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Interface wave diffraction by a thin vertical barrier submerged in the lower fluid

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This paper is concerned with interface wave diffraction by a thin vertical barrier which is completely submerged in the lower fluid of two superposed infinite fluids and which extends infinitely downwards into the lower fluid. By a suitable application of Green's integral theorem in the two fluid regions, the problem is formulated in terms of a hypersingular integral equation for the difference of potential across the barrier. A numerical procedure is utilized to evaluate the reflection and transmission coefficients directly from this hypersingular integral equation.

Also, an integro-differential equation formulation of the problem is considered, wherein the equation is solved approximately up to $O(s)$, s being the ratio of the densities of the upper and lower fluids. Utilizing this approximate solution, the reflection and transmission coefficients are also obtained up to $O(s)$. Numerical results illustrate that the reflection coefficient up to $O(s)$ thus obtained is in good agreement with the same evaluated directly from the hypersingular integral equation for $0 < s \leq 0.5$. The advantage of the hypersingular integral equation formulation is that the reflection and transmission coefficients can be evaluated for any value of s such that $0 \leq s < 1$. It is observed that the presence of the upper fluid reduces the reflection coefficients from their exact values for a single fluid significantly.

1 INTRODUCTION

Problems involving two superposed fluids are in general complicated because of the coupled boundary conditions at the interface of the two fluids [cf eqn (3)]. The literature on two-fluid problems is rather limited. Gorgui and Kassem,[1] Rhodes-Robinson[2] and Kassem[3,4] have studied the problem of generation of waves at the interface of two superposed fluids due to various types of singularities submerged in either of the two fluids by methods similar to those which were used for the corresponding one-fluid problems. A study of the problem of interface wave diffraction by a thin vertical barrier is initiated here by assuming the barrier to be submerged in the lower fluid of two superposed fluids. The corresponding problem for a single fluid bounded by a free surface is well studied in the literature for a number of positions of the barrier. This problem is an extension of the submerged barrier problem, considered by Dean,[5] Ursell[6] and others for deep water, to two superposed fluids wherein the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards, the barrier being submerged in the lower fluid and extending infinitely downwards.

By suitable application of Green's integral theorem in the two fluid regions, and taking care of the coupled interface conditions in an appropriate manner, the problem under consideration is formulated here in terms of a hypersingular integral equation for the difference of potential across the barrier. This type of hypersingular integral equation arises in acoustics, hydrodynamics and elastostatics[7–9] in a natural manner. Parsons and Martin[8] have studied the problem of water wave scattering by a plate submerged in a single fluid, after formulating it in terms of a hypersingular integral equation for the difference of potential across the barrier by a finite series of suitably chosen orthogonal polynomials multiplied by an appropriate weight function. This was then utilized to evaluate the reflection and transmission coefficients numerically. Very recently, they have also considered water wave scattering by submerged curved plates and surface-piercing flat plates by the same procedure.[9] The numerical procedure of Parsons and Martin[8] is adopted in this paper to solve the hypersingular integral equation by using a collocation method. This was then utilized to compute the reflection and transmission coefficients directly.

We have also adopted an alternative approach to solving this problem approximately. Instead of the aforesaid hypersingular integral equation formulation, an integro-differential equation formulation similar to that used for the corresponding single-fluid case is utilized.[10] It may be noted that for the single-fluid case, the corresponding integro-differential equation possesses an exact solution. However, this is not so for the present two-fluid case. Here, the integro-differential equation is solved approximately by exploiting the fact that its kernel has some sort of series representation involving the parameter s ($0 \le s < 1$), the ratio of the densities of upper and lower fluids. Assuming a similar series representation for the potential difference across the barrier, we obtain, up to $O(s)$, two integro-differential equations whose solutions are known explicitly. The reflection and transmission coefficients are assumed to have similar representation in terms of s and are obtained up to $O(s)$ analytically. The numerical results illustrate that the reflection coefficients evaluated directly from the hypersingular integral equation for various values of the wave number and for $s = 0.0013$ (for an air–water combination), 0.01, 0.1, 0.25, 0.5 coincide with the reflection coefficients evaluated by the second method up to three decimal places in most cases. This also demonstrates the utility of the approximate solution.

As mentioned by LaFond,[11] internal waves (here termed as interface waves) exist between subsurface water layers of varying density; they exist in all oceans, and probably in most bays and lakes. It was earlier reported[12] that in the mouths of some of the Norwegian fjords there exists a layer of fresh water over salt water. These remarks form a basis for the practical interest in the problem considered here, wherein a two-fluid model is constructed to investigate the effect of the upper fluid on the reflection and transmission coefficients for the classical problem of water wave diffraction by a submerged vertical barrier. Also, this model may be used for practical purpose to find the effect of air on the reflection and transmission coefficients by interpreting the two-fluid problem as an atmosphere–ocean system. However, as the ratio of the densities of air and water is 0.0013, this is too small to produce any appreciable effect on the reflection coefficients, as the numerical results presented later in this paper are very close to those for $s = 0$.

