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LAW OF ITERATED LOGARITHM FOR FLUCTUATION OF POSTERIOR DISTRIBUTIONS FOR A CLASS OF DIFFUSION PROCESSES AND A SEQUENTIAL TEST OF POWER ONE*

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(Translated by Bernard Seckler)

1. Introduction. In [7], Prakasa Rao has proved the Bernstein–von Mises Theorem concerning the asymptotic behavior of posterior distribution for a class of diffusion processes defined by a linear stochastic differential equation. Here the law of iterated logarithm for fluctuations of the distributions around the true parameter for the class of diffusion processes defined by (2.1) is obtained. As a consequence of this result, a sequential test of power one is developed for testing the hypothesis of the form $H_0: \theta = \theta_0$ against the alternative $H_1: \theta \neq \theta_0$, where θ is a parameter involved in the drift coefficient. Results obtained here are analogous to those in Lerche [3], [4] in his study of the fluctuations of the posterior distributions for the case of independent and identically distributed random variables. For a discussion related to the Bernstein–von Mises Theorem, see Baswa and Prakasa Rao [1].

2. Main result. Consider the stochastic differential equation

$$d\xi(t) = [a(t, \xi) + \theta b(t, \xi)]\, dt + \sigma(t, \xi)\, dW(t), \qquad \xi(0) = \xi_0, \quad t \geqq 0, \tag{2.1}$$

where $a(t, x)$, $b(x, t)$ are real-valued functions defined on $R_+ \times R$, $\sigma(t, x)$ is a positive function defined on $R_+ \times R$, W is the standard Wiener process on R such that $a(t, \xi)$, $b(t, \xi)$, and $\sigma(t, \xi)$ are $\mathcal{F}_t$-measurable where $\mathcal{F}_t = \sigma\{\xi(s): 0 \leqq s \leqq t\}$. Suppose the stochastic differential equation (2.1) has a unique solution on $[0, T]$ for every $T > 0$ and for every $\theta \in \Theta$ open in R. Let μ_θ^T be the probability measure on $C[0, T]$ corresponding to the solution of (2.1) on $[0, T]$. Suppose that $\mu_{\theta_1}^T$ and $\mu_{\theta_2}^T$ are mutually absolutely continuous for all θ_1, θ_2 in Θ. It is well known that the Radon–Nikodym derivative of $\mu_{\theta_1}^T$ with respect to $\mu_{\theta_2}^T$ is given by

$$\log \frac{d\mu_{\theta_1}^T}{d\mu_{\theta_2}^T} = (\theta_1 - \theta_2) \int_0^T \frac{b(t, \xi)}{\sigma(t, \xi)}\, dW(t) - \frac{1}{2}(\theta_1 - \theta_2)^2 \int_0^T \left\{\frac{b(t, \xi)}{\sigma(t, \xi)}\right\}^2 dt. \tag{2.2}$$

(A_1) Suppose Λ is a prior probability measure on $(\theta, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of Θ. Assume that Λ has density $\lambda(\cdot)$ with respect to the Lebesgue measure and that the density $\lambda(\cdot)$ is continuous and positive in an open neighborhood of θ_0, the true parameter.

Suppose there exists a continuous positive monotonic increasing nonrandom function $Q(T)$ of T such that $Q(T) \to \infty$ as $T \to \infty$ and a positive constant β such that

$$\beta_T = \frac{1}{Q^2(T)} \int_0^T \left\{\frac{b(s, \xi)}{\sigma(s, \xi)}\right\}^2 ds \xrightarrow{\text{a.s.}} \beta \quad \text{as } T \to \infty. \tag{A_2}$$

Let

$$\alpha_T = \frac{1}{Q(T)} \int_0^T \frac{b(s, \xi)}{\sigma(s, \xi)}\, dW(s). \tag{2.3}$$

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It is easy to see from (2.2) that the maximum likelihood estimator $\hat{\theta}_T$ of θ based on $\xi_0^T \equiv \{\xi(t), 0 \leqq t \leqq T\}$ satisfies the relation

$$\alpha_T = (\hat{\theta}_T - \theta_0)\beta_T Q(T). \tag{2.4}$$

The posterior density of θ given ξ_0^T is

$$p(\theta \mid \xi_0^T) = \frac{d\mu_\theta^T}{d\mu_{\theta_0}^T}(\xi_0^T)\lambda(\theta) \Big/ \int_\Theta \frac{d\mu_\theta^T}{d\mu_{\theta_0}^T}(\xi_0^T)\lambda(\theta)\, d\theta. \tag{2.5}$$

Let us write $v = Q(T)(\theta - \hat{\theta}_T)$ and

$$p^*(v \mid \xi_0^T) = Q(T)^{-1} p\left(\hat{\theta}_T + \frac{v}{Q(T)} \,\middle|\, \xi_0^T\right). \tag{2.6}$$

Note that $p^*(v \mid \xi_0^T)$ is the posterior density of $Q(T)(\theta - \hat{\theta}_T)$ given ξ_0^T.

