On Weakly Completely Mixed Bimatrix Games

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ABSTRACT

Weakly completely mixed bimatrix games are defined to be games with a completely mixed Nash component. For these games this component turns out to consist of only one point, which is isolated. Special classes of these games are completely mixed matrix and bimatrix games, the first introduced by Kaplansky, the latter by Raghavan. We give a characterization of these games, which can be used for completely mixed matrix games also. Given a completely mixed strategy pair, we are able to construct a (weakly) completely mixed bimatrix game having this pair as an equilibrium. We derive interesting results for the case where the payoff matrices have a nonnegative and irreducible inverse.

1. INTRODUCTION

An interesting subclass of matrix games is the class of completely mixed matrix games. By definition all equilibria of such games are completely mixed, i.e. give to every pure strategy of both players some relevance. Kaplansky [7] was able to calculate the value of an arbitrary matrix game by considering completely mixed subgames. Blackwell [1] gave an alternative proof of the Perron-Frobenius theorem by constructing special completely mixed games. Since then many authors have studied the relationship between completely mixed matrix games and the eigenvalues of the matrix involved [12, 15–17]. Raghavan showed that many of the characterizations of M-matrices mentioned in the literature are consequences of the theory of completely mixed matrix games and the Perron-Frobenius theorem [14].

These results for completely mixed matrix games are a good motivation to consider completely mixed equilibria in bimatrix games. Raghavan [13] extended the definition of completely mixed games to bimatrix games and found that a completely mixed bimatrix game posesses a unique and completely mixed equilibrium (also cf. [5] and [11]). Earlier Kaplansky [7] had shown that a similar result holds for completely mixed matrix games.

In general the set of equilibria of a bimatrix game consists of a finite number of convex components [6, 8]. In this paper we introduce and study *weakly completely mixed* bimatrix games; a game is defined to be weakly completely mixed if at least one convex component of the set of equilibria consists of only completely mixed equilibria. This definition implies that completely mixed bimatrix games are weakly completely mixed.

The organization of the paper is as follows. In Section 2 we recall some basic facts of the theory of bimatrix games, and we concentrate on maximal Nash subsets. In Section 3 we define and characterize weakly completely mixed bimatrix games. We find that such games have a unique completely mixed equilibrium. The difference from completely mixed bimatrix games is that other equilibria may exist. Furthermore we sum up some properties of weakly completely mixed bimatrix games. In Section 4 we show that for a given pair of completely mixed strategies it is possible to construct a weakly completely mixed bimatrix game having this pair as an equilibrium. We define a special type of games for this purpose. For games in this typical form we are able to decide under what conditions they are completely mixed. Finally, in Section 5, we consider bimatrix games consisting of two matrices that have nonsingular irreducible and nonnegative inverse matrices. These games are weakly completely mixed, and for each player the payoffs yielded by the completely mixed equilibrium are strictly smaller than the payoffs yielded by any other equilibrium. Furthermore we derive that the absolute value of any eigenvalue of the matrices is strictly larger than the payoff yielded by the completely mixed equilibrium, i.e. with respect to the matrix in question.

NOTATION. By $\mathbf{N}$ we denote the set of positive integers. $\mathbf{R}^n$ is the set of n-tuples of real numbers. For a convex set V we denote by relint$(V) \neq \varnothing$ the interior of V with respect to its affine hull. For a finite set V we denote by $|V|$ the number of elements in V. The rank of a matrix A is denoted by rank(A). A matrix A is called strictly positive, and we write $A > 0$, if all the entries of A are positive. A matrix A is called positive or nonnegative if all its entries are nonnegative. Similar definitions are used for vectors in $\mathbf{R}^n$, $n > 1$. We let $e_1, e_2, \dots, e_n$ be the standard basis in $\mathbf{R}^n$. By 1_n we mean the vector in $\mathbf{R}^n$ with the integer 1 in all its entries. For $x, y \in \mathbf{R}^n$, $(x \cdot y) = \sum_{i=1}^{n} x_i y_i$.