2 FORMULATION OF THE PROBLEM

We consider motion in two immiscible, inviscid, homogeneous and incompressible superposed fluids. Let ρ_1 be the density of the lower fluid occupying the region $y \ge 0$ and $\rho_2 (< \rho_1)$ be the density of the upper fluid occupying the region $y \le 0$. The y-axis is taken vertically downwards with the plane $y = 0$ as the mean interface. A thin vertical barrier is completely submerged in the lower fluid and extends infinitely downwards so that it occupies the position $x = 0$, $a \le y < \infty$. Assuming linear theory and irrotational motion, a train of time-harmonic progressive waves propagating at the interface from negative infinity is represented by $\phi_0(x, y)$ in the lower fluid and $\psi_0(x, y)$ in the upper fluid, where

$$\phi_0(x, y) = \exp(-My + \mathrm{i}Mx)$$
$$\psi_0(x, y) = -\exp(My + \mathrm{i}Mx) \tag{1}$$

Here, $M = (1 + s/1 - s)K$, where $K = \sigma^2/g$, σ is the circular frequency of the incoming interface waves with time dependence $\exp(-\mathrm{i}\sigma t)$ (dropped throughout the analysis), and g is the acceleration due to gravity. Due to the presence of the barrier, the incident wave train is partially reflected by and transmitted over the barrier. If the resulting motion is described by the velocity potentials $\phi(x, y)$ and $\psi(x, y)$ in the lower and upper fluids, respectively, then ϕ and ψ satisfy

$$\nabla^2\phi = 0 \text{ for } y \ge 0,\ \nabla^2\psi = 0 \text{ for } y \le 0 \tag{2}$$

$$K\phi + \phi_y = s(K\psi + \psi_y),\ \phi_y = \psi_y \text{ at } y = 0 \tag{3}$$

$$\phi_x = 0 \text{ at } x = 0,\ a < y < \infty \tag{4}$$

$$r^{1/2}\nabla\phi \text{ is bounded as } r = \{x^2 + (y - a)^2\}^{1/2} \to 0 \tag{5}$$

Also, ϕ and ψ are to satisfy the following requirements as $|x| \to \infty$,

$$\begin{bmatrix} \phi(x, y) \\ \psi(x, y) \end{bmatrix} \to \begin{cases} T \begin{bmatrix} \phi_0(x, y) \\ \psi_0(x, y) \end{bmatrix} & \text{as } x \to \infty \\ \begin{bmatrix} \phi_0(x, y) \\ \psi_0(x, y) \end{bmatrix} + R \begin{bmatrix} \phi_0(-x, y) \\ \psi_0(-x, y) \end{bmatrix} & \text{as } x \to -\infty \end{cases} \tag{6}$$

where T and R are, respectively, the unknown transmission and reflection coefficients and are to be determined.

To solve the boundary value problem described by eqns (2)–(5), we require two-dimensional source potentials due to a line source submerged in either of the two fluids. Let $G(x, y; \xi, \eta)$ and $H(x, y; \xi, \eta)$ be the source potentials in the lower and upper fluids, respectively, due to a line source submerged in the lower fluid at (ξ, η) $(\eta > 0)$, and $\overline{G}(x, y; \xi, \eta)$ and $\overline{H}(x, y; \xi, \eta)$ be the same due to a line source submerged in the upper fluid at (ξ, η) $(\eta < 0)$. These source potentials were obtained earlier by Gorgui and Kassem[1] and are given in the Appendix. We now apply Green's integral theorem to $\phi_1(x, y) = \phi(x, y) - \phi_0(x, y)$ and $G(x, y; \xi, \eta)$ in the region bounded externally by the lines $y = 0$, $-X \le x \le X$; $x = \pm X$, $0 \le y \le Y$; $y = Y$, $-X \le x < 0$; $x = 0-$, $a < y \le Y$; $x = 0+$, $a < y \le Y$; $y = Y$; $0 < x \le X$ and internally by a circle of small radius ϵ with centre at (ξ, η), and ultimately we make $X, Y \to \infty$ and $\epsilon \to 0$. We then find

$$2\pi\phi_1(\xi, \eta) = -\int_a^\infty f(y)G_x(0, y, \xi, \eta)\,dy + \int_{-\infty}^\infty [\phi_1 G_y - G\phi_{1y}]_{y=0}\,dx \qquad (7)$$

where

$$f(y) = \phi_1(+0, y) - \phi_1(-0, y) = \phi(+0, y) - \phi(-0, y), \ (a < y < \infty) \qquad (8)$$

Again we apply Green's integral theorem to $\psi_1(x, y) = \psi(x, y) - \psi_0(x, y)$ and $H(x, y; \xi, \eta)$ in the region bounded by the lines $y = 0, -X \le x \le X$; $x = \pm X, -Y \le y \le 0$; and $y = -Y, -X \le x \le X$. We then obtain

$$\int_{-\infty}^\infty \left[\psi_{1y}H - \psi_1 H_y\right]_{y=0} dx = 0 \qquad (9)$$

Multiplying eqn (9) by s and subtracting from eqn (7), we find

$$2\pi\phi_1(\xi, \eta) = -\int_a^\infty f(y)G_x(0, y; \xi, \eta)\,dy + \int_{-\infty}^\infty [G_y\phi_1 - s\psi_1 H_y - G\phi_{1y} + sH\psi_{1y}]_{y=0}\,dx \qquad (10)$$

