THE ALFSEN-EFFROS STRUCTURE TOPOLOGY IN THE THEORY OF COMPLEX L^1-PREDUALS

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C O N T E N T S

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INTRODUCTION

A complex Banach space X is said to be an L^1-predual if X^* is isometric to $L^1(\mu)$ for some non-negative measure μ. Well known examples of L^1-preduals include the space $C(X)$ of complex-valued continuous functions on a compact Hausdorff space and the abstract M-spaces of Kakutani. In [19], Grothendieck introduced a class of L^1-preduals, now known as G-spaces, and conjectured that those are all the L^1-preduals. In his 1964 memoir [35], Lindenstrauss settled this conjecture by exhibiting a wide class of Banach spaces, other than G-spaces, which are L^1-preduals. He also gave several characterizations and interesting properties of L^1-preduals in terms of intersection properties of balls and extensions of operators. Since that time, the theory of L^1-preduals has attracted wide attention. L^1-preduals are now sometimes called Lindenstrauss spaces.

Let $P: X \longrightarrow X$ be a linear projection. We call P an L-projection if $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$. The range of an L-projection is called an L-ideal. As a consequence of the results of Alfsen-Effros [2] and Hirsberg [22], one knows that a norm closed subspace $J \subseteq A(K)^*$ (where $A(K)$, the space of continuous complex-valued affine functions on a compact convex set K is equipped with the supremum norm) is an L-ideal iff J is the linear span of a split face of the image of K in $A(K)^*$ under the evaluation map. Through the combined efforts of Lindenstrauss [35], Semadeni [45], Hirsberg and Lazar [21], it is known that a Lindenstrauss space whose unit ball has an extreme point, can be realised isometrically as the space $A(K)$ for some compact Choquet simplex K. Now, Ellis [14] proved that a compact convex set is a Choquet simplex iff every closed face of it is split. In view of the one-to-one correspondence between L-ideals and split faces mentioned at the beginning of this paragraph, all these results suggest the possibility of characterising <u>general</u> L^1-preduals in terms of L-ideals in their dual spaces. Several such characterizations are obtained in the first three sections of the present thesis.

The following theorem of Lima [28, Theorem 5.8] plays an important role in the proofs of our results : Let E denote the set of extreme points of the unit ball of the dual of a complex Banach space X. Then the statements (1) and (2) are equivalent

(1) X is an L^1-predual

(2) line $\{f\}$ is an L-ideal for each $f \in E$ and if $f \in X^*$, $\|f\| = 1$ and $P(f) = 0$ or f for all L-projections P on X^* then $f \in E$.

A closed subspace $M \subset X$ is said to be an M-ideal if $M^o = \{f \in X^* : f(x) = 0 \ \forall x \in M\}$ is an L-ideal. M-ideals were introduced by Alfsen-Effros [2]. In that paper Alfsen-Effros introduce a topology on E, called the structure topology, whose closed sets are obtained by intersecting E with w^*-closed

L-ideals in X^*. This topology is the analogue for general Banach spaces of the facial topology on the extreme boundary of a compact convex set [1] and of the biface topology on the extreme boundary of the dual unit ball of a Lindenstrauss space, introduced by Effros [12]. When the scalar field is real, we characterize L^1-preduals in terms of this structure topology on E and in the complex case obtain some general structural characterizations of various sub-classes of L^1-preduals.

We now give a brief sectionwise summary of this thesis.

In section 1, for a complex Banach space X, we define the concepts of T-faces and T-dilated sets, analogous to bifaces and symmetrically dilated sets of E.G. Effros [12]. We prove that when X is a Lindenstrauss space, the linear span of the w^*-closed convex hull of a dilated set is a w^*-closed L-ideal. As a consequence of this result we get that line F is a w^*-closed L-ideal for any w^*-closed face F of the dual unit ball X_1^* and line c(D) (c(D) stands for the w^*-closed convex hull of D) is a w^*-closed L-ideal for any w^*-compact set $D \subseteq E$.

For a Banach space X such that X_1^* with w^*-topology is a standard compact convex set in the sense of Rogalski [40], we show in section 2 that if line c(D) is an L-ideal for all w^*-compact $D \subseteq E$, then X is an L^1-predual. We give an example to show that this in general does not characterize L^1-preduals. In terms of the Alfsen-Effros, structure topology on E, we characterize <u>real</u> L^1-preduals as those real Banach spaces X for which the sets $\{f \in E : |f(x)| = 1\}$ are structurally closed for all $x \in X$ with $\|x\| = 1$. At the end of the section we give an example to show that this structural characterization <u>does not</u> extend to complex Banach spaces.

Section 3 deals with the characterizations of Lindenstrauss spaces using w^*-closed faces of the dual unit ball. Using results of Hirsberg and Lazar [21] and the parallel face characterization of simplexes due to Briem [7] we show that if $A \subset C(Y)$ is a closed subspace containing constants and separating points of the compact Hausdorff space Y, then the assumptions that line F is an L-ideal and (line F) $\cap X_1^* = c(TF)$ (where T is the unit circle) for all peak faces of the state space of A, imply that A is an L^1-predual. For a general Banach space X we show that if line F is an L-ideal for any w^*-closed face F of X_1^* then X is a Lindenstrauss space. The Bishop-Phelps theorem [5] plays a crucial role here as it does in an analogous characterization of simplexes due to A.J. Ellis [14]. We also give a characterization in terms of the M-sets, first defined by Hirsberg in [22]. We then give the partial complex analogue of the structural characterization given for real Banach spaces in section 2 by showing that if $\{f \in E : |f(x)| = 1\}$ is structurally closed and $F = \{f \in X_1^* : f(x) = 1\}$ is split in $co(F \cup -iF)$ for all $x \in X$ with $\|x\| = 1$, then X is an L^1-predual.

As an application of the last theorem, in section 4 we give a new proof of a theorem of Lima [29] which asserts that complex $E(4)$ - spaces are L^1-preduals. (For the definition of $E(4)$ - spaces, see [25]).

In section 5, we study complex G-spaces (for the definition see [36]). We characterize them as those Banach spaces X for which the functions $|x| : E \longrightarrow \mathbb{R}$ defined by $|x|(f) = |f(x)|$ are structurally upper semi-continuous for all $x \in X$. We prove that for a complex G-space, the intersection of any family of M-ideals is an M-ideal, a result proved for the real scalars by U. Uttersrud [47] and P.D. Taylor [46]. We then attempt to solve a problem of Uttersrud [47], of characterizing G-spaces as those Banach spaces X for which the intersection of M-ideals is an M-ideal and line $\{f\}$ is an L-ideal for all $f \in E$. We show that if $\{f \in E : \text{line } \{f\} \text{ is an L-ideal}\}$ is w^*-sequential dense in the w^*-closure of E and intersection of any countable number of M-ideals in X is an M-ideal then line $\{f\}$ is an L-ideal for all f in the w^*-closure of E. This enables us to settle positively Uttersrud's question for the class of separable Banach spaces whose unit ball has an extreme point and for separable L^1-preduals. We also give simple and transparent proofs of results which are more general than those obtained by N. Roy in [43] and A. Gleit in [20] in this context.

In section 6 we consider C_σ and $c_o(\Gamma)$ spaces (for the definitions of these spaces see section 6). We characterize C_σ - spaces as those complex Banach spaces X for which $|x| : E \longrightarrow \mathbb{R}$ is lower semi-continuous in the structure topology for all $x \in X$ or, equivalently, any relatively w^*-closed T-invariant subset of E is structurally closed. We show that C_σ - spaces are those G-spaces X which have the following property : for any L-ideal N in X^* the unit ball of $\overline{N}$ (closure taken in w^*-topology) equals the w^*-closure of the unit ball of N. If every T-invariant subset of E happens to be structurally closed, or, every L-ideal in X^* is w^*-closed and line $\{f\}$ is an L-ideal for all $f \in E$, then we show that X is isometric to $c_o(\Gamma)$.

Some of our results when specialised to $A(K)$ spaces, yield characterizations of Bauer simplexes which are sharper than some of the ones existing in the literature.

In section 7, we give new and simple proofs of some results of Wulbert [48] on L^1-preduals which are isometric to closed, self-adjoint subspaces of $C(Y)$ for a compact Hausdorff space Y. Using Bednar and Lacey's characterization of real L^1-preduals (see [3]) in terms of barycentric maps and the corresponding complex analogues due to A.K. Roy [42], we show that if the real part of a closed, self-adjoint subspace A of $C(Y)$ is an L^1-predual then A is an L^1-predual. We give an example of a closed subspace of a C(X) which is an L^1-predual but the real part is not an L^1-predual thus proving the falsity of Proposition 3.5 in [48].

In section 8 we consider isometries of A(K)-spaces. We describe a class of isometries for A(K) and give a sufficient condition on an isometry, in terms of facially continuous functi on the extreme boundary of K so that the isometry in question is in the prescribed class. We then give a complete descriptio of isometries of A(K) when K is a Choquet simplex. Our resu extend the classical Banach-Stone theorem for C(Y) and a theo of A.J. Lazar [27]. When K is a simplex we also completely describe isometries of $A^o(K) = \{a \in A(K) : a(p) = 0\}$, where is a fixed extreme point of K. We use these results to obtain a complete description of bi-contractive projections (i.e. proj tions P with $\|P\| \leq 1$ and $\|I - P\| \leq 1$) in A(K) for a simplex K.

Notation and terminology :

Let $\mathbb{C}$ denote the complex plane and T, the unit circle in $\mathbb{C}$.

For a Banach space X, let $X_1 = \{x \in X : \|x\| \leq 1\}$ and $S = \{x \in X : \|x\| = 1\}$. Let E denote the extreme points of X_1^*. For any $D \subseteq X^*$, let $c(D)$ denote the w^*-closed convex hull of D.

All closures unless otherwise mentioned are taken in the w^*-topology. Let $\xrightarrow{*}$, $\xrightarrow{s}$ denote convergence in w^* and structure topologies respectively.

For a compact Hausdorff space Y and a probability measure μ on Y let $\operatorname{Supp}\mu$ denote the topological support of μ. For $y \in Y$, let $\delta(y)$ denote the Dirac measure at y.

For a compact convex set K (always considered in a locally convex Hausdorff topological vector space) let $E(K)$ denote the extreme points of K. For $D \subseteq K$, we denote by $CO(D)$ the convex hull of D. For a probability measure μ an K, let $\gamma(\mu)$ denote the resultant of μ. We make free use of standard concepts and notations of convexity theory from [1].

We now recall briefly some notations and definitions of complex Choquet theory (see [13, 37]).

For any compact, absolutely convex set K and $t \in T$, let σt denote the homeomorphism $x \longrightarrow tx$. For $g \in C(K)$ define a

T-homogeneous function hom g on K by

$$(\text{hom } g)(L) = \int_T \bar{t}(g\circ\sigma_t)(L)\,dt \quad \text{for} \quad L \in K$$

where dt is the Haar measure on T.

For a complex Borel measure μ on K, define hom μ : C(K) $\longrightarrow$ C(K) by (hom μ)(f) = μ (hom f), f $\in$ C(K). Easy to see that hom μ is a linear map and $\|\text{hom } \mu\| \leq \|\mu\|$. Also $\mu \longrightarrow$ hom μ is w^*-continuous from $C(K)^*$ onto $C(K)^*$.

If $d\mu = h\, d|\mu|$ where h is a Borel measurable function of modulus one, define $R : C(K)^* \longrightarrow C(K)^*$ by

$$(R\mu)(g) = \int_K g(h(L).L)\,d|\mu|(L), \quad \mu \in C(K)^*, \quad g \in C(K).$$

Then we have $\|R\mu\| = \|\mu\|$, hom $(R(\mu))$ = hom μ.

Let $A \subseteq C(Y)$ be a closed subspace separating points of Y but not ingeneral containing constants. Let $e : Y \longrightarrow A_1^*$ denote the evaluation map. A complex measure μ on Y is called a boundary measure if $|\mu| \circ e^{-1}$ is a maximal measure on A_1^*.

SECTION 1

T-faces and T-dilated sets

Let X be a complex Banach space. For any $p, q \in X$, write $p \measuredangle q$ if $\|q\| = \|q-p\| + \|p\|$. The operation $\measuredangle$ was introduced in [2] and is a partial ordering on X.

In this section we develop complex analogues of some of the results obtained in [12], Section 5.

Definition : A w^*-closed T-invariant convex set $H \subseteq X_1^*$ is called a T-face if

1) $\forall\, p \in H, p \neq 0, p/\|p\| \in H$

2) $p \in H,\ q \measuredangle p \Longrightarrow q \in H$ (hereditary property).

Examples : For any $p \in E$, $\{\alpha p : |\alpha| \leq 1, \alpha \in \mathbb{C}\}$ is a T-face.

If $N \subseteq X^*$ is a w^*-closed hereditary subspace then N_1 is a T-face.

Let H_p denote the smallest T-face containing p.

Definition: We say a w^*-closed set $D \subseteq X_1^*$ is T-dilated if for all $p \in D$, $E(H_p) \subseteq D$.

Remark : For $p \in E$, $H_p = \{\alpha . p : |\alpha| \leq 1, \alpha \in \mathbb{C}\}$. Any w^*-closed T-invariant $D \subseteq E$, is T-dilated.

If $p \in X_1^*$, $E(H_p) \subseteq E$ and for $0 < \lambda < 1$, $H_p = H_{\lambda p}$.

We now quote a charactorization of complex L^1-preduals from Effros [13].

A complex Banach space X is an L^1-predual iff hom μ_1 = hom μ_2 whenever μ_1, μ_2 are maximal probability measures on X_1^* with common resultant.

For the rest of this section we assume that X is a Lindenstrauss space. Let $K = X_1^*$.

Definition : For any $p \in K$, let $\theta_p = R$ (hom μ) where μ is a maximal probability measure with $\gamma(\mu) = p$.

Inview of the result quoted above it is easy to see that θ_p is a well-defined map and θ_p represents p.

Lemma 1.1. 1) If $\|p\| = 1$ then $\theta(p) = \mu$

2) For $p \in K$, $p \neq 0$, $\theta(p) = \|p\| . \theta(p/\|p\|)$

3) For $p, q \in K$, $q \prec p$, $\theta(q) \leq \theta(p)$

4) Supp $\theta(p) \subseteq H_p \; \forall \; p \in K$

5) A w^*-closed T-invariant convex set H is a T-face iff $\forall \; p \in H$, $p \neq 0$, Supp $\theta(p) \subseteq H$.

Proof : 1) Follows from Lemma 3.6 of [37], since R(hom μ) = μ in this case.

2) Let $p \in K$, $0 < \|p\| < 1$ and μ, μ' be maximal with $\gamma(\mu) = p$, $\gamma(\mu') = p/\|p\|$. Fix $q \in E$.

If $\lambda = \|p\| \mu' + (1 - \|p\|)(\frac{1}{2}\delta(q) + \frac{1}{2}\delta(-q))$.

Then λ is a maximal measure with $\gamma(\lambda) = p$.

By Effros characterization, hom λ = hom μ. However hom $\lambda = \|p\|$ hom μ', because

$$(\text{hom }\delta(q))(g) = (\text{hom } g)(q) = -(\text{hom } g)(-q) = -\text{hom }\delta(-q)(g)$$

for $g \in C(K)$.

Therefore we have

$$\theta(p) = R\ (\text{hom }\lambda) = R(\|p\| \text{ hom }\mu') = \|p\| . \theta(p/\|p\|).$$

3) Follows from the observation $\theta(p) = \theta(p-q) + \theta(q)$ which is easy to deduce from 1 and 2.

4) Enough to show that for $p \in K$, $\|p\| = 1$, Supp $\theta(p) \subseteq H_p$.

Let μ be maximal with $\gamma(\mu) = p$. From Proposition 1.2.3 of [1], we get a net $\left\{ \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta(p_i^\alpha) \right\}$ where $\lambda_i^\alpha \in [0,1]$, $p_i^\alpha \in K\ \forall\ \alpha$ and i; $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1$, $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha p_i^\alpha = p\ \forall\ \alpha$ and such that $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta(p_i^\alpha) \xrightarrow{w^*} \mu$. Since $\|p\| = 1$, $\lambda_i^\alpha p_i^\alpha \prec p\ \forall\ i$ and α, so that $p_i^\alpha \in H_p\ \forall\ i$ and α. Hence Supp μ = Supp $\theta(p) \subseteq H_p$.