THEOREM 2.1. *Suppose the assumptions* (A_1) *and* (A_2) *stated above hold and that*

$$(A_3) \qquad \hat{\theta}_T \to \theta_0 \text{ a.s. } [P_{\theta_0}] \quad \text{as } T \to \infty.$$

Then

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \left| p^*(v \mid \xi_0^T) - \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-(1/2)\beta v^2} \right| dv = 0 \quad \text{a.s. } [p_{\theta_0}]. \tag{2.7}$$

A more general version of the above theorem is proved by Prakasa Rao in [7] when the functions $a(t,\cdot)$, $b(t,\cdot)$ and $\sigma(t,\cdot)$ depend on $\xi(t)$ alone. However the same proof with slight modifications goes through for this general case. We omit the details.

Let

$$\tau(T) = \int_0^T \left\{\frac{b(s,\xi)}{\sigma(s,\xi)}\right\}^2 ds. \tag{2.8}$$

Let P_{T,ξ_0^T} denote the posterior probability measure of θ given the process ξ_0^T. Note that, for any $d > 0$,

$$P_{T,\xi_0^T}\left(\theta_0 - d\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2} \leqq \theta \leqq \theta_0 + d\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)$$

$$= P_{T,\xi_0^T}\left(-dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2} \leqq v + Q(T)(\hat{\theta}_T - \theta_0) \leqq dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)$$

(since $v = Q(T)(\theta - \hat{\theta}_T)$)

$$= P_{T,\xi_0^T}\left(-\frac{\alpha_T}{\beta_T} - dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2} \leqq v \leqq -\frac{\alpha_T}{\beta_T} + dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right) \tag{2.9}$$

(this follows from (2.4))

$$= N_\beta\left(-\frac{\alpha_T}{\beta_T} - dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}, -\frac{\alpha_T}{\beta_T} + dQ(T)\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)$$

$$+ o(1).$$

Almost surely $[P_{\theta_0}]$ as $T \to \infty$ where $N_\beta(x, y)$ denotes the probability of the interval (x, y) corresponding to the normal density with mean 0 and variance β^{-1}. This follows from Theorem 2.1. Observe that

$$\begin{aligned} (2.10)\quad & N_\beta\left(-\frac{\alpha_T}{\beta_T} - dQ(T)\left\{\frac{\log\log \tau(T)}{\tau(T)}\right\}^{1/2}, -\frac{\alpha_T}{\beta_T} + dQ(T)\left\{\frac{\log\log \tau(T)}{\tau(T)}\right\}^{1/2}\right) \\ &= \mathbf{P}\left\{-d \leqq (Y + \sqrt{\beta}\alpha_T/\beta_T)\left(\sqrt{\beta}\, Q(T)\left\{\frac{\log\log \tau(T)}{\tau(T)}\right\}\right)^{1/2} \leqq d\right\}, \end{aligned}$$

where Y is $N(0, 1)$ and this in turn can be written as

$$\mathbf{P}\{-d \leqq Z \leqq d\},$$

where Z is

$$(2.11)\qquad N\left(\frac{\alpha_T \tau(T)^{1/2}}{Q(T)\beta_T(\log\log \tau(T))^{1/2}}, \frac{\tau(T)^{1/2}}{\beta^{1/2}Q(T)(\log\log \tau(T))^{1/2}}\right).$$

Relations (2.9)-(2.11) imply that, for any $d > 0$,

$$\begin{aligned} (2.12)\quad & P_{T,\xi_0^T}\left(\theta_0 - d\left\{\frac{\log\log \tau(T)}{\tau(T)}\right)^{1/2} \leqq \theta \leqq \theta_0 + d\left(\frac{\log\log \tau(T)}{\tau(T)}\right\}^{1/2}\right) \\ &= \mathbf{P}\{-d \leqq Z \leqq d\} + o(1) \quad \text{a.s.}\,[P_{\theta_0}] \end{aligned}$$

as $T \to \infty$, where Z is as defined by (2.11). Note that

$$(2.13)\qquad \frac{\tau(T)^{1/2}}{\beta^{1/2}Q(T)(\log\log \tau(T))^{1/2}} \to 0 \quad \text{a.s.}\,[P_{\theta_0}] \quad \text{a.s. } T \to \infty$$

since

$$\beta_T = \frac{\tau(T)}{Q^2(T)} \to \beta > 0 \quad \text{a.s.}\,[P_{\theta_0}] \quad \text{as } T \to \infty$$