The interface conditions satisfied by ϕ_1, ψ_1 and G, H imply

$$\left.\begin{aligned} G_y\phi_1 - s\psi_1 H_y &= \frac{s-1}{K}\phi_{1y}G_y \text{ at } y = 0 \\ G\phi_{1y} - sH\psi_{1y} &= \frac{s-1}{K}G_y\phi_{1y} \text{ at } y = 0 \end{aligned}\right\} \qquad (11)$$

so that the term in the square bracket in the right-hand side of eqn (10) vanishes identically. This results in the representation

$$2\pi\phi_1(\xi, \eta) = -\int_a^\infty f(y)G_x(0, y; \xi, \eta)\,dy \qquad (12)$$

To find $\psi_1(\xi, \eta)$ $(\eta < 0)$, we similarly apply Green's theorem to $\psi_1(x, y)$ and $\overline{H}(x, y; \xi, \eta)$ in the region bounded externally by the lines $y = 0, -X \le x \le X; x = \pm X, -Y \le y \le 0$ and $y = -Y, -X \le x \le X$ and internally by a small circle of radius ϵ and centre at (ξ, η), and ultimately we make $X, Y \to \infty$ and $\epsilon \to 0$. We thus find

$$2\pi\psi_1(\xi, \eta) = -\int_a^\infty \left[\overline{H}\psi_{1y} - \psi_1\overline{H}_y\right]_{y=0} dx \qquad (13)$$

We again apply Green's integral theorem to $\phi_1(x, y)$ and $\overline{G}(x, y; \xi, \eta)$ in the region bounded by the lines $y = 0$, $-X \le x \le X$; $x = \pm X, 0 \le y \le Y$; $y = Y, -X \le x < 0$; $x = 0-, a < y \le Y$; $x = 0+, a < y \le Y$, and $y = Y, 0 < x \le X$, and ultimately we make $X, Y \to \infty$. We then find

$$\int_a^\infty f(y)\overline{G}_x(0, y; \xi, \eta)\,dx - \int_{-\infty}^\infty \left[\overline{G}\phi_{1y} - \phi_1\overline{G}_y\right]_{y=0} dx = 0 \qquad (14)$$

Multiplying eqn (13) by s and subtracting from eqn (14), we find

$$2\pi s\psi_1(\xi, \eta) = -\int_a^\infty f(y)\overline{G}_x(0, y; \xi, \eta)\,dy + \int_{-\infty}^\infty \left[\overline{G}\phi_{1y} - s\overline{H}\psi_{1y} - \phi_1\overline{G}_y + s\overline{H}_y\psi_1\right]_{y=0} dx \qquad (15)$$

The term in the square bracket of eqn (15) vanishes identically because of the interface conditions. Thus

$$2\pi s\psi_1(\xi, \eta) = -\int_a^\infty f(y)\overline{G}_x(0, y; \xi, \eta)\,dy \qquad (16)$$

Thus, the velocity potentials $\phi(\xi, \eta)(\eta > 0)$ in the lower fluid and $\psi(\xi, \eta)(\eta < 0)$ in the upper fluid are obtained in terms of the unknown difference of potentials $f(y)$ across the barrier submerged in the lower fluid.

From the condition in eqn (4), we have

$$\phi_{1\xi}(0, \eta) = -\phi_{0\xi}(0, \eta) = -iM\exp(-M\eta), \ a < \eta < \infty \qquad (17)$$

Using eqn (17) in eqn (12), we obtain the integro-differential equation for $f(y)$ as

$$\frac{\partial}{\partial\xi}\int_a^\infty f(y)G_x(0, y; 0, \eta)\,dy = 2\pi iM\exp(-M\eta), a < \eta < \infty \qquad (18)$$

This is to be solved subject to the condition that $f(a) = 0$ and $f(y)$ is bounded as $y \to \infty$.

The order of integration and differentiation in eqn (18) can be interchanged provided the integral is interpreted as a finite part integral. This leads to the hypersingular integral equation

$$⨎_a^\infty f(y)G_{x\xi}(0, y; 0, \eta)\,dy = 2\pi iM\exp(-M\eta), \ a < \eta < \infty \qquad (19)$$

As in the work of Parsons and Martin,[8] the cross on the integral sign indicates that it is to be interpreted as a two-sided finite part integral of order two.

In the next section, we solve this hypersingular integral equation [eqn (19)] numerically, utilizing the method used by Parsons and Martin[8] and also calculate the reflection and transmission coefficients.

3 THE HYPERSINGULAR INTRGRAL EQUATION AND THE METHOD OF SOLUTION

In eqn (19), $G_{x\xi}(0, y; 0, \eta)$ is given by

$$G_{x\xi}(0,y;0,\eta) = -\frac{1}{(y-\eta)^2} - \frac{1}{(y+\eta)^2} - \frac{2M}{(1+s)(y+\eta)} - \frac{2M^2}{1+s}\int_0^\infty \frac{\exp[-k(y+\eta)]}{k-M}\,dk \tag{20}$$

Then, eqn (19) becomes

$$\fint_a^\infty \left[\frac{1}{(y-\eta)^2} + \frac{1}{(y+\eta)^2} + \frac{2M}{(1+s)(y+\eta)} + \frac{2M^2}{1+s}\int_0^\infty \frac{\exp[-k(y+\eta)]}{k-M}\,dk\right] f(y)\,dy = -2\pi i M \exp(-M\eta),\ a < \eta < \infty. \tag{21}$$