<u>Lemma 1.2</u>. For any T-face H of K, N = line H (i.e. the complex linear span of H) is a w^*-closed L-ideal.

<u>Proof</u> : We first claim that $N_1 = H$.

Let $0 \neq p \in N_1$, then $p = \sum_{i=1}^{n} \alpha_i q_i$, $q_i \in H$, $\alpha_i \neq 0\ \forall\ i$.

So $p = \lambda \sum_{i=1}^{n} \lambda_i t_i q_i$ where $\lambda = \sum_{i=1}^{n} |\alpha_i|$, $t_i = \alpha_i / |\alpha_i|$, $\lambda_i = |\alpha_i| / \lambda$ for $1 \leq i \leq n$. Since $q = \sum_{i=1}^{n} \lambda_i t_i q_i$ is in H, $q / \| q \| \in H$. Therefore $p = \lambda . \| q \| . q / \| q \|$ is in H, since $\lambda \| q \| \leq 1$. Hence $N_1 = H$. Now by the Krein-Smulian theorem, N is w^*-closed.

To show that N is an L-ideal, we need to show that $CO(N') \cap N = 0$ (see [2] page 110) where

$$N' = \left\{ q : p \prec q,\ p \in N \implies p = 0 \right\}.$$

Suppose $0 \neq x = \alpha r_1 + (1-\alpha) r_2$, $x \in N$, $0 \neq r_i \in N'$, $0 < \alpha < 1$. (To simplify matters we are considering convex combination of only two elements. However, it will be clear from the arguments below that the same proof works for arbitrary, finite, convex combinations).

Since N' is closed under multiplication by +ve scalars we get that $r_i / \| r_i \|$ is in N'. Let $p = x / \| x \|$ and $\beta = \alpha \| r_1 \| + (1 - \alpha) \| r_2 \|$. Then we have

$$\frac{1}{\beta}(\alpha \|r_1\| (r_1 / \|r_1\|) + (1-\alpha) \|r_2\| (r_2 / \|r_2\|)) = x/\beta$$

$$= \|x\| / \beta . p + (1 - \|x\| / \beta).0$$

Choose a maximal probability measure μ with Supp $\mu \subseteq N_1$, $\gamma(\mu) = p$ (such a choice is possible because of 1) and 4) of Lemma 1.1).

Also choose maximal measures μ_j on K, supported by N' with $\gamma(\mu_j) = r_j / \|r_j\|$ (see [2], page 113, Lemma 4.3). By Effros' characterization $\alpha \|r_1\|$ hom $\mu_1 + (1-\alpha)\|r_2\|$ hom $\mu_2 = \|x\|$ hom μ and since hom μ_j is supported by $TN' = N'$ and hom μ is supported by N we must have hom $\mu = 0$. Hence $x = 0$ since hom μ represents x, giving the required contradiction.

<u>Lemma 1.3</u>. For any w^*-closed T-invariant set $D \subseteq K$ such that Supp $\theta(p) \subseteq c(D)$ $\forall\, p \in D$, $c(D)$ is a T-face.

<u>Proof</u> : Inview of 5), Lemma 1.1, we need only to verify that Supp $\theta(p) \subseteq c(D)$ $\forall\, p \in c(D)$. Let $p \in c(D)$, $p \neq 0$. Let μ be a probability on D with $\gamma(\mu) = p$. Let μ' be a maximal measure on K, dominating μ in the Choquet ordering. Then by Theorem 2.1 of [12], $\exists$ nets of measures $\{\mu_\alpha\}$ an $\{\mu'_\alpha\}$ such that $\mu_\alpha \xrightarrow{w^*} \mu$, $\mu'_\alpha \xrightarrow{w^*} \mu'$, where $\mu_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha \delta(p_i^\alpha)$, $\mu'_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha \lambda_i^\alpha$ $\forall\, \alpha$ and λ_i^α maximal with $\gamma(\lambda_i^\alpha) = p_i^\alpha$, $p_i^\alpha \in D$, $\{c_i^\alpha\}_{i=1}^{n_\alpha} \subseteq [0,1]$, $\sum_{i=1}^{n_\alpha} c_i^\alpha = 1$ $\forall\, \alpha$

Since $p_i^\alpha \in D$, Supp $\theta(p_i^\alpha) \subseteq c(D)$ i.e.

Supp R(hom λ_i^α) $\subseteq c(D)$ and hence hom λ_i^α has its support in $c(D)$ as hom (R(hom λ_i^α)) = hom (hom λ_i^α) = hom λ_i^α. So hom $\mu'_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha$ hom λ_i^α has its support in $c(D)$. Since $\mu'_\alpha \xrightarrow{w^*} \mu'$ and 'hom' is w^*-continuous we get hom $\mu'_\alpha \xrightarrow{w^*}$ hom μ' so that hom μ' has its support in $c(D)$.

Therefore Supp $\theta(p)$ = Supp $(R(\text{hom } \mu'))$ $\subseteq$ $c(D)$.

Corollary 1.4. For any dilated set D, $H = c(D)$ is a T-face and $D \cap E^{\cdot}$ is a structurally closed set.

Proof : Let $p \in D$, by Lemma 1.1, 4), Supp $\theta(p) \subseteq H_p \subseteq H$. Therefore by above lemma, H is a T-face and if N = line H, since $N \cap E = H \cap E = D \cap E$, apply Lemma 1.2 to complete the proof.

Corollary 1.5. If $D \subseteq E$ is a w^*-compact T-invariant set then line $c(D)$ is a w^*-closed L-ideal and (line $c(D))_1 = c(D)$. If $F \subseteq K$ is a w^*-closed face then line F is a w^*-closed L-ideal and (line $F)_1 = c(T.F)$.

Proof : First part is easy to see.

Let $F \subseteq K$ be a w^*-closed face and put $H = c(T F)$. If $p = t.q$, $q \in F$, $t \in T$ and μ is a maximal measure with $\gamma(\mu) = p$ then the measure $\mu o \sigma_t$ represents q and since F is a closed face, Supp $(\mu o \sigma_t) \subseteq F$. Therefore Supp $\theta(p) \subseteq H$. Hence by Lemma 1.3, H is a T-face.

Clearly line $F \subseteq$ line H. Since F is a w^*-compact convex set it is easy to see that $G = \{\alpha x : 0 \leq \alpha \leq 1, x \in F\}$ is a w^*-compact convex set and that $TF \subseteq G - G + i(G - G)$. Consequently $c(TF) \subseteq$ line F. Therefore line F is a w^*-closed L-ideal and (line $F)_1 = c(TF)$.

SECTION 2

Some characterizations of complex L^1-preduals

Let X be a complex Banach space. We continue to denote X_1^* by K. In the previous section, we proved that if X is a Lindenstrauss space then for any w^*-compact T-invariant set $D \subseteq E$, line $c(D)$ is an L-ideal. We now show that this property actually characterizes L^1-preduals among a certain class of Banach spaces which includes the separable Banach spaces.

Theorem 2.1. Let X be a complex Banach space such that E is w^*-Borel and for any maximal measure μ on K, $\mu(E) = 1$ (K is a standard compact convex set in the sense of Rogalski [40]). Assume that for any w^*-compact and T-invariant $D \subseteq E$, line $c(D)$ is an L-ideal. Then X is an L^1-predual.

Proof : We first claim that line $c(D)$ is w^*-closed and $(\text{line } c(D))_1 = c(D)$.

Since $c(D)$ is a w^*-compact convex set and line $c(D)$ is norm closed by [9] v. 5.9 we get that line $c(D)$ is w^*-closed. If $c(D)$ is properly contained in $(\text{line } c(D))_1$ then there is an extreme point p of K in $(\text{line } c(D))_1$ such that $p \notin c(D)$. By hypothesis line $\{p\}$ is an L-ideal. Let N be the L-ideal complementary to line $\{p\}$. Since $D \subseteq E$ and $p \notin D$, D is T-invariant, we get that $D \subseteq N$. Let $q \in c(D)$ and let μ be a probability on D with $\gamma(\mu) = q$. If P denotes the L-projection associated with line $\{p\}$ then since line $\{p\}$ is w^*-closed and $f \longrightarrow P(f)(x)$ is w^*-Borel on K (for any $x \in X$), satisfying the bary centric calclulus we get (see [2] page 113)

$$P(q)(x) = \int_D P(z)(x)d\mu(z) = 0 \quad (\text{since } D \subseteq N) \; \forall x .$$

Therefore $P(q) = 0$ i.e. $q \in N$. So $c(D) \subseteq N$ and hence line $c(D) \subseteq N$. A contradiction.

Therefore $(\text{line } c(D))_1 = c(D)$.

If we can show that whenever $\| p \| = 1$ and $P(p) = 0$ or p for all L-projections P on X^* then $p \in E$, we will have verified all the conditions of Lima's characterization of L^1-preduals quoted in the introduction and it will follow that X is an L^1-predual. Fix such a p and let μ be a maximal probability measure with $\gamma(\mu) = p$. We will show that μ is supported by a single point and thus $p \in E$.

Let D be a w^*-compact subset of E and let $N = \text{line } c(TD)$ and let P_D denote the L-projection associated with N. The condition on p implies either $p \in N$ or $p \in N'$ (the L-ideal complementary to N). Write $\mu_1 = \mu/TD$, $\mu_2 = \mu/E - TD$ (here we make use of the fact that E is w^*-Borel). If $\mu_j \neq 0$, let $q_j = \gamma(\mu_j / \| \mu_j \|)$ and let $r_j = \| \mu_j \| q_j$; otherwise put $r_j = 0$. Then we have $p = r_1 + r_2$ and since $\|r_j\| \leq 1$, $1 = \|p\| = \|r_1\| + \|r_2\|$. Since $r_1 \in c(TD)$, we get that $r_1 \in N$.

If $p \in N'$ then $p = (I - P_D)(p) = (I - P_D)(r_2)$ and since $\| (I - P_D)(r_2) \| \leq \| r_2 \|$ we must have $r_1 = 0$. Therefore $\mu(TD) = 0$.

On the otherhand if $p \in N$ then $p = P_D(p) = r_1 + P_D(r_2)$ implies $r_2 \in N$. If $r_2 \neq 0$ then since μ is supported by E, we must have a w^*-compact set $D' \subset E - TD$ such that $\mu(D') \neq 0$. Let $s = \gamma(\frac{\mu/D'}{\mu(D')})$ and $u = \mu(D').s$. We can write $p = r_1 + u + v$ for some $v \in K$ where $1 = \|r_1\| + \|u\| + \|v\|$. This implies that $u \in N$. Since $N_1 = c(TD)$, there exists a maximal probability measure λ on K supported by TD such that $\gamma(\lambda) = u$. Then if $\mu' = \mu/D'$, the measure $\lambda - \mu'$ is a boundary measure annihilating $A_o(K) = \{a \in A(K) : a(o) = 0\}$. Since line $c(TD')$ is a w^*-closed L-ideal by Theorem-4.5 of [2], we get that $\lambda - \mu'/TD' \in A_o(K)^o$. Since λ is supported by TD we get that μ' and hence λ are in $A_o(K)^o$ and so $u = \gamma(\lambda) = 0$. This contradiction shows that $r_2 = 0$ and hence $\mu(E - TD) = 0$.

Suppose now that $\operatorname{Supp} \mu \cap E$ contains two linearly independent vectors z_1, z_2. We can find disjoint open neighbourhoods of $T.z_1$ and $T.z_2$ and hence we can find a compact subset D of E with $\mu(TD) > 0$ and $\mu(E - TD) > 0$. The previous reasoning shows that these inequalities holding simultaneously is impossible. Therefore μ is supported by $T.z$ for some $z \in E$. Since $p = \gamma(\mu)$ and $\|\mu\| = 1$, it is evident that μ is supported by a singleton as required.

<u>Remark</u> 1) For a 'standard' compact convex set K, Rogalski has proved in [40] that if for all $D \subseteq E(K)$, D w^*-compact, $\overline{co}(D)$ is a split face then K is a simplex. Since K is a simplex iff $A_{\mathbb{R}}(K)$ is a Lindenstrauss space (see [45]), using the correspondence between w^*-closed L-ideals in $A_{\mathbb{R}}(K)^*$ and closed split faces of K, one can see that what we have proved above is an extension of Rogalski's result to those complex Banach spaces X for which X_1^* is standard.

2) When X is a real Banach space the second condition in Theorem 2.1 reads : line c(D) is an L-ideal for all w^*-compact $D \subseteq E$. For in this case it is easy to see that line c(D) = line $c(D \cup -D)$.

In [15], A.J. Ellis and A.K. Roy gave an example to show that Rogalski's result does not extend to the general non-standard case. We now adapt that example to show that the hypothesis 'standard' cannot be dropped from Theorem 2.1.

<u>Example 2.2</u>. Let $Y = \bigcup \{Y_\alpha : \alpha \in [0,1]\}$, where the disjoint sets Y_α each consist of three points $\{r_\alpha, s_\alpha, t_\alpha\}$. Topologise Y so that each r_α, t_α is isolated and such that each s_α has a neighbourhood base consisting of the sets $\{s_\alpha\} \bigcup \{Y_\beta : 0 < |\alpha - \beta| < \varepsilon\}$, $\varepsilon > 0$. Y is a compact Hausdorff space.

Let λ denote the Lebesgue measure on $[0,1]$ and let $\lambda_1 = \lambda / [0, \frac{1}{2}]$, $\lambda_2 = \lambda / [\frac{1}{2}, 1]$.

$$\text{Let}\quad A = \left\{ f \in C(Y) : f(s_\alpha) = \frac{f(r_\alpha) + f(t_\alpha)}{2} \text{ and } \int f(s_\alpha)\, d\lambda_1(\alpha) = \int f(s_\alpha)\, d\lambda_2(\alpha) \right\}$$

Let K be the state space of A. From Theorem 1 of [15], we have that K is not a simplex and not standard. Since A is self-adjoint using results of Hirsberg and Lazar [21] we get that A is not a Lindenstrauss space.

The map $f \longrightarrow f|f(1)$ is w^*-continuous from $E(A_1^*)$ onto $E(K)$. If $D \subset E(K)$ is compact then since D is finite, say $D = \{x_1,\ldots,x_n\}$, we have that $F = co\{x_1,\ldots,x_n\}$ is a split face. Using the self-adjointness of A once again, we deduce that (see Corollary 2.7 [22]) $co(F \cup -iF)$ is split in $co(K \cup -iK)$ and consequently line F is a w^*-closed L-ideal.

<u>Proposition 2.3</u>. Let X be a complex Banach space and an L^1-predual. If $x \in S$ then $\{f \in E : |f(x)| = 1\}$ is closed in the structure topology.

<u>Proof</u> : Let $F = \{f \in K : f(x) = 1\}$, $x \in S$. Since F is a w^*-closed face by Corollary 1.5, $N = \text{line } F$ is a w^*-closed L-ideal and $(N)_1 = c(TF)$. Hence $E(N_1) = E(c(TF)) = \{f \in E : |f(x)| = 1\}$. Therefore $\{f \in E : |f(x)| = 1\}$ is a structurally closed set.

The above proposition was proved when X is a real Banach space by Effros in [12]. However the proof given there contains an error (specifically, the set H considered in

Proposition 4.9 of [12] is not convex). Interms of the structure topology on E, Theorem 2.1 can be phrased as 'if X is a complex Banach space with X_1^*, standard and any w^*-compact T-invariant subset of E is structurally closed, then X is an L^1-predual'. We now give a characterization of real Lindenstrauss spaces interms of the structure topology.

<u>Theorem 2.4</u>. A real Banach space X is an L^1-predual iff $\forall\ x \in S$, $\{f \in E : |f(x)| = 1\}$ is a structurally closed set.

<u>Proof</u> : Fix $x_o \in S$ and let $F = \{f \in K : f(x_o) = 1\}$. Since $\{f \in E : |f(x_o)| = 1\} = E(N_1)$ for some w^*-closed L-ideal N, we get that $N = \text{line } F$ and $N_1 = CO(F \cup -F)$. We claim that N is an L-space. Let $J = \{x \in X : f(x) = 0\ \forall\ f \in F\}$.