2. BIMATRIX GAMES AND MAXIMAL NASH SUBSETS

An $m \times n$ game is a two-person game defined by a pair (A, B) of real $m \times n$ matrices. In a play of such a game each of two players chooses a (mixed) strategy, which is an element of the set Δ_m for the first player and Δ_n for the second player. Here for $t \in \mathbf{N}$

$$\Delta_t = \left\{ p \in \mathbf{R}^t \,\middle|\, p \geqslant 0,\ \sum_{i=1}^{t} p_i = 1 \right\}.$$

Corresponding to the strategy pair $(p, q) \in \Delta_m \times \Delta_n$, the payoffs in this game to player 1 and 2 are pAq and pBq respectively. A pair $(\bar{p}, \bar{q}) \in \Delta_m \times \Delta_n$ is called an *equilibrium* of the bimatrix game (A, B) if $\bar{p}A\bar{q} = \max_{p \in \Delta_m} pA\bar{q}$ and $\bar{p}B\bar{q} = \max_{q \in \Delta_n} \bar{p}Bq$. $E(A, B)$ denotes the set of all equilibria of the game (A, B). The set $E(A, B)$ is nonempty by a theorem of Nash [9, 10].

For a strategy $p \in \Delta_t$ we define its *carrier* by

$$C(p) = \{i \in \{1, \ldots, t\} \mid p_i > 0\}.$$

The *set of pure best answers* for player 1 to the strategy q of player 2 is defined by

$$\mathrm{PB}_1(q) = \left\{ i \in \{1, \ldots, m\} \,\middle|\, e_i A q = \max_{k \in \{1, \ldots, m\}} e_k A q \right\},$$

and

$$\mathrm{PB}_2(p) = \left\{ j \in \{1, \ldots, n\} \,\middle|\, pBe_j = \max_{k \in \{1, \ldots, n\}} pBe_k \right\}$$

denotes the set of pure answers for player 2 to the strategy p of player 1. It is a well-known result that, for a given pair (p, q) of strategies, $(p, q) \in E(A, B)$ if and only if $C(p) \subset \mathrm{PB}_1(q)$ and $C(q) \subset \mathrm{PB}_2(p)$.

Let $S \subset E(A, B)$. Two points (p, q) and (p', q') in S are called S-interchangeable if $(p, q') \in S$ and $(p', q) \in S$. We call S a *Nash subset* for the game (A, B) if every pair of equilibria in S is S-interchangeable. A Nash subset S is called *maximal* if there does not exist a Nash subset $T \subset E(A, B)$ such that S is properly contained in T. Note that a maximal Nash subset is convex.

The following two theorems concerning maximal Nash subsets are due to Jansen [6]:

THEOREM 1. *The set of equilibria of a bimatrix game is the not necessarily disjoint union of a finite number of maximal Nash subsets.*

THEOREM 2. *Let (A, B) be an $m \times n$ bimatrix game, and let S be a maximal Nash subset for this game. Suppose that $(\hat{p}, \hat{q}) \in \mathrm{relint}(S)$. Then $S = K(\hat{q}) \times L(\hat{p})$. Here $K(\hat{q}) = \{p \in \Delta_m \mid (p, \hat{q}) \in E(A, B)\}$ and $L(\hat{p}) = \{q \in \Delta_n \mid (\hat{p}, q) \in E(A, B)\}$.*

REMARK 1. A matrix game A can be seen as a bimatrix game $(A, -A)$. It is well known that $E(A, -A)$ is the only maximal Nash subset for $(A, -A)$ and that for each player the payoff is independent of the equilibrium.

3. WEAKLY COMPLETELY MIXED BIMATRIX GAMES

In this section we define and give a characterization of weakly completely mixed bimatrix games. Furthermore we give some properties of these games.

DEFINITION. For $t \in \mathbf{N}$, a strategy $x \in \Delta_t$ is said to be completely mixed if $x \in \mathring{\Delta}_t := \text{relint}(\Delta_t)$.

A game (A, B) is called *completely mixed* if for all $(p, q) \in E(A, B)$ both p and q are completely mixed strategies. In this case we call such an equilibrium (p, q) a *completely mixed equilibrium*. We define a game (A, B) to be *weakly completely mixed* if there exists a maximal Nash subset $S \subset E(A, B)$ such that all equilibria in S are completely mixed.