Substituting $y = 2a/(1+p)$ and $\eta = 2a/(1+q)$ in eqn (21), we find

$$\fint_{-1}^{1} \left[\frac{1}{(p-q)^2} + L(q,p)\right] f_1(p)\,dp = h(q),\ -1 < q < 1 \tag{22}$$

where

$$L(q,p) = \frac{1}{(p+q+2)^2} + \frac{4Ma}{(1+s)(1+p)(1+q)(p+q+2)} + \frac{8M^2a^2}{(1+s)(1+p)^2(1+q)^2}\psi(p,q) \tag{23}$$

and

$$\psi(p,q) = \int_0^\infty \frac{\exp(-\mu k)}{k-M}\,dk \tag{24}$$

with $\mu = [2a(p+q+2)]/[(p+1)(q+1)]$, and the path in the integral being indented below the point $k = M$,

$$h(q) = -4\pi M a i \frac{\exp\left(-\frac{2Ma}{1+q}\right)}{(1+q)^2},\ -1 < q < 1 \tag{25}$$

$$f_1(p) = f\left(\frac{2a}{1+p}\right) \tag{26}$$

so that $f_1(\pm 1) = 0$.

Following Yu and Ursell,[13] it can be shown that

$$\psi(p,q) = -\exp(-\mu M) \times \left[\ln \mu M + \gamma - i\pi + \sum_{m=1}^{\infty} \frac{(M\mu)^m}{m.m!}\right] \tag{27}$$

where $\gamma = 0.5772$ is Euler's constant and μ is given above.

Since $f_1(\pm 1) = 0$, following Parsons and Martin,[8] we assume

$$f_1(p) = (1-p^2)^{1/2} g(p) \tag{28}$$

where $g(p)$ is a bounded function. $g(p)$ is now approximated as

$$g(p) = \sum_{n=0}^{N} a_n U_n(p) \tag{29}$$

where $U_n(p)$ is the Chebyshev polynomial of the second kind given by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{30}$$

and the values of a_n are to be determined.

The use of eqns (28) and (29) in eqn (22) produces

$$\sum_{n=0}^{N} a_n A_n(q) = h(q),\ -1 < q < 1 \tag{31}$$

where

$$A_n(q) = -\pi(n+1)U_n(q) + \int_{-1}^{1} (1-p^2)^{1/2} U_n(p) L(q,p)\,dp \tag{32}$$

To find the unknown constants $a_n (n = 0, 1, 2, \ldots, N)$ we put $q = q_j (j = 0, 1, 2, \ldots, N)$ in eqn (31) to obtain the linear system

$$\sum_{n=0}^{N} a_n A_n(q_j) = h(q_j),\ j = 0, 1, 2, \ldots, N \tag{33}$$

which can be solved by standard methods. In eqn (33), the values of q_j are collocation points and can be chosen suitably. Parsons and Martin[8] suggested these as

$$q_j = \cos\left\{\frac{(j+1)\pi}{N+2}\right\},\ j = 0, 1, 2, \ldots, N \tag{34}$$

They also suggested another possible choice as

$$q_j = \cos\left\{\frac{(2j+1)\pi}{2N+2}\right\},\ j = 0, 1, 2, \ldots, N \tag{35}$$

In our numerical calculations we have used both these choices.

The reflection and transmission coefficients R and T can be obtained by making $\xi \to \pm\infty$ in eqn (12) or eqn

(16) and noting the requirements at infinity given in eqn (6). Then,

$$R = -\frac{M}{1+s}\int_a^\infty f(y)\exp(-My)\,dy \tag{36}$$

and

$$T = 1 + \frac{M}{1+s}\int_a^\infty f(y)\exp(-My)\,dy \tag{37}$$

Substituting $y = 2a/(1+p)$, eqns (36) and (37) become

$$R = -\frac{2Ma}{1+s}\int_{-1}^{1}\frac{f_1(p)\exp\left(-\frac{2Ma}{1+p}\right)}{(1+p)^2}\,dp \tag{38}$$

and

$$T = 1 + \frac{2Ma}{1+s}\int_{-1}^{1}\frac{f_1(p)\exp\left(-\frac{2Ma}{1+p}\right)}{(1+p)^2}\,dp \tag{39}$$

Substituting $f_1(p)$ into eqns (38) and (39), we find

$$R = -\frac{2Ma}{1+s}\sum_{n=0}^{N} a_n \int_{-1}^{1}\frac{(1-p^2)^{1/2}}{(1+p)^2} \times \exp\left(-\frac{2Ma}{1+p}\right)U_n(p)\,dp \tag{40}$$

and

$$T = 1 + \frac{2Ma}{1+s}\sum_{n=0}^{N} a_n \int_{-1}^{1}\frac{(1-p^2)^{1/2}}{(1+p)^2} \times \exp\left(-\frac{2Ma}{1+p}\right)U_n(p)\,dp \tag{41}$$

These can be evaluated numerically after the values of a_n are obtained from the linear system [eqn (33)].

It may be noted that R and T are obtained here for any value s such that $0 \le s < 1$. However, for small values of s, a perturbation technique can be utilized to obtain T and R up to $O(s)$ analytically. This is carried out in the next section.