Define $\Phi : X/J \longrightarrow A_{\mathbb{R}}(F)$ by $\Phi(x + J)(f) = f(x)\ \forall\ x \in X,\ f \in F$

Φ is a well defined linear map. Since $(X/J)^* = \text{line } F$ and $N_1 = CO(F \cup -F)$ we get that

$$\| x + J \| = \sup_{f \in N_1} |f(x)| = \sup_{f\ \in\ F} |f(x)| = \| \Phi(x + J) \| .$$

Since $\Phi(x_o + J) = 1$ and $\Phi(X/J)$ separates points of F, by a well known argument in convexity theory we get that $\Phi(X/J) = A_{\mathbb{R}}(F)$.

Let $a \in A_{\mathbb{R}}(F)$, $\| a \| = 1$ and let $G = \{f \in F : a(f) = 1\}$. Let $x_1 \in X$, $\| x_1 + J \| = 1$ and $\Phi(x_1 + J) = a$. Since J is an M-ideal there exists $x_2 \in J$ such that $\|x_1 + x_2\| = \|x_1 + J\| = 1$.

Put $x' = \frac{x_0 + x_1 + x_2}{2}$

For any $f \in G$ since $f(x_0) = 1$ and $f(x_2) = 0$,

$$f(x') = \tfrac{1}{2}(1 + f(x_1)) = \tfrac{1}{2}(1 + \Phi(x_1 + J)(f)) = \tfrac{1}{2}(1 + a(f)) = 1$$

If $f \in K$, $f \notin G$ then $f(x_1) \neq 1$ if $f \in F$ and $f(x_0) \neq 1$ if $f \notin F$ so that $f(x') \neq 1$. Hence $G = \{f \in K : f(x') = 1\}$.

By hypothesis line G is an L-ideal in X^* and hence in line F (see [2], Part II, Proposition 1.4). It is easy now to conclude that G is a split face of F.

Since any peak face of F is split it follows from [14] that F is a simplex and consequently line F is an L-space.

To conclude that X is an L^1-predual we will now verify the conditions in Lima's characterization.

Let $f_0 \in E$, $0 < \varepsilon < 1$.

Using the Bishop-Phelps theorem [5], get $g_0 \in X^*$ and $y \in S$ such that $g_0(y) = \| g_0 \|$ and $\| f_0 - g_0 \| < \varepsilon$. By what we have seen above if we let $G = \{f \in K : f(y) = 1\}$ then line G is an L-ideal and a dual L-space and $g_0 \in$ line G. If P_0 denotes the L-projection corresponding to line G then since $\| (I - P_0)(f_0 - g_0) \| < \varepsilon$ and $(I - P_0)(f_0) = 0$ or f_0, we get that $P_0(f_0) = f_0$ i.e. $f_0 \in$ line G. Since f_0 is an extreme point of $(\text{line } G)_1$ and as line G is a dual L-space by Lima's characterization we get that line $\{f_0\}$ is an L-ideal in line G and hence line $\{f_0\}$ is an L-ideal in X^*.

If $\|f_o\| = 1$ and $P(f_o) = 0$ or f_o for all L-projections P on X^*, get g_o and G as before and observe that $f_o \in$ line G. Since L-ideals in line G are precisely those L-ideal of X^* which are contained in line G (see Prop. 1.4 [2] Part II) we get that $P(f_o) = 0$ or f_o for all L-projections P on line G. Since line G is an L-space by Lima's result again, we get that f_o is an extreme point of $(\text{line } G)_1$ and hence $f_o \in E$.

Therefore X is an L^1-predual.

Remark: It is clear from the above proof that the condition in the theorem is equivalent to saying line F_x is an L-ideal and $(\text{line } F_x)_1 = \text{co}(F_x \bigcup -F_x)$ for all $x \in S$, where $F_x = \{f \in K : f(x) = 1\}$. For a general Banach space X and a w^*-closed face F of K, $(\text{line } F)_1$ may not be equal to $\text{co}(F \bigcup -F)$ even when line F is an L-ideal (see the example below) though such a thing is always true when $X = A_{\mathbb{R}}(H)$ for some compact convex set H (and hence our result is analogous to peak face characterization of simplexes). We do not know whether X will be an L^1-predual or not if one merely stipulates that line F_x is an L-ideal for all $x \in S$.

Example 2.5. Let $X = \mathbb{R}^2$ with the norm

$$\|(x,y)\| = \max\{|x|, |y|, |x-y|\}.$$

Since $X_1 = \text{co}\{\pm(0,1), (1,1), (1,0)\}$ it is easy to see that the dual norm is given by

$$\| (x,y) \|^* = \max \{|x|, |y|, |x+y|\}.$$

Take $(1,1) \in X_1$ and let $F = \{(x,y) : \| (x,y) \|^* \leq 1, \ x+y = 1\}$

$$= \{(x,y) : x+y = 1, \ x,y \geq 0\}$$

line $F = \mathbb{R}^2$ but $(1,-1) \notin co(F \cup -F)$.

We now give an example to show that Theorem 2.4 is not true when X is a complex Banach space.

Let A denote the disc algebra on T i.e.

$$A = \left\{ f \in C(T) : \begin{array}{l} f \text{ has a continuous extension to the closed unit} \\ \text{disc which is analytic in the interior.} \end{array} \right\}$$

It is well known that A is not a Lindenstrauss space (see [21]). Let $a_o \in A$, $\| a_o \| = 1$, $F = \{f \in A_1^* : f(a_o) = 1\}$ without loss of generality we may assume that $F \cap S_A \neq \emptyset$ where S_A is the state space of A. Put $D = \{x \in T : a_o(x) = 1\}$. Since $b = \frac{1+a_o}{2}$ peaks on D, it follows from Theorem 2.8 of Hirsberg [22] that line $c(e(D))$ is a w^*-closed L-ideal ($e : T \to S_A$ is the evaluation map) and $(\text{line } c(e(D)))_1 = c(Te(D))$.

For any $x \in D$, $e(x)(a_o) = a_o(x) = 1$ implies line $c(e(D))$ is contained in line F. If $f \in E(F)$ and $f = te(x)$, $x, t \in T$ then $1 = ta_o(x) = tb(x)$ implies $a_o(x) = 1 = t$ so that $E(F) \subseteq e(D)$. Hence line $c(e(D))$ = line F and $(\text{line } F)_1 = c(TF)$. As remarked earlier this means that the set $\{f \in E(A_1^*) : |f(a_o)| = 1\}$ is structurally closed.

SECTION 3

Facial characterizations

Let X be a complex Banach space. In this section, we show that if line F is an L-ideal for all w^*-closed faces F of X_1^* then X is an L^1-predual. We first consider the situation for uniformly closed subspaces of $C(Y)$, containing constants, where Y is a compact Hausdorff space.

Suppose $A \subseteq C(Y)$ is a closed subspace, containing constants and separating points. Let $S_A = \{f \in A_1^* : f(1) = 1\}$ (with w^*-topology) denote the state space of A. For an arbitrary compact convex set H say that a closed face F of H is a peak face of H if $F = a^{-1}(0)$ for some non-negative function a in $A_{\mathbb{R}}(H)$. Denote by K the unit ball of A^*.

<u>Theorem 3.1</u>. Let A and S_A be as above. Then A is a complex Lindenstrauss space if $J = \text{line } F$ is an L-ideal in A^* and $J_1 = c(TF)$ when ever F is a peak face of S_A.

<u>Proof</u> : To prove that A is an L^1-predual it will be sufficient, using the results of Hirsberg and Lazar ([21], Theorem 7) to show that $Z = CO(S_A \cup -iS_A)$ is a simplex. By Briem's characterization of simplexes [7], it will be enough to verify that each peak face of Z is parallel.

Let $G = co(F_1 \bigcup -iF_2)$ be a peak face of Z, where $F_1 = G \bigcap S_A$, $F_2 = iG \bigcap S_A$ are peak faces of S_A. We first show that $(\text{line } F_1) \bigcap S_A = F_1$ and $(\text{line } F_2) \bigcap (-iS_A) = -iF_2$. Put $J = \text{line } F_1$.

Since J is w^*-closed ([9],v.5.9), $G' = J \bigcap S_A$ is w^*-closed and clearly $G' \supseteq F_1$. If $\lambda p_1 + (1-\lambda)p_2 \in G'$, $0 < \lambda < 1$, $p_j \in S_A$ then since $\lambda p_1, (1-\lambda)p_2 \measuredangle \lambda p_1 + (1-\lambda)p_2$ and J is an L-ideal we get that $p_1, p_2 \in G'$. Therefore G' is a face of S_A and hence a face of $J_1 = c(TF_1)$. Now if $f \in E(G')$ then $f \in E(S_A)$ and $f \in E(c(TF_1))$. By Milman's theorem we get that $f \in TF_1$. Therefore $f \in F_1$, since $tF_1 \bigcap S_A = \emptyset$ for $t \in T$, $t \neq 1$. Hence $E(G') \subseteq F_1$ and since G' and F_1 are w^*-closed we get that $G' = F_1$. A similar argument shows $(\text{line } F_2) \bigcap -iS_A = -iF_2$.

Since J is a w^*-closed L-ideal and $J \bigcap S_A = F_1$ a result of Hirsberg [22] shows that $J \bigcap Z = co(F_1 \bigcup -iF_1)$ is a split face of Z. If μ is a real boundary measure in $A_{\mathbb{R}}(Z)^o$ then $\mu / co(F_1 \bigcup -iF_1)$ belongs to $A_{\mathbb{R}}((co(F_1 \bigcup -iF_1))^o$. But since F_1 is a parallel face of $co(F_1 \bigcup -iF_1)$ we get that $\mu(F_1) = 0$ (see Hirsberg [23]). Similarly we can show that $\mu(-iF_2) = 0$ and hence $\mu(G) = 0$. Therefore G is a parallel face of Z, so that Z is a simplex.

<u>Remark</u> : In [6]. Briem has proved that if every peak set for Re A (real parts of functions in A) is a split set for A then A is self-adjoint and S_A is a simplex. Imitating the arguments

given in the above proof one obtains a different proof of that theorem.

We now consider the general case.

Let $A \subseteq C(Y)$ be a closed subspace separating points of Y and let $e : A \longrightarrow K = A_1^*$, be the evaluation map.

<u>Definition</u> : A closed subset D of Y of the form $e^{-1}(TF)$ where F is a closed face of K (w.r.t. w^*-topology) is called an <u>M-set</u> for A if $\mu \in A^o$ is a boundary measure on Y then $\mu/D \in A^o$.

Note that if $1 \in A$ and if F is a closed face of S_A then the set D is an M-set for A in the usual sense (see Hirsberg [22]).

<u>Theorem 3.3</u>. For A and K as above the following are equivalent 1) A is a complex Lindenstrauss space

2) Whenever F is a w^*-closed face of K, the set $D = e^{-1}(TF)$ is an M-set for A

3) line F is an L-ideal for all w^*-closed face F of K.

<u>Proof</u> : $1 \Longrightarrow 2$: Suppose $D = e^{-1}(TF)$ for a w^*-closed face F of K. Let μ be a boundary measure on Y, $\mu \in A^o$ and $\|\mu\| = 1$. We must show that $\mu/D \in A^o$. The measure $\mu' = \mu o e^{-1}$ is a boundary measure on K with resultant zero and consequently (see Effros [13], Lemma 4.2) $R\mu'$ is a maximal probability

measure on K with $\gamma(R\mu') = 0$. Since A is an L^1-predual hom $(R\mu') = \text{hom}(\delta(0)) = 0$ and therefore hom $R\mu'/T.e(D) = 0$. Since $Te(D)$ is T-invariant it follows that hom $(R\mu'/T.e(D)) = 0$ and hence $\gamma(R\mu'/T.e(D)) = 0$ so that $\int_{Te(D)} f\, d R\mu' = 0 \; \forall f \in A$.

Choose functions $\{g_\alpha\}$ in $C(K)$, real valued, $0 \leq g_\alpha \leq 1$ and g_α's pointwise decrease to $I_{Te(D)}$ (indicator of $T.e(D)$). Then for $f \in A$

$$0 = \int_{Te(D)} f\, d R\mu' = \lim_\alpha \int_{Te(D)} g_\alpha f dR\mu'$$

$$= \lim_\alpha \int_Y f(h(y)e(y)) g_\alpha (h(y)e(y)) d|\mu|(y) \longrightarrow (*)$$

where $d\mu = hd|\mu|$ is the polar decomposition of μ. Now if $y \in D$ then $h(y)e(y) \in Te(D)$ so that $g_\alpha(h(y)e(y)) \longrightarrow 1$. If $y \notin D$ then $h(y)e(y) \notin Te(D)$ and hence $g_\alpha(h(y)e(y)) \longrightarrow 0$. Therefore (*) gives

$$0 = \int_D f(h(y)e(y)) d|\mu|(y) = \int_D f d\mu \;\; \forall f \in A$$

Consequently D is an M-set for A.

$2 \Longrightarrow 3$: Let F be any closed face of K. Put $D = e^{-1}(T.F)$. We define a linear mapping $P : A^* \longrightarrow A^*$ by $P(p)(f) = \int_D f d\mu$ for $f \in A$, $p \in A^*$ where μ is a boundary measure on Y representing p. The fact that D is an M-set implies that P is well defined.

We claim that line F = Range P. If $p \in F$ and λ a maximal measure supported by F with $\gamma(\lambda) = p$ then $\lambda o e$ is a boundary measure representing p and supported by D. Consequently $P(p)(f) = \int f d(\lambda o e) = f(p) \; \forall f \in A$ so that $P(p) = p$. This implies line F $\subseteq$ Range P.

Let $0 \neq p \in$ Range P, $\| p \| = 1$. By Hustad's theorem (see Phelps [37] Theorem 2.2) we can find a boundary measure on Y with $\| \mu \| = 1$, representing p. Thefore $p(f) = P(p)(f) = \int_D f d\mu \; \forall f$ implies that $\gamma(R(\mu/D)) = p$ so that $p \in c(TF)$. Hence line F $\subseteq$ Range P $\subseteq$ line c(TF) $\subseteq$ line F (see proof of Corollary 1.5). Therefore line F = Range P.

To complete the proof we need to show that P is an L-projection. For $p \in A$, choose boundary measure μ on Y, representing p with $\| p \| = \| \mu \|$.

For $f \in A$, $p(f) = \int_Y f d\mu = \int_D f d\mu + \int_{Y-D} f d\mu$

$$= P(p)(f) + (p - P(p))(f) = \int_D f d\mu + (p - P(p))(f)$$

so that $\|P(p)\| + \|p - P(p)\| \leq \|\mu/D\| + \|\mu/Y\text{-}D\| = \|\mu\| = \|p\|$.

Hence P is an L-projection.

3 $\Longrightarrow$ 1 : We first show that for any w^*-closed face F of K, $(\text{line } F)_1 = c(TF)$. If not there is a p in line $F \cap E(K)$ such that $p \notin c(TF)$, because line F is an L-ideal and is w^*-closed by v.5.9[9]. By hypothesis line $\{p\}$ is an L-ideal so that

$A^* = \text{line}\{p\} \oplus N$ where N is the L-ideal complementary to line $\{p\}$. Since F is a face and $p \notin TF$ we get that $F \subseteq N$ so that line $F \subseteq N$. This contradiction shows that $(\text{line } F)_1 = c(TF)$.

If $x_0 \in A$, $\|x_0\| = 1$ and $F = \{f \in K : \text{Re}(f(x_0)) = 1\}$ then since F is a w^*-closed face if we let $J = \{x \in A : f(x) = 0 \ \forall f \in F\}$ as before (see the proof of Theorem 2.4), the natural map $\Phi : A/J \longrightarrow A(F)$ is an isometry since $(\text{line } F)_1 = c(TF)$. So $\Phi(A/J)$ is a closed subspace of $A(F)$ separating points of F and containing the constant function 1. Easy to see that the state space of $\Phi(A/J)$ is F. If $G \subseteq F$ is a w^*-closed face of F then line G is a w^*-closed L-ideal in A^* and hence in line F and moreover $(\text{line } G)_1 = c(TG)$. Therefore by Theorem 3.1, we get that $\Phi(A/J)$ is an L^1-predual i.e. line F is an L-space.

Since by hypothesis line $\{p\}$ is an L-ideal for all $p \in E(K)$, to conclude that A is a Lindenstrauss space we need only to verify the second condition in Lima's characterization. But this can be done by an application of the Bishop-Phelps theorem and proceeding exactly as in the last part of the proof of Theorem 2.4.