Raghavan [13] found that a completely mixed $m \times n$ bimatrix game (A, B) has the properties $m = n$, $\text{rank}(A), \text{rank}(B) \in \{n-1, n\}$, and $|E(A, B)| = 1$. So the set of equilibria of a completely mixed bimatrix game only contains one maximal Nash subset, which consists of only one (completely mixed) equilibrium. This implies that a completely mixed bimatrix game is weakly completely mixed. In view of Remark 1 we find that, if we restrict to matrix games, the definitions of weakly completely mixed and completely mixed coincide.

In the following theorem we characterize the class of weakly completely mixed bimatrix games. Since the equilibrium set of a bimatrix game does not change if the same constant is added to all the entries of one of the matrices, we may suppose without loss of generality that both matrices are strictly positive.

THEOREM 3. *Let $A > 0$ and $B > 0$. Then (A, B) is a weakly completely mixed bimatrix game if and only if:*

(1) *A and B are square matrices, say $n \times n$;*
(2) *A and B have full rank;*
(3) *all entries of both $\mathbf{1}_n B^{-1}$ and $A^{-1}\mathbf{1}_n$ are positive.*

Proof. $\Rightarrow$: Let (A, B) be a weakly completely mixed bimatrix game, and let S be a completely mixed minimal Nash subset for (A, B). Take $(p, q) \in \text{relint}(S)$.

Suppose $\text{rank}(B) \leqslant m - 2$. Then there are two linearly independent solutions of $xB = 0$. At least one of these, say z, is linearly independent of p. First suppose $\sum_{i=1}^m z_i \neq 0$. Without loss of generality we may suppose $\sum_{i=1}^m z_i = 1$. Now let $p' = (\lambda + 1)p - \lambda z$, with $\lambda > 0$. We want p' to be a strategy with at least one coordinate equal to zero. Therefore we take λ such that $\lambda^{-1}(\lambda+1) = \max_{i \in \{1,\dots,m\}} z_i p_i^{-1}$. Then $p'Aq = pAq$. Furthermore $p'Bq = (1+\lambda)pBq \geqslant (1+\lambda)pB\hat{q} = p'B\hat{q}$ for all $\hat{q} \in \mathring{\Delta}_n$. Hence $p' \in K(q)$. According to Theorem 2 we now have $(p', q) \in S$. This contradicts S being completely mixed. Thus $\operatorname{rank}(B) \geqslant m-1$ if $\sum_{i=1}^m z_i \neq 0$. a similar proof handles the case for which $\sum_{i=1}^m z_i = 0$, if one chooses $p' = p - \lambda z$ for a suitable λ. We conclude $\operatorname{rank}(B) \geqslant m-1$.

Suppose $\operatorname{rank}(B) = m-1$. Then there exists a $y \in \mathbf{R}^m$, $y \neq 0$, such that $yB = 0$. However if y is linearly independent of p, we can again construct a p' that is not completely mixed while $(p', q) \in S$. This contradicts S being completely mixed. Therefore $pB = 0$. However, this is impossible, since $B > 0$ and $p > 0$. Hence $\operatorname{rank}(B) = m$. Similarly one proves $\operatorname{rank}(A) = n$. Since $\operatorname{rank}(B) = m$ implies that $n \geqslant m$ and $\operatorname{rank}(A) = n$ implies that $m \geqslant n$, we have $m = n$. Thus we have proved (1) and (2).

Then, since $pB = \lambda \mathbf{1}_n$ and $Aq = \mu \mathbf{1}_n$ for some positive reals λ and μ, we obtain $\mathbf{1}_n B^{-1} = \lambda^{-1} p$ and $A^{-1}\mathbf{1}_n = \mu^{-1} q$. Hence also (3) is fulfilled.

$\Leftarrow$: Assume that (1), (2), and (3) are fulfilled. Define $p = (\mathbf{1}_n B^{-1} \mathbf{1}_n)^{-1} \mathbf{1}_n B^{-1} > 0$ and $q = (\mathbf{1}_n A^{-1} \mathbf{1}_n)^{-1} A^{-1} \mathbf{1}_n > 0$. Then $pBe_j = (\mathbf{1}_n B^{-1} \mathbf{1}_n)^{-1}$ for all j, and $e_i Aq = (\mathbf{1}_n A^{-1} \mathbf{1}_n)^{-1}$ for all i. Hence (p, q) is a completely mixed equilibrium of the game (A, B). Since A and B are nonsingular, and $\sum_{i=1}^n p_i = 1$ and $\sum_{j=1}^n q_j = 1$, we find that (p, q) is the only completely mixed equilibrium. Moreover, (p, q) is isolated or else other completely mixed equilibria exist. So $\{(p, q)\}$ is a completely mixed maximal Nash subset. This shows that (A, B) is weakly completely mixed. ∎

Now we sum up some results for "general" bimatrix games, i.e. not necessarily with two strictly positive matrices.