4 APPROXIMATE SOLUTION UP TO $O(S)$

The integro-differential equation, eqn (18), can be reduced to another integro-differential equation by using the procedure adopted in the corresponding one-fluid problem.[10] Noting

$$G_{x\xi}(0, y; 0, \eta) = G_{\eta\eta}(0, y; 0, \eta) \tag{42}$$

eqn (18) can be written as

$$\frac{d^2}{d\eta^2}\int_a^\infty f(y)G(0, y; 0, \eta)\,dy = 2\pi i M\exp(-M\eta),\ a < \eta < \infty \tag{43}$$

Our integration of eqn (43) yields

$$\frac{d}{d\eta}\int_a^\infty f(y)G(0, y; 0, \eta)\,dy = -2\pi i M\exp(-M\eta),\ a < \eta < \infty \tag{44}$$

where

$$\frac{d}{d\eta}\int_a^\infty f(y)\{MG + G_\eta\}(0, y; 0, \eta)\,dy = 0 \tag{45}$$

Now,

$$\begin{aligned}(MG + G_\eta)(0, y; 0, \eta) &= E(y, \eta) + \frac{2sM}{1+s}\ln|y+\eta| \\ &= E(y, \eta) + 2sM\ln|y+\eta| \\ &\quad + O(s^2)\end{aligned} \tag{46}$$

where

$$E(y, \eta) = M\ln\left|\frac{y-\eta}{y+\eta}\right| - \frac{1}{y-\eta} + \frac{1}{y+\eta} \tag{47}$$

Equation (46) gives some sort of expansion for the kernel of the integro-differential equation [eqn (45)] for small s. We assume a similar expansion for $f(y)$ given by

$$f(y) = f_0(y) + sf_1(y) + O(s^2) \tag{48}$$

Using eqns (46) and (48) in eqn (45) and equating the coefficients of s^0 and s from both sides, we find that $f_0(y)$ and $f_1(y)$ satisfy the following integro-differential equations.

$$\frac{d}{d\eta}\int_a^\infty f_0(y)E(y, \eta)\,dy = 0,\ a < \eta < \infty \tag{49}$$

and

$$\frac{d}{d\eta}\int_a^\infty f_1(y)E(y, \eta)\,dy = -2M\frac{d}{d\eta}\int_a^\infty f_0(y)\ln|y+\eta|\,dy,\ a < \eta < \infty \tag{50}$$

The integro-differential equations [eqns (49) and (50)] can be reduced to the following singular integral equations

$$\int_a^\infty \frac{y\lambda_0(y)}{\eta^2 - y^2}\,dy = 0,\ a < \eta < \infty \tag{51}$$

and

$$\int_a^\infty \frac{2y\lambda_1(y)}{\eta^2 - y^2}\,\mathrm{d}y = r(\eta) \tag{52}$$

where the integrals are in the sense of Cauchy principal value and

$$\lambda_0(y) = Mf_0(y) + f_0'(y) \tag{53}$$

$$\lambda_1(y) = Mf_1(y) + f_1'(y) \tag{54}$$

and

$$r(\eta) = -2M\frac{\mathrm{d}}{\mathrm{d}\eta}\int_a^\infty f_0(y)\ln|y+\eta|\,\mathrm{d}y \tag{55}$$

The functions $\lambda_0(y)$ and $\lambda_1(y)$ have integrable singularities at $y = a$. The edge condition [eqn (5)] suggests that

$$\begin{aligned}\lambda_0(y) &= O(|y^2 - a^2|^{-1/2})\\ \lambda_1(y) &= O(|y^2 - a^2|^{-1/2}) \text{ as } y \to a\end{aligned} \tag{56}$$

Solutions of eqns (51) and (52) satisfying eqn (56) are well known[6] and are given by

$$f_0(y) = c_o\exp(-My)\int_a^y \frac{\exp(Mu)}{(u^2 - a^2)^{1/2}}\,\mathrm{d}u \tag{57}$$

and

$$f_1(y) = \exp(-My)\int_a^y \frac{\exp(Mu)}{(u^2 - a^2)^{1/2}}\{c_1 - B(u)\}\,\mathrm{d}u \tag{58}$$

where

$$B(u) = \frac{2}{\pi^2}\int_a^\infty \frac{\eta r(\eta)(\eta^2 - a^2)^{1/2}}{\eta^2 - y^2}\,\mathrm{d}\eta \tag{59}$$

and c_0 and c_1 are arbitrary constants to be determined.

From the Appendix, we have

$$\begin{aligned}G(0,y;0,\eta) = G_0(0,y;0,\eta) + s\{&\ln|y^2 - \eta^2|\\ &- G_0(0,y;0,\eta)\} + O(s^2)\end{aligned} \tag{60}$$

where

$$\begin{aligned}G_0(0,y;0,\eta) &= -2\pi\mathrm{i}\exp\{-M(y+\eta)\}\\ &\quad - 2\int_0^\infty \frac{L(M,y)L(M,\eta)}{k(k^2 + M^2)}\,\mathrm{d}k\end{aligned} \tag{61}$$

with $L(M,t) = k\cos kt - M\sin kt$.

