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A class of *E*-optimal proper efficiency-balanced designs

By ASHISH DAS
Statistics and Mathematics Division, Indian Statistical Institute, Calcutta 700035, India

AND SANPEI KAGEYAMA
Department of Mathematics, Hiroshima University, Hiroshima 734, Japan

SUMMARY

Proper efficiency-balanced designs are considered. Upper bounds are given for the efficiency factor of proper efficiency-balanced designs. A class of *E*-optimal efficiency-balanced designs is observed as designs which attain such an upper bound.

Some key words: Balanced incomplete block design; Canonical efficiency factor; *C*-matrix; Efficiency-balanced design; Efficiency factor; *E*-optimality; Nonbinary.

1. INTRODUCTION

We consider a block design with v treatments and b blocks of size k each, which is also called proper. For it, let $R = \mathrm{diag}(r_1, \ldots, r_v)$, $r = (r_1, \ldots, r_v)'$ and $N = (n_{ij})$, the $v \times b$ incidence matrix of the design, where n_{ij} is the number of times the ith treatment occurs in the jth block and r_i is the replication number of the ith treatment, for $i = 1, \ldots, v$ and $j = 1, \ldots, b$. Under the usual fixed effects, additive, homoscedastic linear model, the coefficient matrix, *C*-matrix, of the reduced normal equations for estimating linear functions of treatment effects is $C = R - k^{-1}NN'$ which is symmetric, nonnegative definite, with zero row and column sums. A design is said to be connected if and only if rank $(C) = v - 1$.

James & Wilkinson (1971) introduced canonical efficiency factors of a block design as the nonzero eigenvalues of $R^{-1}C$. Let e_i $(i = 1, \ldots, v-1)$ be the canonical efficiency factors of a connected design. It is clear that $0 < e_i \leq 1$ $(i = 1, \ldots, v-1)$ for any connected design. The statistical role of e_i's in terms of basic contrasts is discussed by Pearce, Caliński & Marshall (1974) and Caliński (1977). The harmonic mean of the e_i's is here denoted by e and called the efficiency factor of the design,

$$e = (v-1) \Big/ \sum_{i=1}^{v-1} e_i^{-1}. \tag{1·1}$$

A design is said to be efficiency-balanced if the e_i's are all equal, that is $e_i = e$ for all i, in which case (1·1) attains its relative maximum. For such a design, every treatment contrast is estimated with the same efficiency factor e (Jones, 1959; Caliński, 1971; Puri & Nigam, 1975). Williams (1975) showed that a necessary and sufficient condition for a proper connected design to be efficiency-balanced is that the incidence matrix N satisfies

$$NN' = k(1-e)R + (e/b)rr'. \tag{1·2}$$

It is known (Kageyama, 1980) that a binary proper efficiency-balanced design is a balanced incomplete block design. Thus, in this paper we deal with nonbinary block designs only.

Alternatively to (1·1), the efficiency factor of a block design has been defined traditionally by $e' = H/\bar{r}$, with H denoting the harmonic mean of the nonzero eigenvalues of the *C*-matrix and with $\bar{r} = bk/v$ (Raghavarao, 1971, § 4.5). Maximizing the efficiency factor e' is equivalent to finding an *A*-optimal design (Shah & Sinha, 1989, p. 45). Evidently, it is not possible to maximize e and

e' simultaneously, unless the design is equireplicate. Instead, however, one can search for designs which maximize the efficiency factor e and, at the same time, are E-optimal, i.e. maximize the minimum positive eigenvalue of the C-matrix (Shah & Sinha, 1989, p. 31).

The object of the present paper is to provide efficiency-balanced designs of unequal replications that still have some desirable optimality properties, and so to extend the optimality results given for equireplicate designs by Shah & Sinha (1989). First, some bounds on the efficiency factor of connected block, including efficiency-balanced, designs have been obtained, particularly applicable for designs with unequal replications. Then a class of E-optimal efficiency-balanced designs which attain an upper bound for the efficiency factor (1·1) is given.