Carefully looking at the proof of Theorem 3, we find that a bimatrix with two strictly positive matrices that is weakly completely mixed posesses a unique and isolated completely mixed equilibrium. However, then also in general

Proposition 1. *A weakly completely mixed bimatrix game has a unique and isolated completely mixed equilibrium.*

In the $\Leftarrow$ part of the proof above we did not use the fact that A and B are strictly positive; hence:

Proposition 2. *If for an ordered pair of matrices (A, B) conditions (1), (2), and (3) of Theorem 3 hold, then (A, B) is a weakly completely mixed bimatrix game.*

Suppose (A, B) is weakly completely mixed. Then we add constants c and d to the entries of A and B respectively, such that the resulting matrices A' and B' are strictly positive. Then A' and B' have properties (1), (2), and (3) of Theorem 3, and consequently $-A'$ and $-B'$ have properties (1) and (2) of Theorem 3. Following the $\Leftarrow$ part of the proof of Theorem 3, we find that $(-A', -B')$ is weakly completely mixed. Then, by adding c to every entry of $-A'$ and d to every entry of $-B'$, we find

PROPOSITION 3. *If (A,B) is weakly completely mixed, then also $(-A,-B)$ is weakly completely mixed.*

The 3×3 bimatrix game given by

$$(A,B)=\begin{bmatrix}(1,0) & (0,0) & (0,1)\\ (0,1) & (1,0) & (0,0)\\ (0,0) & (0,1) & (1,0)\end{bmatrix}$$

is completely mixed, since the only equilibrium is $((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3}))$. This strategy pair is also an equilibrium of $(-A,-B)$. However, $(-A,-B)$ is not completely mixed, since e.g. (e_1, e_2) is also equilibrium of this game. So in general Proposition 2 does not hold for completely mixed bimatrix games.

Let B' be the matrix obtained from B above by exchanging the first and the second column of B. Then (e_2, e_2) is an equilibrium of (A, B'), and hence (A,B') is not completely mixed. However, (A,B') is weakly completely mixed, as the next proposition tells us. First we need a definition.

DEFINITION. An $n\times n$ matrix P obtained from the $n\times n$ identity matrix by permuting its columns is called a *permutation matrix*.

With this definition at hand it follows easily that

PROPOSITION 4. *If (A,B) is weakly completely mixed, then the permuted game (P_1AP_2, P_3BP_4) is also weakly completely mixed, where P_i is a permutation matrix for all i.*

Form the $\Leftarrow$ part of the proof of Theorem 3 we obtain that the completely mixed equilibrium (p,q) of a weakly completely mixed bimatrix game (A,B) with two strictly positive matrices satisfies the equations

$$p=\frac{1_n B^{-1}}{1_n B^{-1} 1_n} \quad \text{and} \quad q=\frac{A^{-1}1_n}{1_n A^{-1} 1_n}.$$

Then, since A and B are nonsingular, it is straightforward to show that the set of all $n\times n$ weakly completely mixed bimatrix games is an open subset of the set of all $n\times n$ bimatrix games.

Hence small perturbations of the matrices of a weakly completely mixed bimatrix game lead to matrices that again form a weakly completely mixed game. The completely mixed equilibrium of the perturbed game can be calculated as in Cohen [3], where perturbations of completely mixed bimatrix games were considered. Perturbations theory for completely mixed matrix games has been studied in Cohen [2].

REMARK 2. Suppose (A, B) is a weakly completely mixed $n \times n$ bimatrix game and that A is strictly positive and B is singular. Closely looking at the proof of Theorem 3 we find that in that case $\operatorname{rank}(B) = n - 1$. Consequently there exists a $z \in \mathbf{R}^n$, $z \neq 0$, such that $Bz = 0$. Let us add to every entry of B a constant c such such that the resulting matrix B' is strictly positive. Then also (A, B') is weakly completely mixed. Hence, by Theorem 3, B' is nonsingular. Consequently $0 \neq B'z = c(\mathbf{1}_n \cdot z)\mathbf{1}_n$. So c is nonzero and $(\mathbf{1}_n \cdot z) \neq 0$.