and $\tau(T) \to \infty$ a.s. $[P_{\theta_0}]$ as $T \to \infty$. Furthermore,

$$\begin{aligned} (2.14)\quad \frac{\alpha_T \tau(T)^{1/2}}{Q(T)\beta_T(\log\log \tau(T))^{1/2}} &= \frac{\alpha_T Q(T)}{(\tau(T)\log\log \tau(T))^{1/2}} \\ &= \int_0^T h(s)\, dW(s)/(\tau(T)\log\log \tau(T))^{1/2}, \end{aligned}$$

where

$$h(s) = \frac{b(s, \xi)}{\sigma(s, \xi)}.$$

By the law of the iterated logarithm for stochastic integrals (cf. McKean [6]), it follows that

$$(2.15)\qquad \overline{\lim_{T \to \infty}} \frac{\left|\int_0^T h(s)\, dw(s)\right|}{[2\tau(T)\log\log \tau(T)]^{1/2}} = 1 \quad \text{a.s.}$$

In particular, it can be seen that, if $d < \sqrt{2}$, then

$$(2.16)\qquad \underline{\lim_{T \to \infty}}\, \mathbf{P}\{-d \leqq Z \leqq d\} = 0 \quad \text{a.s.}\,[P_{\theta_0}]$$

from (2.13) and (2.15) and, if $d > \sqrt{2}$, then

$$\overline{\lim_{T\to\infty}} P\{-d \leqq Z \leqq d\} = 1 \quad \text{a.s.} [P_{\theta_0}]. \tag{2.17}$$

Let $B_{\theta_0}(a)$ denote the closed sphere with radius a and center θ_0. Relations (2.16) and (2.17) prove that

$$\overline{\lim_{T\to\infty}} P_{T,\xi_0^T}\left\{B^c_{\theta_0}\left(d\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)\right\} = \begin{cases} 0 & \text{if } d > \sqrt{2}, \\ 1 & \text{if } d < \sqrt{2}, \end{cases} \tag{2.18}$$

P_{θ_0}-a.s., where A^c denotes complement of a set A. Suppose $d = \sqrt{2}$. Note that

$$P_{T,\xi_0^T}\left(|\theta - \theta_0| > \sqrt{2}\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right) = \mathbf{P}\{|Z| > \sqrt{2}\} + o(1) \quad \text{a.s.}$$

In view of (2.15), it follows that, for any given $\varepsilon > 0$,

$$\frac{|\int_0^T h(s)\, dW(s)|}{(\tau(T)\log\log\tau(T))^{1/2}} \geqq \sqrt{2} - \Phi^{-1}(\varepsilon)\frac{1}{(\log\log\tau(T))^{1/2}} \tag{2.19}$$

infinitely often P_{θ_0}-almost surely, where Φ is the standard normal distribution function. This implies that

$$\overline{\lim_{T\to\infty}} P(|Z| > \sqrt{2}) \geqq 1 - \varepsilon \quad \text{a.s.} [P_{\theta_0}]$$

for every $\varepsilon > 0$ from (2.13), (2.11) and the representation (2.14). Hence

$$\overline{\lim_{T\to\infty}} \mathbf{P}(|Z| > \sqrt{2}) = 1 \quad \text{a.s.} [P_{\theta_0}],$$

which shows that

$$\overline{\lim_{T\to\infty}} P_{T,\xi_0^T}\left[B^c_{\theta_0}\left(\sqrt{2}\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)\right] = 1 \quad \text{a.s.} (P_{\theta_0}]. \tag{2.20}$$

Combining (2.18) and (2.20), we have the following theorem.

THEOREM 2.2. *Suppose the assumptions* (A_1), (A_2), *and* (A_3) *hold. Then*

$$\overline{\lim_{T\to\infty}} P_{T,\xi_0^T}\left[B^c_{\theta_0}\left(d\left\{\frac{\log\log\tau(T)}{\tau(T)}\right\}^{1/2}\right)\right\} = \begin{cases} 0 & \text{if } d > \sqrt{2}, \\ 1 & \text{if } d \leqq \sqrt{2}, \end{cases} \tag{2.21}$$

a.s. $[P_{\theta_0}]$ *where*

$$\tau(T) = \int_0^T \left\{\frac{b(s,\xi)}{\sigma(s,\xi)}\right\}^2 ds.$$

The next result is a consequence of Theorem 2.1 and the properties of the normal distribution function.

