operator associate with it a doubly infinite sequence $\{\alpha_{\pm j}; j \in \mathbb{N}\}$ satisfying the following conditions

(i) $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0,$

$\alpha_{-1} \leqslant \alpha_{-2} \leqslant \cdots \leqslant 0;$

(ii) if A has infinitely many positive and infinitely many negative eigenvalues then the sequence $\{\alpha_{\pm j}\}$ contains only these numbers each repeated as often as its multiplicity as an eigenvalues of A (0 is not included in the sequence in this case even if it is an eigenvalue of A);

(iii) if A has only finitely many positive eigenvalues then the sequence $\{\alpha_j\}$ contains these repeated according to their multiplicities and an infinite number of zero terms; and if A has only finitely many negative eigenvalues then the sequence $\{\alpha_{-j}\}$ contains these repeated according to their multiplicities and an infinite number of zero terms.

With this notation Markus proves a result from which it follows that if A and B are compact Hermitian operators and if $\{\alpha_{\pm j}\}$ and $\{\beta_{\pm j}\}$ are sequences associated with them according to the above rules then

$$\left[\sum_{j=1}^{\infty} \{|\alpha_j - \beta_j|^p + |\alpha_{-j} - \beta_{-j}|^p\}\right]^{1/p} \leqslant \|A - B\|_p \quad \text{for } 1 \leqslant p \leqslant \infty. \tag{27}$$

This device of adding zeroes to make both the positive and the negative eigenvalues of A infinite in number achieves exactly what our extended enumeration did. One can easily see that the "optimal matching" of the eigenvalues of A and B is achieved by the pairing in (27). If both A and B have infinitely many positive and negative eigenvalues then extending the enumerations by adding zeroes does not affect the sums involved. So, for the Hermitian case our Theorem 2 is included in this result of Markus. The $p=2$ case of (27) is also proved in Friedland [12].

Kato [14] has proved a similar result in the more general situation when A and B are any two bounded Hermitian operators whose difference is compact. Let $\sigma(A)$ denote the spectrum of a Hermitian operator A. An isolated point of $\sigma(A)$ is always an eigenvalue of A; if it has finite multiplicity call it a *discrete eigenvalue*. Let $\sigma_d(A)$ be the collection of all such points. The complement of $\sigma_d(A)$ in $\sigma(A)$ is called the *essential spectrum* of A and is denoted as $\sigma_{ess}(A)$. Eigenvalues of A that have infinite multiplicity are in $\sigma_{ess}(A)$ whether they are isolated points of $\sigma(A)$ or not. The set $\sigma_{ess}(A)$ is a closed subset of $\mathbb{R}$ and so its complement in $\mathbb{R}$ is a countable union of open intervals I_n. Kato defines an *extended enumeration of discrete eigenvalues* of A to be a sequence $\{\alpha_j\}$ with the following properties

(i) every discrete eigenvalue of A appears in this sequence as often as its multiplicity,
(ii) all other points of the sequence $\{\sigma_j\}$, belong to the set of boundary points of the open intervals I_n mentioned above.

We should add here an explanatory note. An extended enumeration $\{\alpha_j\}$ according to the above definition need not include all the boundary points of all the intervals I_n and those that are included may be counted as often as one wishes.

With this definition Kato proves that if A and B are Hermitian operators such that $A-B$ is compact then there exist extended enumerations $\{\alpha_j\}$ and $\{\beta_j\}$ of discrete eigenvalues of A and B such that

$$\left[\sum_{j=1}^{\infty}|\alpha_j-\beta_j|^p\right]^{1/p}\leqslant\|A-B\|_p \text{ for } 1\leqslant p\leqslant\infty. \tag{28}$$

The result of Markus can be derived from this.

We should add that all the inequalities (26)–(28) are true for the larger class of symmetric norms.

Our Theorem 3 is proved using the above results for the Hermitian case. We will need the following facts. Let

$$T=T_1+iT_2=\frac{T+T^*}{2}+i\frac{T-T^*}{2} \tag{29}$$

be the Cartesian decomposition of any operator T. Then.

$$\|T\|_2^2=\|T_1\|_2^2+\|T_2\|_2^2. \tag{30}$$

$$\|T_1\|_\infty\leqslant\|T\|_\infty,\ \|T_2\|_\infty\leqslant\|T\|_\infty. \tag{31}$$

If T is normal then the eigenvalues of T_1 and T_2 are the real and the imaginary parts of the eigenvalues of T. We will use the Clarkson–McCarthy inequalities which say that if T and S are in the Schatten class $\mathcal{I}_p$ then

$$2(\|T\|_p^p+\|S\|_p^p)\leqslant\|T+S\|_p^p+\|T-S\|_p^p \quad \text{for } 2\leqslant p\leqslant\infty \tag{32}$$

$$2^{p-1}(\|T\|_p^p+\|S\|_p^p)\leqslant\|T+S\|_p^p+\|T-S\|_p^p \quad \text{for } 1\leqslant p\leqslant 2. \tag{33}$$

See [9] or [19]. We will also use the elementary inequalities:

$$|x+iy|^p \leqslant 2^{p/2-1}(|x|^p+|y|^p) \qquad \text{for } 2 \leqslant p \leqslant \infty, \tag{34}$$

$$|x+iy|^p \leqslant |x|^p+|y|^p \qquad \text{for } 1 \leqslant p \leqslant 2, \tag{35}$$

valid for all real numbers of x and y.