Now suppose B is a singular $n \times n$ matrix with the property $\operatorname{rank}(B) = n - 1$, which, for eigenvalue zero, has a left eigenvector $p > 0$ and a right eigenvector that is *not* perpendicular to $\mathbf{1}_n$. Let us add to every entry of B a constant c such that the resulting matrix B' is strictly positive. Then $\tilde{p}B' = c\mathbf{1}_n$, where $\tilde{p} = (p \cdot \mathbf{1}_n)^{-1}p \in \mathring{\Delta}_n$. Suppose that $B'u = 0$ for some $u \in \mathbf{R}^n$ with $u \neq 0$. Then $0 = \tilde{p}B'u = c(\mathbf{1}_n \cdot u)$, or $(\mathbf{1}_n \cdot u) = 0$. However, this implies $0 = B'u = Bu + c(\mathbf{1}_n \cdot u) = Bu$, which contradicts our assumption. Hence B' is nonsingular. Now take a nonsingular $n \times n$ matrix A satisfying $A^{-1}\mathbf{1}_n > 0$. Then (A, B') satisfies all three conditions of Theorem 3 and is therefore weakly completely mixed. This implies that also (A, B) is weakly completely mixed.

Thus, by the arguments above, we have found a characterization similar to the one in Theorem 3 for the case of a singular matrix B. Conditions (2) and (3), with respect to B, are altered into:

(2) $\operatorname{rank}(B) = n - 1$.

(3) For eigenvalue zero, B has a left eigenvector that is strictly positive and a right eigenvector that is not perpendicular to $\mathbf{1}_n$.

A similar characterization can be given in the case of a singular matrix A.

4. CONSTRUCTION PROBLEMS

From the previous section we learn that weakly completely mixed bimatrix games have square matrices and possess a unique completely mixed equilibrium. Furthermore, completely mixed bimatrix games are weakly completely mixed games. It is then natural to ask the following questions.

Given a pair $(p,q)\in \mathring{\Delta}_n \times \mathring{\Delta}_n$, can we construct a weakly completely mixed $n\times n$ bimatrix game for which (p,q) is an equilibrium? If we can do this, then under what conditions is the game completely mixed? Or equivalently, under what conditions is (p,q) then the only equilibrium of the game?

In this section we construct a class of bimatrix games with help of which we can obtain an answer to the first question. It is also possible for games in this class to give an exact answer to the second question.

DEFINITION. A bijection $\pi:\{1,\dots,n\}\to\{1,\dots,n\}$ is called *irreducible* if there exists no proper subset $I\subset\{1,\dots,n\}$ such that $\pi(I)=I$.

THEOREM 4. *Let* $(p,q)\in \mathring{\Delta}_n \times \mathring{\Delta}_n$ *be given. Let* π *be a bijection on* $\{1,\dots,n\}$. *Let A and B be the $n\times n$ matrices defined by*

$$Be_j = p_j^{-1}e_j$$

for all j, and

$$e_iA = q_{\pi(i)}^{-1}e_{\pi(i)}$$

for all .

Then the bimatrix game (A,B) is weakly completely mixed. It is completely mixed if and only if π is irreducible. For all bijections π the unique completely mixed equilibrium is (p,q).

Proof. The proof consists of four steps.

First we prove that (A,B) is weakly completely mixed. Note that A and B are nonsingular and square. In fact $1_nB^{-1}e_j=p_j>0$ for all j, and $e_iA^{-1}1_n=q_i>0$ for all i. By Proposition 2 the game (A,B) is weakly mixed.

Secondly we show that $(p,q)\in E(A,B)$, which assertion is, by Proposition 1, equivalent to the statement that (p,q) is the unique completely mixed equilibrium of (A,B). We have $pBe_j=1$ for all j and $e_iAq=1$ for all i. Hence $C(p)=\mathrm{PB}_1(q)$ and $C(q)=\mathrm{PB}_2(p)$. Consequently $(p,q)\in E(A,B)$.