2. Bounds on efficiency factor

Upper bounds are derived in terms of the design parameters only. The efficiency factor of a connected block design with v treatments and b blocks of size k each has, as in (1·1),

$$e = (v-1) \Big/ \sum_{i=1}^{v-1} e_i^{-1} \leq (v-1)^{-1} \sum_{i=1}^{v-1} e_i$$
$$= (v-1)^{-1} \operatorname{tr}(R^{-1}C) \leq v(k-1)/\{k(v-1)\},$$

since

$$\operatorname{tr}(R^{-1}C) \leq v - k^{-1} \textstyle\sum_i \sum_j n_{ij}/r_i = v(k-1)/k.$$

Thus, one has the following.

THEOREM 2·1. *In a connected design with v treatments and b blocks of size k each,*

$$e \leq (v-1)^{-1} \operatorname{tr}(R^{-1}C) \leq v(k-1)/\{k(v-1)\},$$

the first equality holding if and only if the design is efficiency-balanced, the second if and only if it is, in addition, binary and thus a balanced incomplete block design.

If bk/v is not an integer, there does not exist a connected design which attains the second bound as in Theorem 2·1. In an efficiency-balanced design with v treatments and b blocks of size k each, by (1·2), the efficiency factor has the form

$$e = (bk - k^{-1} \textstyle\sum_i \sum_j n_{ij}^2)/\{bk - (bk)^{-1} \sum_i r_i^2\},$$

which cannot be expressed in terms of the design parameters only, unless the design is binary. Thus the following result is of interest, especially from an application point of view.

THEOREM 2·2. *For an efficiency-balanced design with v treatments and b blocks of size k each, in which $r_1 \leq r_2 \leq \ldots \leq r_v$,*

$$e \leq b(k-1)/(bk - r_1) = e_I,$$

say.

Proof. By (1·2), $(e/b)r_i^2 + k(1-e)r_i \geq r_i$, that is $e \leq b(k-1)/(bk - r_i)$ for all $i = 1, \ldots, v$, which completes the proof. □

Note that a less sharp bound on e can be obtained by replacing r_1 in e_I by $[bk/v]$, the largest integer not exceeding bk/v. Hence one has the following.

COROLLARY 2·1. *In an efficiency-balanced design with v treatments and b blocks each of size k, the inequalities*

$$e \leq b(k-1)/(bk - [bk/v]) = e_I^*,$$

say and

$$e \leq e_I \leq e_I^* \leq v(k-1)/\{k(v-1)\}$$

hold, $e_I = e_I^*$ if $r_1 = [bk/v]$, the last inequality becoming an equality if bk/v is an integer.

Note that unlike for e_t in Theorem 2·2, the bound e_t^* in Corollary 2·1 is expressed in terms of the given parameters v, b and k only.

3. E-OPTIMALITY

A class of designs, each with v treatments and b blocks of size k, is now considered. A proper efficiency-balanced design attaining the bound e_t^*, as in Corollary 2·1, can be found in a class of E-optimal designs for given v, b and k.

Das & Dey (1991) showed that for any integer p such that $0 < p < \frac{1}{2}v$ there exists a proper efficiency-balanced design d with parameters $v-p$, b, $r_1 = r$, $r_2 = 2r$, k and $e = \lambda v/(rk)$ in a class of block designs with $v-p$ treatments and b blocks of size k each. That design is also E-optimal, provided

(i) $v - p(1+r) \geq 2$,

(ii) $v - p - (v-p)(v-p-pr)^{-1} \geq p\lambda$.

Here v, b, r, k, λ are the parameters of an existing balanced incomplete block design. In fact, the two conditions (i) and (ii) are required only for E-optimality in the class of all connected block designs with given $v-p$, b and k. Their method of constructing the above E-optimal proper efficiency-balanced design is to merge p pairs of treatments in a balanced incomplete block design with parameters v, b, r, k and λ, while the remaining $v-2p$ treatments are unaltered. This yields our proper efficiency-balanced designs which may be mostly nonbinary.