In the following proposition we again consider L-ideals generated by a subclass of all w^*-closed faces of the dual unit ball. Part of the assertion is a partial complex analogue of Theorem 2.4.

Proposition 3.4. Let A and K be as in Theorem 3.3. The following are equivalent

1) For all $a_o \in A$, $\| a_o \| = 1$, $\{f \in E(K) : |f(a_o)| = 1\}$ is a structurally closed set and if $F = \{f \in K : f(a_o) = 1\}$ then F is split in $co(F \cup -iF)$.

2) For all $a_o \in A$, $\| a_o \| = 1$, $D = \{y \in Y : |a_o(y)| = 1\}$ is an M-set and if $B = \bar{a}_o . A$ ($\bar{a}_o$ stands for the complex conjugate of a_o) then B/D is a closed self-adjoint subspace of $C(D)$.

3) A is an L^1-predual.

Proof : We shall prove that $3 \Longrightarrow 2 \Longrightarrow 1 \Longrightarrow 3$.

$3 \Longrightarrow 2$: It follows from the above theorem that D is an M-set. Since A is a Lindenstrauss space if we let $F = \{f \in K : f(a_o) = 1\}$, then it follows from the results in Section 4 of [36] that the natural map $\Phi : A/J \longrightarrow A(F)$ is onto (we are making free use of notations from the proof of Theorem 3.3).

Let $a \in A$ then there is a b in A such that $\Phi \overline{(a + J)} = \Phi (b + J)$. If $y \in D$, and $e(y) = tf$ for $f \in F$, $t \in T$ then $a_o(y) = t$. Also

$$\overline{a_o(y)} . b(y) = f(b) = \Phi(b + J)(f) = \Phi \overline{(a + J)}(f) = \overline{f(a)} = a_o(y) . \overline{a(y)} .$$

Therefore $\bar{a}_o . b$ is the conjugate of $\bar{a}_o . a$ on D. Hence B/D is self adjoint.

It is trivial to verify that A/D is isometrically isomorphic to A/J and hence B/D is closed.

2 $\Longrightarrow$ 1 : If we define $P : A^* \longrightarrow A^*$ as in the proof of Theorem 3.3, we get that line F is a w^*-closed L-ideal. We claim that $(\text{line } F)_1 = c(TF)$.

Enough to show that for $0 \neq q \in c(TF)$, $q/\|q\| \in c(TF)$ (see the proof of Lemma 1.2). Let $p = q/\|q\|$ and choose a boundary measure μ on Y with $\|\mu\| = 1$, representing p. Since $P(p) = p$, for any $f \in A$

$$|p(f)| = |\int_D f d\mu| \leq \|f\| \, |\mu|(D)$$

$$\Longrightarrow 1 = \|p\| \leq |\mu|(D) \leq 1$$

So μ is supported by D consequently $\mu' = R(\mu o e^{-1})$ is supported by T.F and represents p. Therefore $p \in c(TF)$.

Hence $\{f \in E(K) : |f(a_0)| = 1\} = (\text{line } F) \cap E(K)$.

Using self-adjointness of B/D, it is easy to see that $\Phi : A/J \longrightarrow A(F)$ has self-adjoint range and hence is onto. Therefore F is split in $CO(F \cup -iF)$ (see [26], page 246, Lemma 12) as F is the state space of $\Phi(A/J)$.

1 $\Longrightarrow$ 3 : A careful observation of the proof of Theorem 2.4 shows that once we establish that the map $\Phi : A/J \longrightarrow A(F)$ is onto for the face F then the arguments there work verbatim in the complex case giving the required conclusion. But that Φ is

onto follows again from Lemma 12, page 246 of [26], since F is split in $co(F \cup -iF)$.

<u>Remark</u> : In the proofs of several of the preceding theorems, a crucial fact was that line F is an L-space for a w^*-closed face F. It may seem that if one assumes the condition line F is an L-space for all w^*-closed faces F of the dual unit ball of X then X would be an L^1-predual. However $X = A_{\mathbb{R}}(K)$ where K is the unit square in $\mathbb{R}^2$ provides an easy counter example.

SECTION 4

An application

Let X be a complex Banach space. A family $\{B(a_i, r_i)\}_{i=1}^n$ of closed balls in X is said to have weak intersection property (w.i.p), if $\bigcap_{i=1}^{n} B(f(a_i), r_i) \neq \emptyset$ for all $f \in X_1^*$.

X is said to be an $E(4)$ space if for every family $\{B(a_i, r_i)\}_{i=1}^4$ of four closed balls with w.i.p, $\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$.

These definitions are due to Hustad [25] ($E(4)$ spaces were defined in a different way and the above definition is a theorem in that paper). Lima has proved that [29, Theorem 4.1] any $E(4)$ - space is an L^1-predual. As an application of Proposition 3.4, we give a different proof of this theorem. Our proof uses an idea from a short proof of that theorem given by A.K. Roy [41].

We need a characterization of $E(4)$-spaces given by Lima [29]. We state it in a different way since we do not use the notations of that paper.

<u>Theorem 1</u> (A. Lima) : Let X be an $E(4)$-space. For any $x_1, x_2, x_3, x_4 \in X^*$ with $\sum_{k=1}^{4} x_k = 0$ there exist y_{ij} in X^* such that

1) $(x_1,x_2,x_3,x_4) = (0,y_{12},y_{13},y_{14}) + (y_{21},0,y_{23},y_{24})$

$$+ (y_{31},y_{32},0,y_{34}) + (y_{41},y_{42},y_{43},0)$$

and $y_{12} + y_{13} + y_{14} = 0 = y_{21} + y_{23} + y_{24}$

$$y_{31} + y_{32} + y_{34} = 0 = y_{41} + y_{42} + y_{43}$$

2) $\|x_1\| = \|y_{21}\| + \|y_{31}\| + \|y_{41}\|$; $\|x_2\| = \|y_{12}\| + \|y_{32}\| + \|y_{42}\|$

$\|x_3\| = \|y_{13}\| + \|y_{23}\| + \|y_{43}\|$; $\|x_4\| = \|y_{14}\| + \|y_{24}\| + \|y_{34}\|$.

<u>Theorem 2</u> : Let $A \subseteq C(Y)$ be a closed subspace separating points of Y. If A is an $E(4)$-space then A is an L^1-predual

<u>Proof</u> : Let $a_o \in A$, $\|a_o\| = 1$, $F = \{f \in A_1^* : f(a_o) = 1\}$. Since A is an $E(4)$-space it is not difficult to see from Lemma 1 of [30] that line $\{p\}$ is an L-ideal for all $p \in E(A_1^*)$.

It is clear from the results in Section 3 that once we verify that F is split in $CO(F \cup -iF)$ and that line F is an L-ideal we will have verified all the conditions in (1) of Proposition 3.4 and the conclusion will follow .

Since F is a peak face it is easy to see that F is a parallel face of $CO(F \cup -iF)$. An argument identical to the one given by A.K. Roy in [41], shows that F is split in $CO(F \cup -iF)$. Rest of the proof proceeds in the following steps.

<u>Step 1</u> : line F is a hereditary (w.r.t the ordering at the beginning of Section 1) subspace.

Let $p \in \text{line}_{\mathbb{R}}(F)$ (the real linear span of F) and $q \measuredangle p$. Let $p = \lambda_1 p_1 - \lambda_2 p_2$, $p_j \in F$, $\lambda_j > 0$.

By Theorem 1, there exists $\{z_{ij}\}$ such that

$$(\lambda_1 p_1, -\lambda_2 p_2, -q, q-p) = (0, z_{12}, z_{13}, z_{14}) + (z_{21}, 0, z_{23}, z_{24}) + (z_{31}, z_{32}, 0, z_{34}) + (z_{41}, z_{42}, z_{43}, 0)$$

and $\lambda_1 = \|z_{21}\| + \|z_{31}\| + \|z_{41}\|$; $\lambda_2 = \|z_{12}\| + \|z_{32}\| + \|z_{42}\|$

$$\|q\| = \|z_{13}\| + \|z_{23}\| + \|z_{43}\|; \ \|q-p\| = \|z_{14}\| + \|z_{24}\| + \|z_{34}\|.$$

Since facial cones are hereditary we get that z_{21}, z_{31}, z_{41}; $-z_{12}, -z_{32}, -z_{42}$ are in cone (F).

Since $\|p\| = \|q\| + \|q-p\|$ we have

$$\|z_{13} + z_{14} + z_{23} + z_{24} + z_{43} + z_{34}\| = \|z_{13}\| + \|z_{14}\| + \|z_{23}\| + \|z_{24}\| + \|z_{43}\| + \|z_{34}\|.$$

So that

$$\|z_{13} + z_{14}\| = \|z_{13}\| + \|z_{14}\|; \ \|z_{23} + z_{24}\| = \|z_{23}\| + \|z_{24}\|.$$

But $z_{13} + z_{14} = -z_{12} \in \text{cone}(F) \Longrightarrow z_{13}, z_{14} \in \text{cone}(F)$.

Similarly $z_{23} + z_{24} = -z_{21}$ implies $-z_{23}, -z_{24} \in \text{cone}(F)$.

Now $q = -(z_{13} + z_{23} + z_{43}) = z_{12} + z_{14} + z_{21} + z_{24} + z_{41} + z_{42}$.

Therefore $q \in \text{line}_{\mathbb{R}} F$.

Now using the fact that $\text{line}_{\mathbb{R}} F$ is a hereditary subspace and proceeding exactly as in the above argument one sees that line F is hereditary.

Step 2 : line F is w^*-closed.

Let $p \in \text{line}_{\mathbb{R}} F$, $\| p \| = 1$, $p = \lambda_1 p_1 - \lambda_2 p_2$ where $\lambda_k \geq 0$ and $p_k \in F$. Since F is a face and A is an E(4)-space it follows from Corollary 8 of [30] that we can write

$$\lambda_1 p_1 = z + u ; \lambda_1 = \| z \| + \| u \|$$

$$\lambda_2 p_2 = z + v ; \lambda_2 = \| z \| + \| v \|$$

such that $u, v \in \text{Cone}\ (F)$ and $1 = \| u - v \| = \| u \| + \| v \|$. Therefore $(\text{line}_{\mathbb{R}} F)_1 = CO(F \bigcup -F)$ and hence by the Krein-Smulian theorem, $\text{line}_{\mathbb{R}} F$ is w^*-closed.

Since F is split in $CO(F \bigcup -iF)$ it is easy to see that $\text{line}_{\mathbb{R}} F \bigcap i\ \text{line}_{\mathbb{R}} F = \{0\}$. We claim that

$$\inf \left\{ \| p + iq \| : \begin{array}{l} p, q \in \text{line}_{\mathbb{R}} F \\ \| p \| = 1 = \| q \| \end{array} \right\} \geq 1 .$$

Then it is not difficult to deduce (see [24] Section 15) that $\text{line}\ F = \text{line}_{\mathbb{R}} F + i\ \text{line}_{\mathbb{R}} F$ is w^*-closed.

Let $p, q \in \text{line}_{\mathbb{R}} F$, $\| p \| = 1 = \| q \|$ and let $r = p + iq$.

Write $p = p_1 - p_2 \ ; \ 1 = \| p_1 \| + \| p_2 \|$,

$q = q_1 - q_2 \ ; \ 1 = \| q_1 \| + \| q_2 \|$,

$p_i , q_i \in \text{cone}\ (F)$. (This follows from what we have done in the first part of Step 2).

Again by Theorem 1, we can write

$$(r,\ -p_1, p_2,\ -i(q_1 - q_2)) = (0, z_{12}, z_{13}, z_{14}) + (z_{21}, 0, z_{23}, z_{24}) + (z_{31}, z_{32}, 0, z_{34}) + (z_{41}, z_{42}, z_{43}, 0)$$

and
$$\|r\| = \|z_{21}\| + \|z_{31}\| + \|z_{41}\|$$
$$\|p_1\| = \|z_{12}\| + \|z_{32}\| + \|z_{42}\| \qquad \longrightarrow (*)$$
$$\|p_2\| = \|z_{13}\| + \|z_{23}\| + \|z_{43}\|$$
$$1 = \|q_1 - q_2\| = \|z_{14}\| + \|z_{24}\| + \|z_{34}\| .$$

Since cone (F) and i $\text{line}_{\mathbb{R}} F$ are hereditary we get z_{13}, z_{23}, z_{43} ; $-z_{12}, -z_{32}, -z_{42} \in$ cone (F) and $z_{14}, z_{24}, z_{34} \in i$ $\text{line}_{\mathbb{R}} F$.

But $p_2 - p_1 = -z_{14} + (z_{32} + z_{23}) + (z_{42} + z_{43})$ and $z_{32} + z_{23} + z_{42} + z_{43}$, $p_2 - p_1 \in \text{line}_{\mathbb{R}} F$ where as $z_{14} \in i$ $\text{line}_{\mathbb{R}} F$. Therefore $z_{14} = 0 = z_{12} + z_{13}$.

Also $\|z_{12}\| + \|z_{13}\| + \|z_{32}\| + \|z_{23}\| + \|z_{42}\| + \|z_{43}\|$

$$= \|p_1\| + \|p_2\| = \|p_1 - p_2\| = \|z_{32} + z_{23} + z_{42} + z_{43}\|$$

$\Longrightarrow z_{12} = z_{13} = 0$. So that the equations in (*) will become

$$\|p_1\| = \|z_{32}\| + \|z_{42}\| ; \ \|p_2\| = \|z_{23}\| + \|z_{43}\| .$$

Since $z_{21} = -z_{23} - z_{24}$ and $z_{23} \in$ cone (F), $z_{24} \in i$ $\text{line}_{\mathbb{R}} F$ we have $\|z_{21}\| \geq |z_{21}(a_o)| = |z_{23}(a_o) + z_{24}(a_o)|$

$$\geq |z_{23}(a_o)| = \|z_{23}\| .$$

Similarly we get $\| z_{31} \| \geq \| z_{32} \|$.

Hence
$$\| r \| = \| z_{21} \| + \| z_{31} \| + \| z_{41} \|$$
$$\geq \| z_{23} \| + \| z_{32} \| + \| z_{42} + z_{43} \|$$
$$= \| p_1 \| + \| p_2 \| = 1$$

Therefore line F is w^*-closed.

Since line F is a w^*-closed, hereditary subspace and A is E(4) it is not difficult to see, using Lemma 1 of [30], that line F is an L-ideal.

Hence A is a Lindenstrauss space.

SECTION 5

G- spaces

For a complex Banach space X, let Z denote the w^*-closure of E.

A linear subspace $A \subseteq C(Y)$ where Y is a compact Hausdorff space is called a complex G-space if

$$A = \{ f \in C(Y) : f(x_\alpha) = \lambda_\alpha t_\alpha f(y_\alpha) \ \forall \ \alpha \in \Sigma \}$$

where Σ is an index set and $\lambda_\alpha \in [0,1]$, x_α, $y_\alpha \in Y$, $t_\alpha \in T \ \forall \ \alpha$.

Complex G-spaces were introduced by G.H. Olsen in [36] and are the complex analogues of the real G-spaces introduced by Grothendieck [19]. We now state a characterization of G-spaces due to Olsen [36] which we shall henceforth refer to as the Olsen's characterization.

A complex Banach space X is isometric to a G-space iff X is an L^1-predual and $Z \subseteq [0,1].E$.

We first give a characterization of complex G-spaces using structurally upper semi-continuous (u.s.c) functions on E. Recall that $\xrightarrow{s}$ denotes convergence in structure topology.

<u>Theorem 5.1</u>. A complex Banach space X is a G-space iff for all $x \in X$, the function $|x|$ defined on E by $|x|(f) = |f(x)|$ is u.s.c in the structure topology.

Proof : Suppose X is a G-space

Let $0 \neq x \in X$, $c > 0$ and $D = \{f \in Z : |f(x)| \geq c\}$. We claim that D is a dilated set.

Let $f \in D$, by Olsen's characterization $f/\|f\| \in E$ and since $\left|\frac{f(x)}{\|f\|}\right| \geq |f(x)| \geq c$, $f/\|f\| \in D$. Since $E(H_f) = E(H_{f/\|f\|}) = T.f/\|f\|$ and D is T-invariant we get that D is a dilated set.

Therefore by Corollary 1.4, $D \cap E = \{f \in E : |f(x)| \geq c\}$ is a structurally closed set.