In the third step we prove that (A,B) is completely mixed for every irreducible π. Let $(x,y)\in E(A,B)$. Then $C(y)\subset \mathrm{PB}_2(x)$ and $C(x)\subset \mathrm{PB}_1(y)$. Furthermore, for this equilibrium we also have

$$\begin{aligned}\mathrm{PB}_2(x) &= \left\{ j\in\{1,\dots,n\} \,\middle|\, xBe_j=\max_{k\in\{1,\dots,n\}} xBe_k\right\}\\ &= \left\{ j\in\{1,\dots,n\} \,\middle|\, x_jp_j^{-1}=\max_{k\in\{1,\dots,n\}} x_kp_k^{-1}\right\}\\ &\subset C(x)\end{aligned}$$

and

$$\mathrm{PB}_1 = \left\{ i \in \{1,\ldots,n\} \,\middle|\, e_i A y = \max_{k \in \{1,\ldots,n\}} e_k A y \right\}$$

$$= \left\{ i \in \{1,\ldots,n\} \,\middle|\, y_{\pi(i)} q_{\pi(i)}^{-1} = \max_{k \in \{1,\ldots,n\}} y_{\pi(k)} q_{\pi(k)}^{-1} \right\}$$

$$\subset \{ i \in \{1,\ldots,n\} \,|\, \pi(i) \in C(y) \}.$$

Hence $C(y) \subset \{i \in \{1,\ldots,n\} \mid \pi(i) \in C(y)\} = \{\pi^{-1}(i) \in \{1,\ldots,n\} \mid i \in C(y)\}$. Since π is bijection, we find $\pi(C(y)) = C(y)$. Now if π is irreducible, this means $C(y) = \{1,\ldots,n\}$. Thus we obtain $\{1,\ldots,n\} = C(y) \subset \mathrm{PB}_2(x) \subset C(x)$. Equivalently, (x,y) is completely mixed. Then in view of the $\Leftarrow$ part of the proof of Theorem 3, we find $(x,y) = (p,q)$.

Finally we have to prove that if (A,B) is a completely mixed game, then π is irreducible. Suppose π is not. Then there is a proper subset $I \subset \{1,\ldots,n\}$ such that $\pi(I) = I$. Define $(x,y) \in \Delta_n \times \Delta_n$ by

$$x_i = \begin{cases} \left(\sum_{k \in I} p_k\right)^{-1} p_i & \text{if } i \in I, \\ 0 & \text{otherwise} \end{cases}$$

and

$$y_j = \begin{cases} \left(\sum_{k \in I} q_k\right)^{-1} q_j & \text{if } j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$xBe_j = \begin{cases} \left(\sum_{k \in I} p_k\right)^{-1} & \text{if } j \in I, \\ 0 & \text{otherwise,} \end{cases}$$

$$e_i A y = \sum_{l \in I} (e_i A e_l) q_l \left(\sum_{k \in I} q_k \right)^{-1}$$

$$= \sum_{l \in I} (e_{\pi(i)} \cdot e_l) q_l q_{\pi(i)}^{-1} \sum_{k \in I} q_k^{-1}$$

$$= \begin{cases} \left(\sum_{k \in I} q_k\right)^{-1} & \text{if } i \in I, \\ 0 & \text{otherwise,} \end{cases}$$

and we see that $C(x) = \mathrm{PB}_1(y)$ and $C(y) = \mathrm{PB}_2(x)$, or equivalently, that $(x, y) \in E(A, B)$. By construction (x, y) is not completely mixed. This contradicts our assumption. Hence π must be irreducible. The proof of the theorem is completed. ∎

Without proof we give an extension of the result in Theorem 4 which tells us that if we also perform a permutation on the above matrix B, we have a completely mixed game iff the composition of the two permutations is an irreducible permutation.

THEOREM 5. *Let $(p, q) \in \mathring{\Delta}_n \times \mathring{\Delta}_n$ be given. Let π and ρ be bijections on $\{1, \ldots, n\}$. Let A and B be the $n \times n$ matrices defined by*

$$Be_j = p^{-1}_{\rho(j)} e_{\rho(j)}$$

for all $j \in \{1, \ldots, n\}$, and

$$e_i A = q^{-1}_{\pi(i)} e_{\pi(i)}$$

for all $i \in \{1, \ldots, n\}$.