Since

$$\frac{\lambda v}{rk} = \frac{b(k-1)}{bk - \lfloor bk/(v-p) \rfloor}$$

holds if and only if $0 \leq p < v/(r+1)$, the following can be obtained.

THEOREM 3·1. *A proper efficiency-balanced design d attaining the bound e_t^* on the efficiency factor can be constructed from a balanced incomplete block design with parameters v, b, r, k, λ, if there exists a positive integer p such that $p < v/(r+1)$.*

If the upper bound as in Theorem 2·2 is taken, similarly, applying the result of Das & Dey (1991) one can show that a proper efficiency-balanced design d with minimum replication number $r_1 = r$ attaining the bound e_t on the efficiency factor can be constructed from a balanced incomplete block design with parameters v, b, r, k, λ for all integers p satisfying $0 < p < \frac{1}{2}v$. Note that all the existing balanced incomplete block designs with $v > 2$ satisfy this statement.

Next the integer p satisfying (i) and (ii) for efficiency-balanced designs in Theorem 3·1 is investigated. We have that (i) implies $p \leq (v-2)/(r+1)$. On the other hand, (ii) yields

$$(\lambda+1)(r+1)p^2 - \{v(r+\lambda+2)-1\}p + v(v-1) \geq 0,$$

that is $p \leq p_-$ or $p \geq p_+$, where

$$p_\pm = \{2(\lambda+1)(r+1)\}^{-1}[v(r+\lambda+2) - 1 \pm [\{v(r+\lambda+2)-1\}^2 - 4v(v-1)(\lambda+1)(r+1)]^{\frac{1}{2}}].$$

Note that $(v-2)/(r+1) \leq p_-$ if and only if $v(r-2\lambda) \geq -2(2\lambda+1)$ which is always valid when $r \geq 2\lambda$. Thus, one has the following.

THEOREM 3·2. *An E-optimal efficiency-balanced design d attaining the bound e_t^* on the efficiency factor can be constructed from a balanced incomplete block design with parameters v, b, r, k, λ satisfying $v(r-2\lambda) \geq -2(2\lambda+1)$ if there exists a positive integer p for $p \leq (v-2)/(r+1)$.*

COROLLARY 3·1. *An E-optimal efficiency-balanced design d attaining the bound e_t^* on the efficiency factor can be constructed from a balanced incomplete block design with parameters v, b, $r \geq 2\lambda$, k, λ if there exists a positive integer p such that $p \leq (v-2)/(r+1)$.*

When s is a prime or a prime power, there exists a balanced incomplete block design with parameters

$$v=b=s^2+s+1, \quad r=k=s+1, \quad \lambda=1$$

(Raghavarao, 1971, p. 77), which satisfies Corollary 3·1 for $p \leqslant s-1$. Thus there always exists for $p \leqslant s-1$ an E-optimal efficiency-balanced design with parameters

$$v=s^2+s+1-p, \quad b=s^2+s+1, \quad r_1=s+1, \quad r_2=2(s+1), \quad k=s+1, \quad e=1-\frac{s}{(s+1)^2},$$

attaining the bound e_1^*, provided s is a prime or a prime power. For example, in a balanced incomplete block design with parameters $v=b=7$, $r=k=3$, $\lambda=1$ whose blocks are

$$(1, 2, 4)(2, 3, 5)(3, 4, 6)(4, 5, 7)(5, 6, 1)(6, 7, 2)(7, 1, 3),$$

merging two treatments 6 and 7 yields a proper efficiency-balanced design with parameters $v=6$, $b=7$, $r_1=3$, $r_2=6$, $k=3$ and $e=\frac{7}{9}$, whose blocks are given by

$$(1, 2, 4)(2, 3, 5)(3, 4, 6)(4, 5, 6)(5, 6, 1)(6, 6, 2)(6, 1, 3),$$

which is E-optimal in the class of all connected proper block designs with given $v=6$, $b=7$ and $k=3$. Furthermore, our bound e_1^* is attained here.

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