Conversely suppose that $|x|$ is structurally u.s.c for all x.

Let $f_o \in Z - E$ and $f_o \neq 0$. Let $\{f_\alpha\}$ be a net in E and $f_\alpha \xrightarrow{w^*} f$. Fix $p_o \in E(N_1)$ where N is the smallest w^*-closed L-ideal containing f_o. We claim that line $\{f_o\}$ = line $\{p_o\}$.

Fix $x \in X$, $x \neq 0$ and let $c > 0$ be such that $|p_o(x)| < c$. By hypothesis $\{f \in E : |f(x)| < c\}$ is a structurally open set containing p_o. Since by Lemma 3.8 of [2], $f_\alpha \xrightarrow{S} p_o$, there is a β_o such that $\alpha \geq \beta_o$ implies $|f_\alpha(x)| < c$. Therefore $|f_o(x)| \leq c$. Hence $|f_o(x)| \leq |p_o(x)|$ and this is true for all $x \in X$, so that $\operatorname{Ker} p_o \subseteq \operatorname{Ker} f_o$ (Ker - stands for the Kernel). Hence line $\{f_o\}$ = line $\{p_o\}$.

Since line $\{f_o\}$ = line $\{p_o\}$ for all $p_o \in E(N_1)$ we get that line $\{f_o\} = N$ and consequently line $\{f_o\}$ is an L-ideal so that $f_o/\|f_o\| \in E$.

Therefore $Z \subseteq [0,1].E$.

If $S^* = \{f \in X_1^* : \|f\| = 1\}$, then since S^* is a G_δ in the w^*-topology we get that $E = Z \cap S^*$ is w^*-Borel. Also if μ is a maximal probability measure on X_1^* then $\mu(Z) = 1$ and $\mu(S^*) = 1$ (see [2]) so that $\mu(E) = 1$. Hence X_1^* is standard in the sense described in Theorem 2.1.

Now let $D \subseteq E$ be a w^*-compact T-invariant set. Let $\{f_\alpha\}$ be a net in D with $f_\alpha \xrightarrow{S} f$. By replacing $\{f_\alpha\}$ by a subnet if necessary we may assume that $f_\alpha \xrightarrow{w^*} g$, $g \in D$. Using the structural upper semi-continuity of $|x|$ as before we can see that $f = \lambda.g$. Since $1 = \|f\| = \|g\|$, $|\lambda| = 1$ and so $f \in D$ because D is T-invariant. Therefore D is a structurally closed set. It now follows from Theorem 2.2 and the remarks preceding Proposition 2.3 that X is an L^1-predual. The conclusion follows from Olsen's characterization.

<u>Corollary 5.2</u>. If X is a complex Banach space and the structure topology on E is such that for any $p_1, p_2 \in E$, p_1, p_2 linearly independent can be separated by disjoint structurally open sets, then X is a G-space.

<u>Proof</u> : Let $x \in X$, $x \neq 0$ and $c > 0$. Let $D = \{f \in E : |f(x)| \geq c\}$. If $\{f_\alpha\}$ is a net in D and $f_\alpha \xrightarrow{S} f$, $f \in E$, we may assume that $f_\alpha \xrightarrow{w^*} g$ for some $g \in Z$. Since $|f_\alpha(x)| \longrightarrow |g(x)|$, we get that $g \neq 0$. Now by Lemma 3.8 of [2] and the separation property assumed in the hypothesis it follows that $f = \lambda.g/\|g\|, \lambda \in T$.

Since $|g(x)| \geq c$, $f \in D$. Therefore D is a structurally closed set. So $|x|$ is structurally u.s.c for all x and hence X is a G-space.

Remark : For a real G-space X, that $|x|$ is structurally u.s.c for all x on E was observed by N. Roy [44]. Corollary 5.2 was proved for real Banach spaces by U. Uttersrud [47] by a different argument.

Corollary 5.3. A compact convex set K is a Bauer simplex iff $|a| : E(K) \longrightarrow \mathbb{R}$ is u.s.c in the facial topology for all $a \in A_{\mathbb{R}}(K)$.

Proposition 5.4. Let X be a complex Banach space. Consider the following statements.

1) X is a G-space
2) for all $x \in X$, $U_x = \{f \in E : f(x) \neq 0\}$ is open in the structure topology
3) $\forall$ $f \in E$, line $\{f\}$ is an L-ideal and the intersection of any family of M-ideals in X is an M-ideal in X.
4) for any $D \subseteq E$, $\overline{\text{line } D}$ is an L-ideal.

Then we have $1 \Longrightarrow 2 \Longleftrightarrow 3 \Longleftrightarrow 4$.

Proof : $1 \Longrightarrow 2$ In view of Corollary 1.4, it is enough to show that $V_x = \{f \in Z : f(x) = 0\}$ is dilated. Use Olsen's characterization.

$2 \Longrightarrow 3$ Let $f \varepsilon E$. If $T.\{f\}$ is not a structurally closed set then there is a $g \varepsilon E$ in the structural closure of $T.\{f\}$ and not in $T.\{f\}$. Let $x \varepsilon X$, $g(x) \neq 0$. Since U_x is a structural neighbourhood of g, $U_x \cap T.\{f\} \neq \emptyset$. Therefore $f(x) \neq 0$ and hence $f = \lambda g$, $\lambda \varepsilon T$. This contradiction shows that $T.\{f\}$ is closed in the structure topology i.e. line $\{f\}$ is an L-ideal.

If $\{M_\alpha\}$ is any family of M-ideals in X let $M = \bigcap M_\alpha$ and $U = \bigcap_{x \varepsilon M} U_x^c$. Then by hypothesis there is a w^*-closed L-ideal N such that $E(N_1) = U$. Since $U \subseteq M^o$, $N \subseteq M^o$. For any α let $f \varepsilon E((M_\alpha^o)_1)$ then $f \varepsilon E$ and $f(x) = 0 \;\forall\, x \varepsilon M$ so that $f \varepsilon U$. Therefore $M_\alpha^o \subseteq N$ for all α and hence $M^o = N$, consequently M is an L-ideal.

Rest of the proposition is easy to see.

<u>Corollary 5.5</u>. Let $A = \{f \varepsilon C(Y) : f(x_\alpha) = \lambda_\alpha t_\alpha f(y_\alpha) \;\forall\, \alpha \varepsilon \Sigma\}$, where $\lambda_\alpha, t_\alpha, x_\alpha, y_\alpha$ are as in the definition of G-space. For any $D \subseteq Y$, $M_D = \{f \varepsilon A : f(D) = 0\}$ is an M-ideal in A.

<u>Proof</u> : Let $e : Y \longrightarrow A_1^*$ be the evaluation map. Now using Theorem 24 of [36], we can see that for all $y \varepsilon Y$, $e(y) \varepsilon [0,1].E(A_1^*)$. Since A is a G-space, line $\{e(y)\}$ is an L-ideal for all y. Hence $M_D = \bigcap_{x \varepsilon D} M_x$ is an M-ideal.

<u>Remark</u> : Proposition 5.4, when X is a real Banach space was observed by N. Roy [44] and U. Uttersrud [47]. Using a characterization of M-ideals in terms of intersection properties

of balls due to Lima, Uttersrud proves Corollary 5.5 for real scalars. That for a real G-space, intersection of any family of M-ideals is an M-ideal was also proved by P.D. Taylor [46]. Both the proofs use the order structure on the real line and therefore do not work for the complex case.

F. Perdrizet and J. Bunce gave examples to show that for a general Banach space intersection of an arbitrary family of M-ideals can fail to be an M-ideal. Those examples were given in response to a question asked by Effros [10], whether or not the intersection of an arbitrary family of ideals (M-ideals in our sense) is an ideal in $A_{\mathbb{R}}(K)$ for a compact simplex K.

It is well known that when K is a Bauer simplex, the intersection of any family of M-ideals in $A_{\mathbb{R}}(K)$ is an M-ideal. In [20] A. Gleit partially answers the question of Effros by showing that if for a metrizable Choquet simplex K, intersection of a countable family of M-ideals is an M-ideal in $A_{\mathbb{R}}(K)$ then K is a Bauer simplex. As a consequence of our next result, we obtain a simple proof of this theorem. Our result (Proposition 5.9) is a substantial generalization of Gleit's result to $A(K)$ spaces.

In [47] Uttersrud formulates the general form of Effros' question by asking whether the property, intersection of M-ideals is an M-ideal and line $\{p\}$ is an L-ideal for all $p \in E(X_1^*)$, characterize G-spaces (i.e. whether $3 \Longrightarrow 1$ or not in Proposition 5.4). Using arguments some what similar to the ones used by

Gleit in the simplex case, N. Roy [43], showed that $3 \Longrightarrow 1$ for separable real Banach spaces which are L^1-preduals. Our next theorem also enables us to give a simple proof of this result in a some what general set up without using L^1-predual theory.

<u>Theorem 5.6</u>. Let X be a complex Banach space such that

1) $A = \{f \in E : \text{line}\{f\} \text{ is an L-ideal}\}$ is sequentially w^*-dense in Z

2) Intersection of any countable family of M-ideals in X is an M-ideal.

Then for any $f \in Z$, line $\{f\}$ is an L-ideal.

<u>Proof</u> : Let $f \in E - A$. Choose a sequence $\{f_n\} \subseteq A$ such that f_n's are linearly independent and $f_n \xrightarrow{w^*} f$ (this can be done since A is T-invariant, $f \neq 0$, for only finitely many n's, $f_n \in T.\{f_i\}_{i=1}^{\infty}$ and redefining the sequence by discarding those n's).

Put $N = \overline{\text{line}}\{f_n\}_{n=1}^{\infty}$ (closure in the norm topology). Since f_n's are independent and line $\{f_n\}$ is an L-ideal $\forall\, n$,

$$N = \left\{\sum_{i=1}^{\infty} \alpha_i f_i : \sum_{i=1}^{\infty} |\alpha_i| < \infty\right\} \text{ and } \left\|\sum_{i=1}^{\infty} \alpha_i f_i\right\| = \sum_{i=1}^{\infty} |\alpha_i| .$$

Let $F = c\{f_n\}_{n=1}^{\infty}$. Since $E(F) = \{f_n\}_{n=1}^{\infty} \bigcup \{f\}$ (by Milman's theorem) it is easy to see that line $F = N \oplus \text{line}\{f\}$. Since N

is norm closed, line F is norm closed and hence is w^*-closed (v.5.9 [9]). Also line $F = (\bigcap_n \text{Ker } f_n)^c$. Therefore by hypotheses line F is a w^*-closed L-ideal. Since f is an extreme point independet of $\{f_n\}_{n=1}^{\infty}$ (as $f \notin A$) we have $f \notin N$ and $\| \alpha f + \sum_{i=1}^{\infty} \alpha_i f_i \| = |\alpha| + \sum_{i=1}^{\infty} |\alpha_i|$.

Therefore line $\{f\}$ is an L-ideal in line F and hence in X^*.

Now let $f \in Z - E$, $f \neq 0$.

Case 1. Assume that X_1^* is a standard compact convex set. If $f/\|f\| \notin E$, choose a maximal probability measure μ with $\gamma(\mu) = f/\|f\|$. Choose $p_1, p_2 \in \text{Supp } \mu \bigcap E$, p_1 and p_2 independent. Get a sequence $\{f_n\} \subseteq E$, $\{f_n\}_{n=1}^{\infty} \bigcap T. \{p_1, p_2\} = \emptyset$, f_n's are all independent and $f_n \xrightarrow{w^*} f$.

As before let $N = \left\{ \sum_{i=1}^{\infty} \alpha_i f_i : \sum_{i=1}^{\infty} |\alpha_i| < \infty \right\}$ and $F = c \{f_n\}_{n=1}^{\infty}$.

If $E(F) = \{f_n\}_{n=1}^{\infty}$ then since $f \in N$ (use Choquet theorem) N = line F so that line F is a w^*-closed L-ideal. On approximating μ by simple measures having resultant $f/\|f\|$, it is easy to see that Supp $\mu \subseteq$ line F (see Lemma 1.1). Hence $p_i \in N$, contradicting the choice of the sequence $\{f_n\}$.

If $E(F) = \{f_n\}_{n=1}^{\infty} \cup \{f\}$, then line $F = N +$ line $\{f\}$ and we may assume that $f \notin N$. Use hypotheses as before to conclude that line F is a w^*-closed L-ideal and hence p_1, p_2 are in line $F = N \oplus$ line $\{f\}$, by the argument we outlined above. Therefore $p_2 \in N \oplus$ line $\{p_1\}$ and since p_1 is an extreme point independent of $\{f_n\}_{n=1}^{\infty}$, this direct sum is an L^1-direct sum. Hence $p_2 \in N$ or $p_2 = \lambda p_1$, $\lambda \in T$. This again contradicts the choice of the sequence $\{f_n\}$ and points p_1, p_2.

Therefore $f/\|f\| \in E$ and hence line $\{f\}$ is an L-ideal.

Case 2. X is arbitrary.

As before choose a sequence of independent vectors $\{f_n\} \subseteq E$ with $f_n \xrightarrow{w^*} f$. Let $F = c\{f_n\}_{n=1}^{\infty}$. Now $N =$ line F is a w^*-closed L-ideal and is a separable Banach space and hence N_1 is a standard compact convex set.

Since $E(N_1) \subseteq E$ and has atmost countably many independent vectors using hypotheses and Proposition 5.4, we see that the separable Banach space $X/o(N)$ satisfies the same hypotheses as the X in case (1). Therefore $f/\|f\| \in E(N_1) \subseteq E$. Hence line $\{f\}$ is an L-ideal.

Corollary 5.7. If X is a complex L^1-predual space with the property that E is sequentially w^*-dense in Z and the intersection of any countable family of M-ideals is an M-ideal then X is a G-space.

Proof : By Lima's characterization of L^1-preduals, line $\{p\}$ is an L-ideal for all $p \varepsilon E$. Hence line $\{p\}$ is an L-ideal $\forall\ p \varepsilon Z$ by above theorem i.e. $Z \subset [0,1].E$. Hence the conclusion follows from Olsen's characterization.

Corollary 5.8. If X is a complex Banach space such that

1) E has at most countably many linearly independent vectors and line $\{f\}$ is an L-ideal $\forall\ f \varepsilon E$.

2) Intersection of any countable family of M-ideals is an M-ideal.

Then X is a G-space.

Proof : We only need to show that X is an L^1-predual and in view of Lima's characterization it suffices to show that if, $f \varepsilon X^*$, $\|f\| = 1$ and $P(f) = 0$ or f for all L-projections P then $f \varepsilon E$.

Fix such an f. By Choquet's theorem we can write

$f = \sum_{i=1}^{\infty} \lambda_i f_i$, $\lambda_i \varepsilon \mathbb{C}$, $f_i \varepsilon E$ and f_i's are linearly independent.

Now either $\lambda_1 = 0$ or $f = \lambda_1 f_1$. For if $\lambda_1 \neq 0$ then $f = \lambda_1 f_1 + \sum_{n=2}^{\infty} \lambda_n f_n$ and $1 = |\lambda_1| + \sum_{n=2}^{\infty} |\lambda_n|$. But the condition on f shows that $f \varepsilon$ line $\{f_1\}$ or f is in the ideal complementary to line $\{f_1\}$. Since $\lambda_1 \neq 0$, f can not be in the complementary ideal. So $f = \lambda_1 f_1$. Proceeding by induction one shows that there is an $i \ni f = \lambda_i f_i$ and $|\lambda_i| = 1$ so that $f \varepsilon E$.

Proposition 5.9. Let K be a compact compact convex set such that 1) $A = \{x \in E(K) : \{x\} \text{ is a split face}\}$ is sequentially dense in $\overline{E(K)}$.

2) Intersection of any family of M-ideals is an M-ideal in $A_{\mathbb{R}}(K)$.

Then K is a Bauer simplex.

Proof : Use Theorem 5.6 and the correspondence between split faces of K and w^*-closed L-ideals of $A_{\mathbb{R}}(K)^*$ to deduce that E(K) is closed (noting that $1 \in A_{\mathbb{R}}(K)$).

Let $a_o \in A_{\mathbb{R}}(K)$, $a_o \geq 0$, $a_o \neq 0$ and let $F = \{x \in K : a_o(x) = 0\}$; $M_F = \{a \in A_{\mathbb{R}}(K) : a(x) = 0 \;\; \forall x \in E(F)\}$.