Then the bimatrix game (A, B) is weakly completely mixed. It is completely mixed if and only if $\pi\rho$ is irreducible. For all bijections π and ρ the unique completely mixed equilibrium is (p, q).

EXAMPLE 1. Let (A, B) be the bimatrix game given by

$$(A, B) = \begin{bmatrix} (0,3) & (3,0) & (0,0) \\ (3,0) & (0,0) & (0,3) \\ (0,0) & (0,3) & (3,0) \end{bmatrix}.$$

In the notation of Theorem 5 we can obtain A and B by defining π by

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and ρ by

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

The permutations π and ρ are both *not* irreducible, but $\pi\rho$ is, since it equals

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Hence (A, B) is completely mixed by Theorem 5.

6. PAYOFFS AND EIGENVALUES

In this section we describe a subclass of weakly completely mixed bimatrix games for which the payoffs yielded by the completely mixed equilibrium are smaller than those yielded by other equilibria. Moreover, for the same subclass this pair of payoffs is also smaller than the absolute value of every eigenvalue of the pair of matrices to which they belong. For the specification of the subclass we need the notion of an irreducible matrix.

DEFINITION. A square matrix A is called *reducible* if there is a permutation matrix P such that

$$PAP = \begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square matrices. Otherwise A is called *irreducible*.

In the proof of the theorem we use the Perron-Frobenius theorem, which says:

Every nonnegative irreducible $n \times n$ matrix A has a positive eigenvalue λ_0 which is at least as large as the absolute value of any other eigenvalue. This eigenvalue is simple, and its eigenvector has all coordinates strictly of the same sign.

(See e.g. [4, p. 53]).

THEOREM 6. *Let A^{-1} and B^{-1} be two nonnegative irreducible $n \times n$ matrices. Then (A, B) and (B, A) are weakly completely mixed bimatrix games.*

Let (p, q) be the completely mixed equilibrium of (A, B). The pAq is strictly smaller than the absolute value of every eigenvalue of A, and pBq is strictly smaller than the absolute value of every eigenvalue of B.

Furthermore, let $(\bar{p}, \bar{q}) \in E(A, B)$ *be another equilibrium. Then* $pAq < \bar{p}A\bar{q}$ *and* $pBq < \bar{p}B\bar{q}$. *Similar statements hold for* (B, A).

Proof. Since A^{-1} and B^{-1} are nonsingular matrices, they cannot have a row or column with just zero entries. They are also nonnegative, and therefore both (A, B) and (B, A) are weakly completely mixed by Proposition 2. We only prove the assertions for the game (A, B), and with respect to this game only for B. According to the Perron-Frobenius theorem we can find a positive $\lambda_0 \in \mathbf{R}$ and a strictly positive $z \in \mathbf{R}^n$ such that $B^{-1}z = \lambda_0 z$. All other eigenvalues can be ordered so that $\lambda_0 \geqslant |\lambda_1| \geqslant \cdots \geqslant |\lambda_s|$, $s \leqslant n-1$. For $i \in \{1, \dots, s\}$ we have $By_i = \lambda_i^{-1} y_i$ if y_i is an eigenvector of B^{-1} belonging to the eigenvalue λ_i. Also $Bz = \lambda_0^{-1} z$. Hence we have the ordering $\lambda_0^{-1} \leqslant |\lambda_1^{-1}| \leqslant \cdots \leqslant |\lambda_s^{-1}|$ for the eigenvalues of B. Since (p, q) is a completely mixed equilibrium, it follows that $pBz = (pBq)(\mathbf{1}_n \cdot z) = \lambda_0^{-1}(p \cdot z) < \lambda_0^{-1}(\mathbf{1}_n \cdot z)$, or $pBq < \lambda_0^{-1}$. This proves the first assertion.