Using eqns (48) and (60) in eqn (44) and equating the coefficients of s^0 and s from both sides we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\eta}\int_a^\infty f_0(y)G_0(0,y;0,\eta)\,\mathrm{d}y\\ &\quad = -2\pi\mathrm{i}\exp(-M\eta),\ a < \eta < \infty\end{aligned} \tag{62}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\eta}\int_a^\infty f_1(y)G_0(0,y;0,\eta)\,\mathrm{d}y = -2\pi\mathrm{i}\exp(-M\eta)\\ &\quad -\frac{\mathrm{d}}{\mathrm{d}\eta}\int_a^\infty f_0(y)\ln|y^2 - \eta^2|\,\mathrm{d}y,\ a < \eta < \infty\end{aligned} \tag{63}$$

where the right-hand side of eqn (63) was simplified using eqn (62).

To evaluate c_0, we substitute $f_0(y)$ from eqn (57) into eqn (62). After evaluating the various integrals, we find

$$c_0 = -\frac{2}{\Delta_0} \tag{64}$$

where

$$\Delta_0 = K_0(Ma) + \mathrm{i}\pi I_0(Ma) \tag{65}$$

K_0 and I_0 being modified Bessel functions.

To evaluate c_1, we substitute $f_1(y)$ from eqn (58) into eqn (63) and the various integrals involved are simplified. The main steps are indicated below.

$$\int_a^\infty f_1(y)\exp(-My)\,\mathrm{d}y = \frac{c_1}{2M}K_0(Ma) + \frac{\alpha_1}{2M} \tag{66}$$

where

$$\alpha_1 = \int_a^\infty \frac{\exp(-My)B(y)}{(y^2 - a^2)^{1/2}}\,\mathrm{d}y \tag{67}$$

and

$$\begin{aligned}\int_a^\infty f_1(y)L(M,y)\,\mathrm{d}y &= -\frac{\pi}{2}c_1 J_0(ka)\\ &\quad - \int_a^\infty \frac{\cos kyB(y)}{(a^2 - y^2)^{1/2}}\,\mathrm{d}y + \frac{1}{\pi}\int_a^\infty r(y)\cos ky\,\mathrm{d}y\end{aligned} \tag{68}$$

Utilizing these results in eqn (63) and after making further simplifications we obtain

$$\begin{aligned}c_1\Delta_0 + \Delta_1 - \mathrm{i}\delta(M) &= -2 + \frac{\mathrm{i}}{\pi}\Bigg\{\int_a^\eta r(t)\exp(Mt)\,\mathrm{d}t\\ &\quad + \exp(M\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\int_a^\infty f_0(y)\ln|y^2 - \eta^2|\,\mathrm{d}y\Bigg\},\\ &\quad a < \eta < \infty\end{aligned} \tag{69}$$

where

$$\Delta_1 = \alpha_1 + \mathrm{i}\beta_1 \tag{70}$$

$$\beta_1 = \int_{-a}^a \frac{\exp(-My)B(y)}{(a^2 - y^2)^{1/2}}\,\mathrm{d}y \tag{71}$$

and

$$\delta(M) = \int_a^\infty \exp(-Mt)r(t)\,dt \tag{72}$$

The right-hand side of eqn (69) can be further simplified. We note that

$$\frac{d}{d\eta}\left\{\int_a^\eta r(\eta)\exp(Mt)\,dt + \exp(M\eta)\int_a^\infty f_0(y)\ln|y^2-\eta^2|\,dy\right\}$$
$$= \exp(M\eta)\frac{d}{d\eta}\left\{M\int_a^\infty f_0(y)\ln\left|\frac{y-\eta}{y+\eta}\right|\,dy - \int_a^\infty f_0(y)\left(\frac{1}{y+\eta}-\frac{1}{y-\eta}\right)dy\right\}$$
$$= \exp(M\eta)\frac{d}{d\eta}\int_a^\infty \{Mf_0(y)+f_0'(y)\}\ln\left|\frac{y-\eta}{y+\eta}\right|\,dy \tag{73}$$

This vanishes identically for $a < \eta < \infty$ due to eqn (51). Hence c_1 is obtained

$$c_1 = \frac{-2-\Delta_1+i\delta(M)}{\Delta_0} \tag{74}$$

The reflection and transmission coefficients R and T can now be obtained up to $O(s)$ by making $\xi \to \pm\infty$ in eqn (12) or eqn (16) and noting the requirements at infinity given by eqn (6). Then,

$$R = -\frac{M}{1+s}\int_a^\infty f(y)\exp(-My)\,dy \tag{75}$$

and

$$T = 1+\frac{M}{1+s}\int_a^\infty f(y)\exp(-My)\,dy \tag{76}$$

If we assume

$$R = R_0 + sR_1 + O(s^2) \tag{77}$$

and

$$T = T_0 + sT_1 + O(s^2) \tag{78}$$

then

$$R_0 = -M\int_a^\infty f_0(y)\exp(-My)\,dy \tag{79}$$

$$T_0 = 1 + M\int_a^\infty f_0(y)\exp(-My)\,dy \tag{80}$$

$$R_1 = -M\int_a^\infty f_1(y)\exp(-My)\,dy - R_0 \tag{81}$$

and

$$T_1 = M\int_a^\infty f_1(y)\exp(-My)\,dy - T_0 - 1 \tag{82}$$

Using $f_0(y)$ and $f_1(y)$ from eqns (57) and (58), respectively, we find

$$R_0 = \frac{K_0(Ma)}{\Delta_0},\quad T_0 = \frac{i\pi I_0(Ma)}{\Delta_0} \tag{83}$$

and

$$R_1 = -T_1 = \frac{i}{2}[\{\beta_1 - \delta(M)\}K_0(Ma) - \pi\alpha_1 I_0(Ma)] \tag{84}$$

where α_1, β_1 and $\delta(M)$ are given in eqns (67), (71) and (72) respectively.