Since for all $x \in E(K)$, $\{x\}$ is split (follows from Theorem 5.6) and F is a face, by hypotheses, M_F is an M-ideal. Hence $M_F = \{a \in A_{\mathbb{R}}(K) : a(G) = 0\}$ for some closed split face G of K. $a_o \in M_F \Longrightarrow a_o(G) = 0 \Longrightarrow G \subseteq F$.

If $x \in E(F)$, $x \notin G$ then since line G is w^*-closed and $e(x) \notin$ line $e(G)$ ($e : K \longrightarrow A_{\mathbb{R}}(K)^*$ is the evaluation map) there is a $a \in A_{\mathbb{R}}(K)$ such that $a(G) = 0$ and $a(x) \neq 0$. This contradiction show that $G = F$.

Hence any peak face of K is a split face. From [14] it follows that K is a simplex. Hence K is a Bauer simplex.

Remark : When K is a metrizable compact convex set, it is easy to see that one need only consider countable intersections in (2) and hence our result extends that of Gleit [20] for A(K)-spaces.

We now give an example of a compact convex set K which is not metrizable but $E(K)$ is sequentially dense in $\overline{E(K)}$.

<u>Example</u> : Let K be the compact convex set considered in Example 2.2. Let $e : Y \longrightarrow K$ be the evaluation map. From [15] we have that e is a homeomorphism and

$$E(K) = \left\{ e(r_\alpha),\ e(t_\alpha) \right\}_{\alpha \in [0,1]},\quad \overline{E(K)} = e(Y).$$

Fix $\alpha \in [0,1]$ and let $\{\alpha_n\}$ be a sequence in $[0,1]$, $\alpha_n \longrightarrow \alpha$. If $\{s_\alpha\} \cup \bigcup \{Y_\beta : 0 < |\alpha - \beta| < \varepsilon\}$ is a neighbourhood of s_α then since there exists an n_0 such that $n \geq n_0 \Longrightarrow |\alpha - \alpha_n| < \varepsilon$, we get that r_{α_n} is in that neighbourhood for all $n \geq n_0$. Hence $r_{\alpha_n} \longrightarrow s_\alpha$ (similary $t_{\alpha_n} \longrightarrow s_\alpha$) so that $e(r_{\alpha_n}) \longrightarrow e(s_\alpha)$. Therefore $E(K)$ is sequentially dense in $\overline{E(K)}$. However K is not metrizable (K is not even 'standard').

<u>Corollary 5.10</u>. Let X be a real Banach space satisfying (1) of Theorem 5.6 and such that X_1 has an extreme point and the intersection of any family of M-ideals is an M-ideal then X is isometric to $C_{\mathbb{R}}(Y)$ for some compact Hausdorff space Y.

<u>Proof</u> : If x_0 is an extreme point of X_1 then since line $\{f\}$ is an L-ideal for all $f \in Z$, it follows from Theorem 3.1 of [31] that $|f(x_0)| = 1 \ \forall f \in Z$. Hence if we put $Y = \{f \in Z : f(x_0) = 1\}$

with the w^*-topology, then using standard arguments in convexity theory and Proposition 5.9, one sees that the natural map $\Phi : X \longrightarrow C_{\mathbb{R}}(Y)$ is an onto isometry.

<u>Remark</u> : Simple proofs of Nina Roy's [43], result have also been given by Uttersrud (unpublished), and also in a joint paper of A. Lima, G.H. Olsen and U. Uttersrud [33]. Uttersrud obtained a proof of Corollary 5.8, using Theorem 3.3 our proof is more direct.

If (*) denotes the condition that E is w^*-sequentially dense in Z then results of this section show that under the condition (*), Uttersrud's problem has a positive answer when $X = A_{\mathbb{R}}(K)$ (equivalently when X_1 has an extreme point) or E has atmost countably many indepent points or X is an L^1-predual. The general problem seems to be still open.

It will be clear from our results in the next section that for a G-space X, the w^*-closure of an L-ideal is an L-ideal. We do not know whether the condition, w^*-closure of an L-ideal is an L-ideal and line $\{f\}$ is an L-ideal for all $f \in E$ is equivalent to condition (3) of Proposition 5.4 (This is the case when X is such that X_1^* is a standard compact convex set).

SECTION 6

C_σ-spaces

A compact Hausdorff space Y is called a T_σ-space if there exists a map $\sigma : T \times Y \longrightarrow Y$ such that

i) σ is continuous

ii) $\sigma(\alpha , \sigma(\beta , y)) = \sigma(\alpha\beta , y)$; $\alpha, \beta \in T$, $y \in Y$

iii) $\sigma(1 , y) = y$.

Let Y be a T_σ space and

$$A = \{ f \in C(Y) : f(\sigma(\alpha,y)) = \alpha . f(y) \ \forall \ y \in Y, \ \alpha \in T \}$$

then A is called a C_σ-space.

Complex C_σ-spaces were introduced and studied by G.H. Olsen in [36] and are the complex analogues of real C_σ-spaces studied by Jerison. We quote a characterization of C_σ-spaces due to Olsen which we shall be using throughout.

A complex Banach space X is (isometric to) a C_σ-space iff X is an L^1-predual and $E \cup \{0\}$ is w^*-closed.

Theorem 6.1. For a complex Banach space X, the following are equivalent :

1) X is a C_σ-space

2) i) $A = \{ f \in E : \text{line } \{f\} \text{ is an L-ideal} \}$ is w^*-dense in Z

ii) for any L-ideal $N \subseteq X^*$, $\overline{N}$ is an L-ideal and $(\overline{N})_1 = \overline{(N_1)}$.

3) Any relatively w^*-closed T-invariant subset of E is structurally closed.

4) For all $x \in X$, $|x| : E \longrightarrow \mathbb{R}$ is lower semi-continuous (l.s.c.) in the structure topology.

<u>Proof</u> : 1 ⟹ 2 Since for all $f \in E$, line $\{f\}$ is an L-ideal, (i) is clear.

Let $N \subseteq X^*$ be any L-ideal and put $D = \bigcup_{p \in N \cap S^*} \text{Supp}\, \mu_p$ where μ_p is the unique maximal probability measure representing p. Clearly $N_1 \subseteq c(TD)$. Fix $p \in N$, $\| p \| = 1$. On approximating μ_p with simple measures having resultant p, since N is an L-ideal we get that $\text{Supp}\, \mu_p \subseteq N_1$ (see Lemma 1.1). Therefore $\overline{(N_1)} = c(TD)$.

Since X is a C_σ-space, $T.\overline{D}$ is a dilated set and hence line $c(TD)$ is a w^*-closed L-ideal. But $\overline{N}$ = line $c(TD)$ and we also know that $(\overline{N})_1 = c(TD)$.

Therefore $\overline{N}$ is an L-ideal and $(\overline{N})_1 = \overline{(N_1)}$.

2 ⟹ 3 Let $f \in E$ and $\{f_\alpha\}$ be a net in A with $f_\alpha \xrightarrow{w^*} f$. Put $D_\alpha = \{f_\beta\}_{\beta \geq \alpha}$ and $N_\alpha = \overline{\text{line } D_\alpha}$ (closure in the norm topology). Since line $\{f_\beta\}$ is an L-ideal for all β, we get that N_α is an L-ideal. So by hypotheses $\overline{N}_\alpha$ is an L-ideal and $(\overline{N}_\alpha)_1 = (\overline{(N_\alpha)_1})$.

Clearly $c(TD_\alpha) \subseteq (\overline{(N_\alpha)_1})$. Let $p \in N_\alpha$, $\| p \| = 1$. Get a sequence $\{p_n\} \subseteq$ line D_α and $p_n \longrightarrow p$ in norm, $\| p_n \| \leq 1$.

Now $p_n \in$ line $D_\alpha \Longrightarrow p_n \in$ line $\{f_{\alpha_i}\}_{i=1}^{k_n} \Longrightarrow p_n \in c(TD_\alpha)$.

Therefore $(\overline{(N_\alpha)_1}) = c(TD_\alpha) = (\overline{N}_\alpha)_1$.

Now let $N = \bigcap_\alpha \overline{N}_\alpha$ then N is an L-ideal. Clearly line $\{f\} \subseteq N$. If $g \in N$, $\| g \| = 1$ then $g \in (\overline{N}_\alpha)_1$ for all α implies $g \in \bigcap_\alpha c(TD_\alpha) \Longrightarrow g \in$ line $\{f\}$. Therefore line $\{f\} = N$ and hence line $\{f\}$ is an L-ideal $\forall f \in E$.

For any $D \subseteq E$, relatively w^*-closed and T-invariant, let $N = \overline{\text{line } D}$ (norm closure). Using the hypotheses and proceeding as before, easy to see that $\overline{N}$ is a w^*-closed L-ideal and $(\overline{N})_1 = c(D)$. Therefore $(\overline{N})_1 \cap E = c(D) \cap E = \overline{D} \cap E = D$.

Hence D is a structurally closed set.

$3 \Longrightarrow 4$ Easy to see.

$4 \Longrightarrow 1$ Let $0 \neq f \in Z$, $f \notin E$. Let $\{f_\alpha\}$ be a net in E with $f_\alpha \xrightarrow{w^*} f$. Fix $p \in N_f \cap E$ (recall that N_f stands for the smallest w^*-closed L-ideal containing f) and $c > 0$. For $x \in X$, if $|p(x)| > c$ then $\{g \in E : |g(x)| > c\}$ is a structurally open set containing p. Since $f_\alpha \xrightarrow{s} p$ we get that $|f(x)| \geq c$. Therefore $|p(x)| \leq |f(x)|$ for all $x \in X$, so that line $\{p\}=$ line $\{f\}$. Hence line $\{f\} = N_f$ and so line $\{f\}$ is an L-ideal.

Now as in the proof of Theorem 5.1, we can see that X is a G-space. Use that theorem once again to conclude that $|x|$ is structurally u.s.c for all x. Therefore $|x|$ is structurally continuous.

If $f \in Z$ and $f \neq 0$, $f_\alpha \xrightarrow{w^*} f$, $\{f_\alpha\} \subseteq E$ then $f_\alpha \xrightarrow{s} f/\|f\|$. Since $|x|$ is structurally continuous we have $|f(x)| = |f(x)|/\|f\|$ for all x. Hence $\|f\| = 1$. Therefore $f \in E$. So $E \bigcup \{0\}$ is w^*-closed.

Hence X is a C_σ-space.

<u>Corollary 6.2</u>. If X is a G-space then for any L-ideal N, $\overline{N}$ is an L-ideal. $(\overline{N})_1 = \overline{(N_1)}$ for all L-ideals N iff X is a C_σ-space.

<u>Proof</u> : For any L-ideal N, let $D = \bigcup_{p \in N \cap S^*} (\text{Supp}\, \mu_p \cap E)$, where μ_p is maximal with $\gamma(\mu_p) = p$. Use Proposition 5.4, to conclude that $\overline{N}$ is an L-ideal.

<u>Corollary 6.3</u>. Let K be a compact convex set. The following are equivalent

1) K is a Bauer simplex

2) i) $A = \{x \in E(K) : \{x\}$ is a split face$\}$ is dense in $\overline{E(K)}$

ii) for any split face F, $\overline{F}$ is a split face

3) For all $a \in A_{\mathbb{R}}(K)$, $|a| : E(K) \longrightarrow \mathbb{R}$ is lower semi-continuous in the facial topology.

Proof : We use the correspondence between split faces of K and L-ideals of $A_{\mathbb{R}}(K)^*$.

If N is an L-ideal in $A_{\mathbb{R}}(K)^*$, then $N = \text{line } F$ where F is a split face of K and $N_1 = CO(F \cup -F)$ (we are suppressing the embedding map). If $\overline{F}$ is a split face then line $\overline{F} = \overline{N}$ and $(\text{line } \overline{F})_1 = CO(\overline{F} \cup -\overline{F}) \Longrightarrow (\overline{N})_1 = \overline{(N_1)}$.

Remark : Corollary 6.3, improves a result of A·Lima [34]. (3) of Corollary 6.3 and Corollary 5.3, together improve a result of Effros [10].

A C_σ-space X is called a C_Σ-space if E is w^*-closed.

If X is a Banach space and $0 \notin Z$, then E is compact in the structure topology. For if $\{f_\alpha\}$ is a net in E, then a sub-net of $\{f_\alpha\}$ converges to a non-zero element, f, of Z and hence this sub-net will converge in the structure topology to any $p \in N_f \cap E$.

Converse holds when X is a C_σ-space.

Corollary 6.4. If X is a C_σ-space and E is compact in the structure topology then $0 \notin Z$ and hence X is a C_Σ-space.

Proof : If $\{f_\alpha\}$ is a net in E and $f_\alpha \xrightarrow{w^*} 0$, use compactness to get a sub-net (still denoted by $\{f_\alpha\}$) so that $f_\alpha \xrightarrow{S} g$, $g \in E$. Since $|x|$ is structurally continuous for all x, $|f_\alpha(x)| \to |g(x)| = 0$ for all x. A contradiction. Hence E is w^*-closed.

Remarks : Theorem 6.1, improves Théorème 13 of [17]. Fakhoury proves the equivalence, for real Banach spaces, of (1),(3) and of the statement : $|x|$ is structurally continuous for all x, under the hypothesis X is an L^1-predual. The L-ideal characterization of C_σ-spaces is new. The concluding argument in the proof of Theorem 6.1 is in Fakhoury [17].

In view of Corollary 6.2, it may seem that in a general G-space, elements in the w^*-closure of an L-ideal can be approximated in w^*-topology by norm bounded nets from the L-ideal (a simple application of Bair-category theorem shows that this is same as the existance of a $\lambda \geq 1$ such that $(\overline{N})_1 \subseteq \lambda.\overline{(N_1)}$ for an L-ideal N). We now give an example of a separable G-space A and an L-ideal N such that for no $\lambda \geq 1$, $(\overline{N})_1$ is contained in $\lambda.\overline{(N_1)}$. This also furnishes an example of a subspace of characteristic zero in the sense of Diximier [8].

<u>Example</u> : Let $A = \left\{ f \in C_{\mathbb{R}}[0,1] : f(\frac{1}{n}) = \frac{1}{n} f(1-\frac{1}{n}) \ \forall\ n \geq 3 \text{ and } f(0) = 0 = f(1) = f(\frac{1}{2}) \right\}$

A is a separable G-space. It is not difficult to see that

$$E(A_1^*) = \pm \left\{ e(x) : x \in (0,1),\ x \neq \frac{1}{n},\ n \geq 2 \right\}.$$

Choose sequences $\{x_n^i\}$ in $(0, \frac{1}{2}) \ni \{x_n^i\}_{n=1}^{\infty} \cap \{\frac{1}{i}\}_{i=3}^{\infty} = \emptyset \ \forall\ i \geq 3$ and $x_n^i \longrightarrow \frac{1}{i} \ \forall\ i \geq 3$. Put $D = \bigcup_{i=3}^{\infty} \{e(x_n^i)\}_{n=1}^{\infty}$. $D \subseteq E(A_1^*)$ and $\overline{D} = D \cup \{\frac{1}{i}\}_{i=3}^{\infty} \cup \{0\}$. Let $N = \overline{\text{linc } D}$ (norm topology) then $\overline{N}$ is an L-ideal and $\overline{(N_1)} = c(\pm D)$.

Let $F = \pm(\bar{D} \bigcup \{e(1-\frac{1}{i})\}_{i=3}^{\infty})$. Since $e(1-\frac{1}{i}) \xrightarrow{w^*} 0$ and $e(\frac{1}{i}) = \frac{1}{i}\, e(1-\frac{1}{i})\ \forall\ i \geq 3$, we get that F is a dilated set. Therefore line $c(F)$ is a w^*-closed L-ideal and

$$(\text{line } c(F))_1 = (\bar{N})_1 = c(F).$$

Suppose there exists an integer $m_o \geq 1$ such that $c(F) \subseteq m_o\, c(\pm D)$. Then $\{e(1-\frac{1}{i})\}_{i \geq 3} \subseteq m_o\, c(\pm D)$. Fix $i \geq 3$, get $f \in A$, $0 \leq f \leq 1$ such that $f(1-\frac{1}{i}) = 1$, $f(\frac{1}{i}) = \frac{1}{i}$, f peaks at $\frac{1}{i}$ in $[0, \frac{1}{2})$.