Let $\alpha = (\bar{p}B\bar{q})\mathbf{1}_n - \bar{p}B$. Then $\alpha \geqslant 0$, and since $(\bar{p}, \bar{q})$ is not completely mixed, it follows that there is at least one $j \in \{1, \dots, n\}$ for which $\alpha_j > 0$. This yields $\bar{p}B = (\bar{p}B\bar{q})\mathbf{1}_n - \alpha$, or $\bar{p} = (\bar{p}B\bar{q})\mathbf{1}_n B^{-1} - \alpha B^{-1}$. Therefore, since $\mathbf{1}_n B^{-1} \mathbf{1}_n = 1/pBq$, we have $1 = (\mathbf{1}_n \cdot \bar{p}) = \bar{p}B\bar{q}/pBq - \alpha B^{-1}\mathbf{1}_n$ or equivalently $\alpha B^{-1}\mathbf{1}_n = \bar{p}B\bar{q}/pBq - 1$. Now since $B^{-1}\mathbf{1}_n > 0$ by an argument above and $\alpha_j > 0$ for some j, we have that $\bar{p}B\bar{q} > pBq$. Hence we have obtained the second assertion.

The statements with respect to A can be proved similarly. ∎

Remark 3. The $n \times n$ bimatrix games of Section 4 have the properties of Theorem 6. The definition of the games in Theorem 4 is based on a completely mixed pair $(p, q) \in \mathring{\Delta}_n \times \mathring{\Delta}_n$ and a permutation π. The inverse matrices are equal to the transposes of the original ones where the q_i^{-1} are replaced by q_i and the p_i^{-1} by p_i. Evidently these are nonnegative, irreducible, and nonsingular. Let us suppose that π is not irreducible, i.e., $\pi(I) = I$ with $|I| < n$. Then we obtain from the last part of the proof of Theorem 4 that $(x, y) \in E(A, B)$ for the pair (x, y) defined there. The payoffs due to this equilibrium are $xAy = (\sum_{k \in I} q_k)^{-1} > 1 = pAq$ and $xBy = (\sum_{k \in I} p_k)^{-1} > 1 = pBq$. Furthermore the eigenvalues of A are the q_i^{-1} and those of B the p_i^{-1} for all i. Clearly $pAq = 1 < q_i^{-1}$ and $pBq = 1 < p_i^{-1}$.

REFERENCES

1 D. Blackwell, Minimax and irreducible matrices, *J. Math. Anal. Appl.* 3:37–39 (1961).

2 J. E. Cohen, Perturbation theory of completely mixed matrix games, *Linear Algebra Appl.* 79:153–162 (1986).

3 J. E. Cohen, Perturbation theory of completely mixed bimatrix games, *Linear Algebra Appl.* 114/115:169–180 (1989).
4 F. R. Gantmacher, *The Theory of Matrices*, Vol. II, Chelsea, New York, 1959.
5 G. A. Heuer, On completely mixed strategies on bimatrix games, *J. London Math. Soc.* (2) 11:17–20 (1975).
6 M. J. M. Jansen, Maximal Nash subsets for bimatrix games, *Naval Res. Logist. Quart.* 28:147–151 (1981).
7 I. Kaplansky, A contribution to von Neumann's theory of games, *Ann. Math.* 46:474–479 (1945).
8 C. B. Millham, On Nash subsets of bimatrix games, *Naval Res. Logist. Quart.* 21:307–317 (1974).
9 J. F. Nash, Equilibrium points in n-person games, *Proc. Nat. Acad. Sci. U.S.A.* 36:48–49 (1950).
10 J. F. Nash, Non-cooperative games, *Ann. of Math.* 54:286–295 (1951).
11 T. Parthasarathy and T. E. S. Raghavan, *Some Topics in Two-Person Games*, American Elsevier, New York 1971.
12 T. E. S. Raghavan, On positive games matrices and their extensions, *J. London Math. Soc.* 40:467–477 (1965).
13 T. E. S. Raghavan, Completely mixed strategies in bimatrix games, *J. London Math. Soc.* (2) 2:709–712 (1970).
14 T. E. S. Raghavan, Completely mixed games and M-matrices, *Linear Algebra Appl.* 21:35–45 (1978).
15 T. E. S. Raghavan, Some remarks on matrix games and nonnegative matrices, *SIAM J. Appl. Math.* 36:83–85 (1979).
16 G. L. Thompson and R. L. Weil, Further relations between game theory and eigensystems, *SIAM Rev.* 11:579–602 (1969).
17 R. L. Weil, Game theory and eigensystems, *SIAM Rev.* 10:360–367 (1968).

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