5 DISCUSSION

The problem of interface wave scattering by a thin vertical barrier which is completely submerged in the lower fluid and which extends infinitely downwards into the lower fluid of two superposed infinite fluids has been studied. Green's integral theorem is suitably utilized to formulate the problem in terms of a hypersingulur integral equation for the unknown difference of potential across the barrier. This unknown difference of potential is approximated by a truncated series of orthogonal polynomials, namely Chebyshev polynomials of the second kind, multiplied by an appropriate weight function, and the zeros of the Chebyshev polynomial are used as collocation points.

The reflection and transmission coefficients are obtained in terms of a truncated series. Again, the problem is formulated in terms of an integro-differential equation. As the kernel of this integro-differential equation has

Table 1. Reflection coefficient

	$s = 0$	
	H.S.	Exact
Ka	$\|R\|$	$\|R\|$
0.001	0.912800	0.912847
0.005	0.864877	0.864939
0.01	0.832431	0.832524
0.05	0.703620	0.703786
0.1	0.610188	0.610408
0.5	0.266129	0.266668
1.0	0.105712	0.105264
1.5	0.041294	0.041293
2.0	0.015903	0.015902
2.5	0.006032	0.006032
3.0	0.002266	0.002266
3.5	0.000846	0.000846
4.0	0.000314	0.000314

Table 2. Reflection coefficient

	s = 0.0013		s = 0.01		s = 0.1	
	H.S.	$O(s)$	H.S.	$O(s)$	H.S.	$O(s)$
Ka	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$
0.001	0.912744	0.912917	0.912366	0.912540	0.908286	0.908485
0.005	0.864766	0.864913	0.864063	0.864208	0.856448	0.856579
0.01	0.832293	0.832425	0.831353	0.831478	0.821075	0.821206
0.05	0.703332	0.703505	0.701332	0.701511	0.679624	0.679804
0.1	0.609794	0.610006	0.607023	0.607253	0.577136	0.577379
0.5	0.265202	0.266013	0.259042	0.261630	0.200335	0.203492
1.0	0.105100	0.104757	0.101057	0.101391	0.064863	0.069601
1.5	0.040949	0.040990	0.038687	0.038993	0.020248	0.021899
2.0	0.015727	0.015743	0.014591	0.014711	0.006195	0.006722
2.5	0.005950	0.005956	0.005420	0.005466	0.001867	0.002031
3.0	0.002229	0.002231	0.001993	0.002011	0.000558	0.000608
3.5	0.000830	0.000831	0.000729	0.000735	0.000166	0.000181
4.0	0.000308	0.000308	0.000265	0.000268	0.000049	0.000054

some sort of expansion in terms of s, the unknown function satisfying the integro-differential equation and the reflection and the transmission coefficients are assumed to have similar expansions. An approximate solution of the integro-differential equation up to $O(s)$ is obtained. Utilizing this solution, the reflection and transmission coefficients are evaluated up to $O(s)$.

For numerical computation of the reflection coefficient based on the hypersingular integral equation, we have used $N = 10$ in eqn (29) and the collocation points given by eqn (34) in the linear system [eqn (33)]. The different integrals are evaluated by using a 24-point Gauss quadrature formula. Almost the same numerical results are obtained if N is taken as 15 and the collocation points are taken as those given by eqn (35).

In Table 1, $|R|$ is tabulated for $s = 0$ obtained directly from the numerical solution of the hypersingular integral equation and from the known exact result given by Ursell[6] for various values of wave number. It is observed that the results coincide within 3–4 decimal places in most cases. This demonstrates the effectiveness of the numerical scheme based on the hypersingular integral equation formulation of the problem.

Table 3. Reflection coefficient

	s = 0.25		s = 0.5	
	H.S.	$O(s)$	H.S.	$O(s)$
Ka	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$	$\lvert R\rvert$
0.001	0.900652	0.900855	0.883432	0.883630
0.005	0.841925	0.842053	0.808378	0.808513
0.01	0.801363	0.801510	0.755370	0.755518
0.05	0.637748	0.637960	0.540326	0.540584
0.1	0.520144	0.520413	0.392536	0.392856
0.5	0.142905	0.143341	0.029553	0.031293
1.0	0.029539	0.030102	0.001577	0.002266
1.5	0.004982	0.006033	0.000080	0.000117
2.0	0.000963	0.001175	0.000004	0.000006
2.5	0.000184	0.000226	0.000000	0.000000
3.0	0.000035	0.000043	0.000000	0.000000
3.5	0.000007	0.000008	0.000000	0.000000
4.0	0.000000	0.000002	0.000000	0.000000

In Tables 2 and 3, $|R|$ is tabulated for $s = 0.0013$, 0.01, 0.1, 0.25, 0.5 obtained directly from the hypersingular integral equation and approximately up to $O(s)$ by the perturbation method for various values of Ka. It is observed that here also these coincide up to 3–4 decimal places. This demonstrates that the perturbation technique also furnishes good results for values of s up to 0.5.