Now there exists a probability measure μ on $\bar{D} \bigcup -\bar{D}$ such that $\frac{1}{m_o} = \frac{e(1-\frac{1}{i})(f)}{m_o} = \int_{\bar{D} \bigcup -\bar{D}} f\, d\mu \leq \frac{1}{i}$. Therefore $\frac{1}{m_o} \leq \frac{1}{i}\ \forall\ i \geq 3$. A contradiction. Hence for no $\lambda \geq 1$ $(\bar{N})_1$ is contained in $\lambda\overline{(N_1)}$.

<u>Theorem 6.6</u>. If X is a complex Banach space such that any T-invariant set $D \subseteq E$ is structurally closed then X is isometric to $c_o(\Gamma)$. (where $c_o(\Gamma)$ is the space of complex valued functions vanishing at infinity on a discrete space Γ, equipped with the supremum norm).

<u>Proof</u> : Use Theorem 6.1, to conclude that X is a C_σ-space. Let F be a maximal face of X_1^*. Then $E = T.\Gamma$ where $\Gamma = F \bigcap E$ (see [36]).

Define $\Phi : X \longrightarrow c_o(\Gamma)$ by $\Phi(x)(f) = f(x)\ \forall\ x \in X$, $f \in \Gamma$. Φ is well-defined (see Proposition 4.8 of [12]) and an isometry.

To see that Φ is onto let $f \varepsilon c_o(\Gamma)$ and define

$f' : E \bigcup \{0\} \longrightarrow ¢$ by $f'(p) = t.f(q)$ if $p \varepsilon E$, $p = t.q$, $q \varepsilon F$, $t \varepsilon T$
$= 0$ if $p = 0$.

It is not hard to see that f' is well defined and is a w^*-continuous, T-homogeneous, function on $E \bigcup \{0\}$. Extend f' by Tietz's theorem to a w^*-continuous function g on X_1^* and let $h = \text{hom } g$.

By Theorem 9 of [36], there exists $v \varepsilon X$ such that $h(p) = p(v)$ $\forall$ $p \varepsilon X_1^*$ and h agrees with f' on $E \bigcup \{0\}$. Hence it follows that $\Phi(v) = f$.

Therefore Φ is onto.

Corollary 6.7. Let X be a complex Banach space such that line $\{f\}$ is an L-ideal for all $f \varepsilon E$ and any L-ideal in X^* is w^*-closed, then X is isometric to $c_o(\Gamma)$.

Proof : Let $D \subseteq E$ be a T-invariant set. Put $N = \overline{\text{line } D}$ (closure in norm topology). Then by an argument used in the proof of $2 \Longrightarrow 3$ of Theorem 6.1 we get that $N_1 = \overline{CO}(D)$ (norm closure). By hypotheses N is w^*-closed and hence $N_1 = c(D)$. If $f \varepsilon E(N_1)$ and $f \notin D$ then if we write $X^* = \text{line}\{f\} \oplus M$, where M is the L-ideal complementary to line $\{f\}$, since D is T-invariant, $D \subseteq M$ and hence $N \subseteq M$. A contradiction. Therefore $E(N_1) = D$. Hence D is a structurally closed set. Conclusion follows from Theorem 6.6.

Remark : That all L-ideals in $c_o(\Gamma)^* = \ell^1(\Gamma)$, are w^*-closed follows from the structure of L-ideals in $\ell^1(\Gamma)$ (see [4]). Hence the conditions in the hypotheses of Theorem 6.6 and Corollary 6.7 actually characterize $c_o(\Gamma)$.

We use Theorem 6.6 to improve a result of Lima from [32].

Corollary 6.8. Let X be a real Banach space such that

1) $\forall f \in E$, line $\{f\}$ is an L-ideal

2) $K(X,X)$ (space of compact operators on X) is an M-ideal in $L(X,X)$ (space of bounded operators)

then X is isometric to $c_o(\Gamma)$.

Proof : For any $x \in S$ and for $\varepsilon > 0$, let $N = \{f \in E : |f(x)| \geq \varepsilon\}$. Proceeding as in Lemma 1 of [32] one shows that N is a finite set. Since we are assuming that $T.\{f\}$ is structurally closed for all $f \in E$, it follows from Theorem 5.1 that X is a G-space. Now it is easy to see that any T-invariant subset of E is structurally closed and hence X is isometric to $c_o(\Gamma)$.

SECTION 7

Real sections in complex L^1-preduals

The purpose of this section is to provide, new, and simple proofs of many of the results from [48]. We also give an example to show that Proposition 3.5 of [48] is false. We first show that if $A \subseteq C(X)$ is a closed self-ajoint subspace with $(\text{Re } A)^*$ isometric to $L^1(\mu, \mathbb{R})$ then A^* is isometric to $L^1(\mu, \mathbb{C})$. Thereby providing a new argument which will prove Proposition 3.4 and Lemma 5.1 of [48] in a single stroke. For this purpose we need the definition of barycentric map introduced by Bednar and Lacey in [3]. Our definition is from [42].

Let Y be a compact Hausdorff space.

A map $\beta : Y \longrightarrow M(Y) = C(Y)^*$ is said to be a barycentric mapping if 1) $\|\beta(y)\| \leq 1 \ \forall y \in Y$

2) $\forall f \in C(Y)$, the function f_β defined on Y by $f_\beta(y) = \int_Y f d\beta(y)$ is Borel measurable and integrable with respect to each $\mu \in M(Y)$.

3) For $\mu, \mu' \in M(Y)$ if $\mu(f) = \mu'(f) \ \forall f \in C(Y)$ such that $f = f_\beta$ then $\mu(f_\beta) = \mu'(f_\beta) \ \forall f \in C(Y)$.

Bednar and Lacey [3] have used the barycentric map to give a characterization of real Lindenstrauss spaces. We now satate the complex analogue of the Bednar and Lacey result obtained by A.K. Roy in [42].

<u>Theorem</u> : A complex Banach space X is an L^1-predual iff there is a compact Hausdorff space Y and a barycentric map $\beta : Y \longrightarrow M(Y)$ such that X is isometric to

$$A_\beta = \{f \in C(Y) : f = f_\beta\}.$$

<u>Lemma 7.1</u>. Let $A \subseteq C(Y_1)$ and $B \subseteq C(Y_2)$ be two closed, self-adjoint subspaces. If $\Phi : \operatorname{Re} A \longrightarrow \operatorname{Re} B$ is an onto, real, isometry then Φ can be extended to a complex isometry Φ' from A onto B.

<u>Proof</u> : Define $\Phi' : A \longrightarrow B$ by $\Phi'(f) = \Phi(\operatorname{Re} f) + i\,\Phi(\operatorname{Im} f)$ for $f \in A$. Clearly Φ' is a linear, one-one, onto map extending Φ. Fix $f \in A$. There exists $y \in Y_2$ such that $\|\Phi'(f)\| = |\Phi'(f)(y)| = t.\Phi'(f)(y) = \Phi'(tf)(y)$ for some $t \in T$. But this implies, $\|\Phi'(f)\| = \Phi(\operatorname{Re}(tf))(y) \leq \|\operatorname{Re} tf\| \leq \|f\|$. Therefore $\|\Phi'(f)\| \leq \|f\|\ \forall f \in A$ and the symmetry of the argument now shows that Φ' is an isometry.

<u>Theorem 7.2</u>. Let $A \subseteq C(Y)$ be a closed, self-adjoint subspace. A is an L^1-predual iff $\operatorname{Re} A$ is an L^1-predual. Moreover if $(\operatorname{Re} A)^* = L^1(\mu, \mathbb{R})$ for some non-negative measure μ then $A^* = L^1(\mu, \mathbb{C})$ (equality stands for isometry).

<u>Proof</u> : Suppose $\operatorname{Re} A$ is an L^1-predual.

By Bednar and Lacey theore (for real scalars) there is a compact Hausdorff space Y' such that $\operatorname{Re} A$ is isometric to

$A_\rho = \{ f \in C_{\mathbb{R}}(Y') : f = f_\rho \}$, where $\rho : Y' \longrightarrow C_{\mathbb{R}}(Y')^*$ is the (real-measure valued) barycentric map.

Now consider ρ as a barycentric map from Y' into $M(Y')$. Since ρ values are only real measures it follows that $\bar{f}_\rho = (\bar{f})_\rho$ for all $f \in C(Y')$ ($\bar{f}$ denotes complex conjugate of f). If we let $B = \{ f \in C(Y') : f = f_\rho \}$ then B is a self-ajoint subspace such that $\operatorname{Re} B = A_\rho$. Also by the complex form of Benar-Lacey theorem, B is a complex L^1-predual. Since $\operatorname{Re} A$ is isometric to $\operatorname{Re} B$ and A, B are self-adjoint subspaces, by Lemma 7.1, we get that A is isometric to B and hence A is an L^1-predual.

While proving that A_ρ is an L^1-predual Bednar and Lacey, observe that $M = \{ \lambda \in C_{\mathbb{R}}(Y')^* : \int f d\lambda = \int f_\rho d\lambda \ \forall f \in C_{\mathbb{R}}(Y') \}$ is isometric to A_ρ^* via the restriction map. A similar argument works in the complex case also giving us that the restriction map is an isometry from

$$M' = \{ \lambda \in C(Y')^* : \int f d\lambda = \int f_\rho d\lambda \text{ for all } f \in C(Y) \} \text{ onto } B^*.$$

Let $\Phi : L^1(\mu, \mathbb{R}) \longrightarrow M$ be any isometry.

Define $\Phi' : L^1(\mu, \mathbb{C}) \longrightarrow M'$ by $\Phi'(f) = \Phi(\operatorname{Re} f) + i\,\Phi(\operatorname{Im} f)$.

Clearly Φ' is a continuous, linear, one-one, onto map extending Φ. To show that Φ' is an isometry, it is enough to show that it is an isometry on simple functions.

For any two disjoint measurable sets C_1, C_2

$$\mu(C_1) + \mu(C_2) = \| I_{C_1} \pm I_{C_2} \| = \| \Phi(I_{C_1}) \pm \Phi(I_{C_2}) \| = \| \Phi(I_{C_1}) \| + \| \Phi(I_{C_2}) \|.$$

So the real measures $\Phi(I_{C_1})$ and $\Phi(I_{C_2})$ are mutually singular and hence

$$\| \alpha \Phi (I_{C_1}) + \beta \Phi (I_{C_2}) \| = |\alpha| \, \| \Phi (I_{C_1}) \| + |\beta| \, \| \Phi (I_{C_2}) \|$$

for any complex scalars α and β.

Hence $\| \Phi'(\alpha I_{C_1} + \beta I_{C_2}) \| = \| \alpha \Phi (I_{C_1}) + \beta \Phi (I_{C_2}) \|$

$$= \| \alpha I_{C_1} + \beta I_{C_2} \|_1 \quad \text{for } \alpha, \beta \in \mathbb{C}.$$

A similar argument shows that Φ' is an isometry on simple functions and hence Φ' is an isometry.

Suppose A is an L^1-predual. Let $\{B(a_i, r_i)\}_{i=1}^{4}$ be four closed balls in A such that a_i's are real-valued and $\| a_i - a_j \| \leq r_i + r_j$ for all i,j. For any $y \in Y$, $|a_i(y) - a_j(y)| \leq r_i + r_j \; \forall \, i,j \implies \bigcap_{i=1}^{4} B(a_i(y), r_i) \neq \emptyset$. So that

$$\left| \sum_{i=1}^{4} z_i a_i (Y) \right| \leq \sum_{i=1}^{4} r_i |z_i| \quad \forall \, y \in Y$$

where $z_i \in \mathbb{C}$ and $\sum_{i=1}^{4} z_i = 0$.

Therefore $\| \sum_{i=1}^{4} z_i a_i \| \leq \sum_{i=1}^{4} r_i |z_i|$.

Hence the balls $\{B(a_i, r_i)\}_{i=1}^{4}$ have weak intersection property (see Theorem 2.1 [28] or [25]). Since A is an L^1-predual there is a $b \in \bigcap_{i=1}^{4} B(a_i, r_i)$, $b \in A$ (see [29]).

Now $\|\operatorname{Re} b - a_i\| = \|\operatorname{Re}(b - a_i)\| \leq \|b - a_i\| \leq r_i \ \forall\, i$.

So $\operatorname{Re} b \in \bigcap_{i=1}^{4} B(a_i, r_i)$. Hence by Lindenstrauss' characterization of L^1-preduals [35] we get that $\operatorname{Re} A$ is an L^1-predual.

Remark : The arguments in the last part of the above proof are due to A. Lima. The above theorem also gives a different proof of Lemma 5.1 [48]. The above arguments can also be used to give a characterization of complex simplex spaces (Section 4 [36]) in terms of barycentric maps taking values in non-negative measures.

The above theorem completely fails in the absence of the assumption self-adjointness on A. It was wrongly stated in [48] (Prosition 3.5) that if a closed subspace $A \subseteq C(Y)$ is a Lindenstrauss space then $\operatorname{Re} A$ is a Lindenstrauss space. We give an example.

Example : Let $Y = \{1,2,3\}$ and

$$A = \left\{ f \in C(Y) : f(1) = if(2) = \frac{1+i}{2} f(3) \right\}.$$

If $f_0 : Y \longrightarrow \mathbb{C}$ is defined by, $f_0(1) = 1$, $f_0(2) = -i$ and $f_0(3) = 1 - i$ then it is easy to see that $A = \text{line}\{f_0\}$. Therefore A is a Lindenstrauss space.

Claim $\operatorname{Re} A = \left\{ f \in C_{\mathbb{R}}(Y) : f(1) + f(2) = f(3) \right\}$.

If $f \in A$ then since $\operatorname{Re} f(2) = \operatorname{Im} f(1)$ and

$\operatorname{Re} f(1) = \frac{1}{2}\left\{ \operatorname{Re} f(3) - \operatorname{Im} f(3) \right\}$, $\operatorname{Im} f(1) = \frac{1}{2}\left\{ \operatorname{Im} f(3) + \operatorname{Re} f(3) \right\}$, we get that $\operatorname{Re} f(1) + \operatorname{Re} f(2) = \operatorname{Re} f(3)$. For $g \in C_{\mathbb{R}}(Y)$, if g' is defined by $g'(1) = g(2)$, $g'(2) = -g(1)$ and $g'(3) = 2g(2) - g(3)$ then it is easy to see that $h = g + ig' \in A$. Hence the claim.

The functions f_1, f_2 defined on Y by

$f_1(1) = 1 = f_1(3)$, $f_1(2) = 0$; $f_2(2) = 1 = f_2(3)$, $f_2(1) = 0$

are in $\operatorname{Re} A$ and span $\operatorname{Re} A$. Easy to see that $\operatorname{Re} A$ is isometric to $\mathbb{R}^2$ with the norm $\|(x,y)\| = \max\{|x|, |y|, |x-y|\}$. Hence it follows from Example 2.5 that $\operatorname{Re} A$ is not an L^1-predual.

We now recall the definition of real section of a complex Lindenstrauss space from [48].

Let X be a complex L^1-predual space. A closed real linear subspace $G \subseteq X$ is said to be a real section of X if

i) $G + iG$ is dense in X

ii) G is a real L^1-predual space.

iii) There is a set of norm one functionals $M \subseteq X^*$ such that every member of M, takes only real values on G and $\|g\| = \sup_{m \in M} |m(g)| \; \forall g \in G + iG$.

Theorem 7.3. Let X be a complex L^1-predual with a real section G. Then there exists a compact Hausdorff space Y and a self-adjoint L^1-predual $A \subseteq C(Y)$ and a complex linear isometry of X onto A, whose restriction to G is a real isometry onto $\operatorname{Re} A$.

Proof : Let $Y = \bar{M}$. Define $\Phi : G + iG \longrightarrow C(Y)$ by $\Phi(g_1 + ig_2)(y) = y(g_1 + ig_2)\ \forall\, y \in Y,\ g_j \in G$. Since each $y \in Y$, takes only real values on G, it is clear that Φ is well defined and for the same reason we have

$$\overline{\Phi(g_1 + ig_2)}(y) = y(g_1) - iy(g_2) = \Phi(g_1 - ig_2)(y) .$$

Clearly Φ is a linear map and it is not difficult to deduce from condition (iii) in the definition of a real section that Φ is an isometry. Also the above remark shows that $\Phi(G + iG)$ is a self-adjoint subspace of $C(Y)$ with $\mathrm{Re}\,\Phi(G + iG) = \Phi(G)$.