Proof of Theorem 3. Let $B = B_1 + iB_2$ be the Cartesian decomposition of B. We shall apply the inequality (27) to the Hermitian operators A and B_1. Let us represent extended enumerations of eigenvalues of A and B in the form of doubly infinite sequences $\{\alpha_{\pm j}\}$ and $\{\beta_{\pm j}\}$ in which

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0, \qquad \alpha_{-1} \leqslant \alpha_{-2} \leqslant \cdots \leqslant 0;$$
$$\operatorname{Re}\beta_1 \geqslant \operatorname{Re}\beta_2 \geqslant \cdots \geqslant 0, \quad \operatorname{Re}\beta_{-1} \leqslant \operatorname{Re}\beta_{-2} \leqslant \cdots \leqslant 0.$$

In all summations the index j will run over positive and negative integers.

The case $p = 2$ is specially simple. We have from (30) and (27)

$$\begin{aligned} \|A-B\|_2^2 &= \|A-B_1\|_2^2 + \|B_2\|_2^2 \\ &\geqslant \sum_j |\alpha_j - \operatorname{Re}\beta_j|^2 + \sum_j |\operatorname{Im}\beta_j|^2 \\ &= \sum_j |\alpha_j - \beta_j|^2, \end{aligned}$$

which is the desired inequality.

The case $p = \infty$ is equally simple. Use (31) instead of (30). For each j we have

$$\begin{aligned} |\alpha_j - \beta_j|^2 &= |\alpha_j - \operatorname{Re}\beta_j|^2 + |\operatorname{Im}\beta_j|^2 \\ &\leqslant \|A-B_1\|_\infty^2 + \|B_2\|_\infty^2 \\ &\leqslant 2\|A-(B_1+iB_2)\|_\infty^2 \\ &= 2\|A-B\|_\infty^2. \end{aligned}$$

For $2 \leqslant p < \infty$ use (32) and (34) together with (27) to get

$$\begin{aligned} 2^{1-p/2}\sum_j |\alpha_j - \beta_j|^p &\leqslant \sum_j |\alpha_j - \operatorname{Re}\beta_j|^p + \sum_j |\operatorname{Im}\beta_j|^p \\ &\leqslant \|A-B_1\|_p^p + \|B_2\|_p^p \\ &\leqslant \tfrac{1}{2}\{\|A-B_1+iB_2\|_p^p + \|A-B_1-iB_2\|_p^p\} \\ &= \tfrac{1}{2}\{\|A-B^*\|_p^p + \|A-B\|_p^p\} \\ &= \|A-B\|_p^p, \end{aligned}$$

which is the desired inequality.

For $1 \leqslant p \leqslant 2$ use (33) and (35) together with (27) to get the result. ∎

Sakai [18] has proved a rather special case of the above Theorem. He proves it for $p=2$ assuming that A and B_1 are both positive operators. In the special case when A is Hermitian and B skew-Hermitian stronger inequalities for all p-norms have been obtained by Ando and Bhatia [1].

We end with some remarks about results which can be easily proved using the same ideas.

Hoffman and Wielandt also proved an inequality complementary to (2). There exists a permutation π such that

$$\|A-B\|_2 \leqslant \left[\sum_{i=1}^{n} |\alpha_i - \beta_{\pi(i)}|^2\right]^{1/2}.$$

Such complementary inequalities for (4) and (5) can also be obtained.

Let $(A^{(1)},\ldots,A^{(m)})$ be an m-tuple of pairwise commuting compact normal operators in $\mathscr{H}$. Then there exists an orthonormal basis e_j, $j=1,2,\ldots$, such that each e_j is a simultaneous eigenvector for all $A^{(k)}$, $1\leqslant k\leqslant m$. Let $A^{(k)}e_j=\lambda_j^{(k)}e_j$, $1\leqslant k\leqslant m$. The points $(\lambda_j^{(1)},\ldots,\lambda_j^{(m)})$ in the space $\mathbb{C}^m$, $j=1,2,\ldots$, can be called the *joint eigenvalues* of the tuple $(A^{(1)},\ldots,A^{(m)})$. The set of these points together with the point 0 in $\mathbb{C}^m$ coincides with the *Taylor spectrum* and the *Harte spectrum* in this case. See, e.g., [17]. In [4] and [11] it was shown that the Hoffman-Wielandt inequality (2) can be extended to give or bound for the distance between the joint eigenvalues of two commuting m-tuples of normal matrices. Following the same ideas our Theorems 1 and 2 can also be generalised to commuting m-tuples of normal Hilbert-Schmidt operators.

A version of Theorem 3 when A and B are not compact but $A-B$ is, can be proved using Kato's Theorem and the ideas of our proof. Note that $A-B=A-B_1-iB_2$. So both $A-B_1$ and B_2 are compact if $A-B$ is. An extended enumeration of discrete eigenvalues of B should now mean a sequence $\{\beta_i\}$ such that $\{\mathrm{Re}\,\beta_j\}$ is such an enumeration for B_1 in the sense of Kato.

In [9] the Clarkson–McCarthy inequalities are generalised to all unitarily invariant norms. These can be used to obtain some results extending Theorem 3 to such norms.

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