Comparing the values of the reflection coefficients for the two-fluid case with single-fluid values (exact) for fixed Ka, it is observed that the density ratio, s, has a significant effect on the reflection coefficient (and also the transmission coefficient). Also, in Fig. 1, $|R|$ is plotted against s in the range (0, 0.5) for Ka = 0.05, 0.1, 0.5 and 1.0, and it is observed that the effect of s is to diminish the reflection coefficient which implies that the upper layer diminishes the penetration of the interface waves into the lower layer to some extent. Again, as Ka becomes large, the reflection coefficient for a fixed s becomes small, which is plausible since for large wave numbers, the waves are confined within a thin layer near the interface and almost the whole wave energy is transmited above the barrier submerged in the lower fluid.

6 CONCLUSION

The numerical method based on the hypersingular integral equation formulation of the problem seems to be

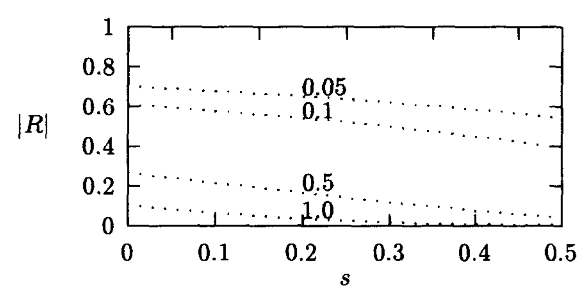

Fig. 1. Graphs of $|R|$ against s for various Ka.

very convenient for handling problems of interface waves involving two superposed fluids. That the method works smoothly is demonstrated here by simply taking $s = 0$ in the numerical scheme. The resulting numerical results for the reflection coefficient coincides up to 3–4 decimal places with the results obtained from the known analytical expression for $|R|$ for a single fluid. This method can be applied successfully when the barrier is inclined with the vertical or is horizontal or even when it is curved. The practical interest in the problem arises when one wants to consider interface wave diffraction by a barrier submerged in salt water over a layer of fresh water in the ocean. For mathematical simplification, the upper layer is assumed to be extending infinitely upwards and the lower layer extending infinitely downwards. An upper layer of finite height with a free surface would have been a better model. We hope to pursue this in the future.

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APPENDIX

The source potentials $G, H, \overline{G}$ and $\overline{H}$ have been obtained by Gorgui and Kassem.[1] These are given below.

For $\eta > 0$,

$$G(x, y; \xi, \eta) = \ln r - \frac{1-s}{1+s} \ln r' - \frac{2}{1+s} \int_0^\infty \frac{\exp\{-k(y+\eta)\}}{k-M} \cos k(x-\xi)\, dk \tag{A1}$$

and

$$H(x, y; \xi, \eta) = \frac{2}{1+s} \left[\ln r + \int_0^\infty \frac{\exp\{k(y-\eta)\}}{k-M} \cos k(x-\xi)\, dk \right] \tag{A2}$$

and for $\eta < 0$,

$$\overline{G}(x, y; \xi, \eta) = \frac{2s}{1+s} \left[\ln r + \int_0^\infty \frac{\exp\{-k(y-\eta)\}}{k-M} \cos k(x-\xi)\, dk \right] \tag{A3}$$

and

$$\overline{H}(x, y; \xi, \eta) = \ln r + \frac{1-s}{1+s} \ln r' - \frac{2s}{1+s} \int_0^\infty \frac{\exp\{k(y+\eta)\}}{k-M} \cos k(x-\xi)\, dk \tag{A4}$$

where the path of integration of each integral is along the positive real axis and indented below the pole at $k = M$, and

$$r^2, r'^2 = (x-\xi)^2 + (y \mp \eta)^2 \tag{A5}$$

Now, it can be shown that

$$(MG + G_\eta)(0, y; 0, \eta) = E(y, \eta) + \frac{2sM}{1+s} \ln |y+\eta| \tag{A6}$$

where $E(y, \eta)$ is given by eqn (47), so that

$$(MG + G_\eta)(0, y; 0, \eta) = E(y, \eta) + 2sM \ln |y+\eta| + O(s^2) \tag{A7}$$

Also, an alternative form for $G(x, y; \xi, \eta)$ is given by

$$G(x, y; \xi, \eta) = \frac{1}{1+s}[G_0(x, y; \xi, \eta) + s\ln(rr')] \quad \text{(A8)}$$

where

$$G_0(x, y; \xi, \eta) = -2\pi \mathrm{i} \exp\{-M(y+\eta) + \mathrm{i}M|x-\xi|\} - 2\int_0^\infty \frac{L(k,\eta)L(k,y)}{k(k^2+M^2)} \exp\{-k|x-\xi|\}\, \mathrm{d}k \quad \text{(A9)}$$

Thus

$$G(0, y; 0, \eta) = G(0, y; 0, \eta) + s\{\ln|y^2 - \eta^2| - G_0(0, y; 0, \eta)\} + O(s^2) \quad \text{(A10)}$$