Since Φ is an isometry and G is a Banach space, we get that $\mathrm{Re}\,\Phi(G + iG)$ is uniformly closed. The self-adjointness of $\Phi(G + iG)$ now implies that $\Phi(G + iG)$ is uniformly closed and hence $G + iG$ is a closed subspace of X.

Hence we have an isometry with all the required properties.

Remark : This is Lemma 4.7 in [48]. Note that the proof makes no use of the hypotheses that X and G are L^1-preduals. Unlike the arguments in the proof of Lemma 4.7 in [48], our proof is completely free of L^1-predual theory. Several of the propositions in [48], can now be easily deduced (viz Lemmas 4.1, 4.4, 4.5).

SECTION 8

Isometries of simplex spaces

Let K be a compact convex set. A set $D \subseteq E(K)$ is said to be facially closed if there exists a closed split face F of K such that $E(F) = D$. The sets D form the closed sets of a topology on $E(K)$ called the facial topology. Following the notations of [1], we denote by $Z(A(K))$ the set of all elements $b \in A(K)$ such that for every $a \in A(K)$ there exists a $c \in A(K)$ satisfying $c(x) = a(x)b(x) \; \forall \; x \in E(K)$.

Since for any $b \in Z(A(K))$, real and imaginary parts of b are in $Z(A_{\mathbb{R}}(K))$, using Corollary II.7.4 and Theorem II.7.10 of [1], we can easily see that for $b \in A(K)$, b is in $Z(A(K))$ if and only if $b/E(K)$ is continuous in the facial topology.

We now describe a class of isometries for $A(K)$ and show that when K is a simplex it gives a complete description of isometries of $A(K)$.

Let $Q : K \longrightarrow K$ be an onto affine homeomorphism and let $a_o \in Z(A(K))$ be such that $|a_o| = 1$ on $E(K)$.

Define $\Phi : A(K) \longrightarrow A(K)$ by $\Phi(a) = c$ for $a \in A(K)$, where c is the unique element of $A(K)$ such that

$$c(x) = a(Q(x))\, a_o(x) \quad \forall \quad x \in E(K).$$

Since $|a_o| = 1$ on $E(K)$ and since $\bar{a}_o \in Z(A(K))$ (which is easy to see from the remarks made above) it is easy to see that

Φ is an onto isometry with $\Phi(1) = a_o$.

Theorem 8.1. Let $\Phi : A(K) \longrightarrow A(K)$ be any onto isometry such that $\Phi(1) \varepsilon Z(A(K))$. Then there exists an affine homeomorphism Q of K such that

$$\Phi(a)(x) = a(Q(x))\Phi(1)(x) \;\forall\; x \varepsilon E(K),\; a \varepsilon A(K).$$

Proof : If $e : K \longrightarrow A(K)^*_1$ denotes the evaluation map then it is well known that e is an affine homeomorphism of K onto $\{f \varepsilon A(K)^*_1 : f(1) = 1\}$, equipped with the w^*-topology and $E(A(K)^*_1) = T\, e(E(K))$.

Since Φ^* is an isometry it is easy to see that $\Phi^*(e(E(K))) \subseteq T\, e(E(K))$. So that if $x \varepsilon E(K)$ then there exists unique $x' \varepsilon E(K)$ and $t \varepsilon T$ (since $A(K)$ separates points of K and $1 \varepsilon A(K)$) such that

$$\Phi^*(e(x)) = t\, e(x').$$

Evaluating at 1, we get, $\Phi(1)(x) = t$. Therefore $|\Phi(1)| = 1$ on $E(K)$.

Since $\Phi(1) \varepsilon Z(A(K)) \Longrightarrow \overline{\Phi(1)} \varepsilon Z(A(K))$, if we define $S : A(K) \longrightarrow A(K)$ by the formula

$$(Sa)(x) = \Phi(a)(x)\, \overline{\Phi(1)(x)} \;\forall\; x \varepsilon E(K),\; a \varepsilon A(K)$$

then since $|\Phi(1)| = 1$ on $E(K)$, by the remarks proceding the theorem it follows that S is an onto isometry. Moreover $S(1)(x) = \Phi(1)(x)\, \overline{\Phi(1)(x)} = 1 \;\forall\; x \varepsilon E(K)$ so that $S(1) = 1$. Therefore S^* maps $e(K)$ onto $e(K)$. Since S^* is a w^*-homeomorphism, we get that $Q = e^{-1} \circ S^* \circ e$ is an affine homeomorphism of K onto K. Also for $x \in E(K)$, $a \in A(K)$

$$\Phi(a)(x) = \Phi(1)(x)\ (Sa)(x) = \Phi(1)(x)\ a(Q(x)).$$

We now show that if K is a Choquet simplex then the isometries of the form considered above, completely describe the isometries of $A(K)$. First we quote a definition and a result due to Effros from [11].

<u>Definition</u> : Say a closed set $D \subseteq K$ is a <u>dilated</u> set if for any maximal probability measure μ with $\gamma(\mu) \in D$, $\operatorname{Supp}\mu \subseteq D$.

<u>Result</u> : If K is a compact Choquet simplex then for any dilated set $D \subseteq K$, $F = \overline{CO}(D)$ is a split face.

<u>Proposition 8.2</u>. Let K be a compact Choquet simplex and let $a_o \in A(K)$ be such that $|a_o| = 1$ on $E(K)$. Then $a_o \in Z(A(K))$.

<u>Proof</u> : It follows from our earlier remarks that we only need to show that $a_o/E(K) \longrightarrow \mathbb{C}$ is facially continuous.

Let $B \subseteq T$ be a closed set and let $B' = \{x \in \overline{E(K)} : a_o(x) \in B\}$. We claim that B' is a dilated set. Let μ be a maximal probability measure with $x_o = \gamma(\mu) \in B'$.

$$1 = |a_o(x_o)| = \left|\int_{\overline{E(K)}} a_o\, d\mu\right| \leq \int_{\overline{E(K)}} |a_o|\, d\mu = 1.$$

Therefore $a_o = a_o(x_o)$ on $\operatorname{Supp}\mu$, so that $\operatorname{Supp}\mu \subseteq B'$ as $\operatorname{Supp}\mu \subseteq \overline{E(K)}$. Hence B' is a dilated set.

From the result of Effros, quoted above, we get that $F = \overline{co}(B')$ is a split face. Therefore

$$\{x \in E(K) : a_o(x) \in B\} = F \cap E(K).$$

Hence a_o is facially continuous and consequently $a_o \in Z(A(K))$.

Remark : For a simplex K, $a \in A(K)_1$ is an extreme point iff $|a| = 1$ on $E(K)$ iff $a \in Z(A(K))_1$ and is an extreme point of $Z(A(K))_1$.

Corollary 8.3. If K is a compact Choquet simplex, then for any isometry Φ of $A(K)$ there exists an affine homeomorphism Q of K such that

$$\Phi(a)(x) = a(Q(x))\,\Phi(1)(x) \quad \forall \quad x \in E(K).$$

Proof : We have observed in the proof of Theorem 8.1 that $|\Phi(1)| = 1$ on $E(K)$ therefore by Proposition 8.2, $\Phi(1) \in Z(A(K))$ and the conclusion follows from Theorem 8.1.

Corollary 8.4. If K_1 and K_2 are compact convex sets and K_2 is a simplex then for any onto isometry $\Phi : A(K_1) \longrightarrow A(K_2)$ there exists an affine homeomorphism Q from K_2 onto K_1 (and hence K_1 is a simplex) such that

$$\Phi(a)(x) = a(Q(x))\,\Phi(1)(x) \;\forall\; x \in E(K_2),\ a \in A(K_1).$$

Proof : Can be easily seen using arguments similar to the ones used in Theorem 8.1.

Remark : When K_1 and K_2 are simplexes and the scalar field is real, A. Lazar [27] proved the above corollary in a different form. If one identifies a compact Hausdorff space X as the extreme boundary of the set of probability measures on X with the w^*-topology, it is not difficult to see that the above corollary is an extension of the classical Banach-Stone theorem for C(X).

Corollary 8.5. Let X be a complex Lindenstrauss space and let $e_1, e_2 \in X_1$, be two linearly independent extreme points. If $K_{e_i} = \{f \in X_1^* : f(e_i) = 1\}$ then K_{e_1} and K_{e_2}, are homeomorphic.

Proof : Use the Hirsberg and Lazar theorem [21] to conclude that K_{e_i} is a simplex and $A(K_{e_1})$ and $A(K_{e_2})$ are isometric and then use Corollary 8.4.

Let K be a compact Choquet simplex and let $A_o(K) = \{a \in A(K) : a(p_o) = 0\}$ where $p_o \in E(K)$ is fixed. Let F be the face complementary to $\{p_o\}$.

If Q is an affine homeomorphism of K, taking p_o into p_o and if $a_o : E(F) \longrightarrow T$ is continuous in the relative facial topology $(E(F) = F \cap E(K))$ then the map $\Phi : A_o(K) \longrightarrow A_o(K)$ defined by $\Phi(a) = c$ where c is the unique element of $A_o(K)$ such that $c(x) = a(Q(x))\, a_o(x)\ \forall\ x \in E(F)$ is an onto isometry (The existance of such a unique c follows from a result of [11] or Theorem 3.8 of [4]).

Theorem 8.6. Isometries of the above form completely describe isometries of $A_o(K)$.

Proof : Let $\Phi : A_o(K) \longrightarrow A_o(K)$ be any isometry. The evaluation map e takes K affine homeomorphically onto $K' = \{f \in A_o(K)_1^* : f(a) \geq 0 \ \forall \ a \in A_o(K)^+\}$ (with w^*-topology) so that p_o goes to zero and F is mapped onto $F' = \{f \in K' : \|f\| = 1\}$.

Φ^* induces a homeomorphism $Q : \overline{E(K)} \longrightarrow \overline{E(K)}$ and a continuous map $\tau : \overline{E(K)} - \{p_o\} \longrightarrow T$ such that

$$\Phi^*(e(x)) = \tau(x)\, e(Q(x)) \ \forall \ x \in E(F).$$

Claim : τ restricted to E(F) is continuous in the relative facial topology.

Once we establish the claim, we can define $S : A_o(K) \longrightarrow A_o(K)$ by $S(a)(x) = \overline{\tau}(x)\, \Phi(a)(x) \ \forall \ x \in E(F)$, $a \in A_o(K)$ and as in the proof of Theorem 8.1, can se that the isometry S, maps non-negative functions to non-negative functions, so that S^* maps K' onto it self and the conclusion can be deduced.

Let $B \subseteq T$ be closed and let $B' = \{x \in \overline{E(K)} - \{p_o\} : \tau(x) \in B\} \cup \{p_o\}$. Then B' is a closed set. Let μ be a maximal measure with $x_o = \gamma(\mu) \in B'$ and $x_o \neq p_o$. Let λ_1 and λ_2 be maximal measures on $A_o(K)_1^*$ such that

$$\gamma(\lambda_1) = \frac{e(Q(x_o))}{\| e(Q(x_o)) \|}, \quad \gamma(\lambda_2) = \frac{e(x_o)}{\| e(x_o) \|}.$$

Clearly Supp λ_1, Supp $\lambda_2 \subseteq F'$ (since F' is a complementary face).

It is easy to see that the measure $\lambda' = \lambda_1 \circ \sigma\, \overline{\tau(x_o)} \circ \Phi^*$ represents $\dfrac{e(x_o)}{\| e(x_o) \|}$.

Since $A_o(K)$ is an L^1-predual, by Effros' characterization we get that $\lambda_2 = \lambda_1'$. Clearly $\mathrm{Supp}\, \mu \subseteq \mathrm{Supp}(\lambda_1' \circ e) \bigcup \{p_o\}$. If $x \in \mathrm{Supp}\, \lambda_1' \circ e$ then $e(x) \in \mathrm{Supp}\, \lambda_1'$ and hence

$$\Phi^*(e(x)) = \tau(x_o) e(x'),\ e(x') \in \mathrm{Supp}\, \lambda_1 .$$

It follows from the definition of τ that $\tau(x) = \tau(x_o)$. Hence $\mathrm{Supp}\, \mu \subseteq B'$. Therefore τ is a facially continuous map. This completes the proof.

Remark : The above theorem extends Theorem 13, page 187 [26] for complex simplex spaces.

If one identifies a locally compact Hausdorff space Y as the extreme boundary of the face complementary to $\{\delta(\infty)\}$ in the set of probability measures on the one point compactification $Y \bigcup \{\infty\}$ of Y, then it is not difficult to see that the above theorem is an extension of the classical Banach-Stone theorem for $C_o(Y)$ (continuous functions vanishing at infinity).

Corollary 8.7. Let X be a complex Lindenstrauss space. Suppose F and G are two maximal faces of X_1^* such that $F_1 = \mathrm{co}(F \bigcup \{0\})$, $F_2 = \mathrm{co}(G \bigcup \{0\})$ are w^*-closed. Then there exists an affine homeomorphism (w.r.t w^*-topology) from F_1 onto F_2 mapping F onto G.

Proof : Using results from Section 4 of [36], it is easy to see that F_1 and F_2 are simplexes and $A_o(F_1)$ and $A_o(F_2)$ are isometric. Now an argument similar to the one given in the proof of the above theorem completes the proof.

We now use the description of isometries, to describe bi-contractive projections in $A(K)$ when K is a Choquet simplex.

Let K be any compact convex set and Q an affine homeomorphism of K such that $Q(Q(x)) = x \ \forall x \in E(K)$. Let $a_o \in Z(A(K))$ be such that $|a_o| = 1$ on $E(K)$ and $a_o \circ Q = \bar{a}_o$.

Define $P : A(K) \longrightarrow A(K)$ by $P(a) = \frac{1}{2}\{a + c\}$ where $c \in A(K)$ agrees with the product $a_o . a \circ Q$ on $E(K)$. Easy to see that P is a bi-contractive projection i.e. $\|P\| \leq 1$ and $\|I - P\| \leq 1$ (I is the identity map).

Proposition 8.8. If K is a simplex then projections of the above form completely describe bi-contractive projections in $A(K)$.

Proof : Let $P : A(K) \longrightarrow A(K)$ be any bi-contractive projection and put $S = 2P - I$. Use Theorem 4.5 of [31] to conclude that S is an isometry. By Corollary 8.3, we get an affine homeomorphism Q of K such that

$$(Sa)(x) = S(1)(x)\, a(Q(x)) \quad \forall \ x \in E(K) .$$

Put $a_o = S(1)$. For any $a \in A(K)$

$$S(S(a)) = 2P(2P(a) - a) - 2P(a) + a$$
$$= 4P(a) - 2P(a) - 2P(a) + a = a .$$

Hence for any $x \in E(K)$, $1 = S(a_o)(x) = a_o(x).a_o(Q(x))$.

Therefore $a_o \circ Q = \bar{a}_o$ (since $|a_o| = 1$ on $E(K)$).

Also for $x \in E(K)$, $a \in A(K)$

$$a(x) = S(S(a))(x) = a_o(x)\ S(a)(Q(x))$$

$$= a_o(x)\ a_o(Q(x))\ a(Q^2(x)).$$

Therefore $Q^2(x) = x \quad \forall \quad x \in E(K)$. Now it is easy to see that P has the form required in the proposition.

We end by giving a simple example of a non-simplicial compact convex set K and an isometry Φ of $A(K)$ which is not of the form described in Theorem 8.1.

Let K be the unit square in $\mathbb{R}^2$ centred at $(0,0)$, so $E(K) = \{(x,y) : |x| = 1 = |y|\}$. K has no proper split faces and hence $Z(A(K)) = \{\alpha.1 : \alpha \in \mathbb{C}\}$. Any $f \in A(K)$ is of the form $f(x,y) = ax + by + c$ where $a,b,c \in \mathbb{C}$. Define $\Phi(f)(x,y) = cx + by + a$. Now $\|f\| = \max |a \pm b \pm c|$ and $\|\Phi(f)\| = \max |c \pm b \pm a|$, hence Φ is an isometry. It is obvious that Φ is onto. But $\Phi(1) = x$, non-constant. Hence Φ is not of the form in Theorem 8.1.

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