THEOREM 2.3. *Suppose the assumptions* (A_1)–(A_3) *hold. Then, for every neighborhood* V *of* θ_0, *there exists a constant* $b_v > 0$ *such that*

$$\lim_{T\to\infty} e^{b_v Q^2(T)} P_{T,\xi_0^T}(V^c) = 0 \quad a.s. [P_{\theta_0}]. \tag{2.22}$$

Proof. Given an open neighborhood V of θ_0, observe that, for sufficiently small $\delta > 0$,

$$\begin{aligned} 0 \leqq P_{T,\xi_0^T}(V^c) &\leqq \int_{|v| > \delta Q(T)} p^*(v \mid \xi_0^T)\, dv \\ &= \left(\frac{\beta}{2\pi}\right)^{1/2} \int_{|v| > \delta Q(T)} e^{-(1/2)\beta v^2}\, dv + o(1) \quad \text{a.s.} [P_{\theta_0}] \quad \text{as } T \to \infty \end{aligned}$$

by Theorem 2.1. But

$$\int_{|v|>\delta Q(T)} \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-(1/2)\beta v^2}\,dv = 2\int_{v>\delta Q(T)} \left(\frac{\beta}{2\pi}\right)^{1/2} e^{(-1/2)\beta v^2}\,dv$$

$$\leqq \frac{2}{\sqrt{2}\pi\delta Q(T)}\, e^{-(1/2)(\delta\sqrt{\beta}Q(T))^2}$$

by Feller [2, p. 175]. Hence

$$0 \leqq e^{(1/2)\delta^2\beta Q^2(T)} P_{T,\xi_0^T}(V^c) \leqq \frac{2}{\sqrt{2}\pi\delta Q(T)}$$

and the last term tends to zero as $Q(T)\to\infty$ as $T\to\infty$. This proves the theorem.

3. Application. We now develop a sequential test with power one for testing the hypothesis $H_0: \theta=\theta_0$ against the alternative $H_1: \theta\neq\theta_0$ for the parameter θ appearing in (2.1). Note that a sequential level α-test with power one is given by a stopping time T satisfying

(i) $P_{\theta_0}(T<\infty)\leqq\alpha$;

(ii) $P_\theta(T<\infty)=1$ for $\theta\neq\theta_0$.

The hypothesis is rejected as soon as $T<\infty$. Otherwise we continue the procedure. We assume that Θ is open. Let $\psi(T)$ be positive increasing such that $\psi(T)=o(\tau(T))$ and $\psi(T)\geqq\gamma_0^2\log\log\tau(T)$, where $\gamma_0>0$ to be chosen later. For any $\varepsilon>0$, define

$$T^* = \inf\left\{T\geqq 0: P_{T,\xi_0^T}\left[B_{\theta_0}^c\left(\left\{\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right)\right]>1-\varepsilon\right\}. \tag{3.1}$$

This stopping time defines a sequential test with power one. In fact Theorem 2.2 implies that

$$P_{\theta_0}(T^*<\infty)\leqq\alpha$$

when $\gamma_0=\sqrt{2}(1+\delta)$ for some $\delta>0$ suitably chosen. Furthermore,

$$P_\theta(T^*<\infty)=1 \quad \text{for } \theta\neq\theta_0,$$

by Theorem 2.3.

In order to compute α approximately for a given δ or vice versa, we can take normal approximation to the posterior distribution in defining T^*. Note that, when θ_0 is the true parameter,

$$P_{T,\xi_0^T}\left\{B_{\theta_0}^c\left(\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right)\right\} = 1-P_{T,\xi_0^T}\left\{|\theta-\theta_0|\leqq\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\}$$

$$=1-P_{T,\xi_0^T}\left\{\left|\frac{v}{Q(T)}+\hat{\theta}_T-\theta_0\right|\leqq\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\}$$

$$=1-P_{T,\xi_0^T}\left\{\left|\frac{v}{Q(T)}+\frac{\alpha_T}{\beta_T Q(T)}\right|\leqq\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\}$$

$$=1-P_{T,\xi_0^T}\left\{|v\beta_T+\alpha_T|\leqq Q(T)\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\}$$

$$\simeq 1-\mathbf{P}\left\{|v\beta_T+\alpha_T|\leqq Q(T)\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\} \quad \text{a.s. } [P_{\theta_0}],$$

where v is $N(Q, \beta^{-1})$. Let

(3.2) $$T^{**} = \inf\left\{T \geqq 0: \mathrm{P}\left\{|v\beta_T + \alpha_T| \leqq Q(T)\left(\frac{\psi(T)}{\tau(T)}\right)^{1/2}\right\} < \varepsilon\right\}.$$

Then

(3.3) $$\alpha^* = P_{\theta_0}(T^{**} < \infty)$$

is an approximation for α.

4. Example. Consider the stochastic differential equation

(4.1) $$dX_t = -\theta X_t\, dt + dW_t, \qquad t \geqq 0,$$

where $\theta > 0$. Suppose the process $\{X_t\}$ is observed up to time T. It is easy to check that the maximum likelihood estimator $\hat{\theta}_T$ of θ satisfies the relation

(4.2) $$\hat{\theta}_T - \theta_0 = \int_0^T X_t\, dW_t \Big/ \int_0^T X_t^2\, dt$$

when θ_0 is the true parameter. Let

(4.3) $$Y_T = \int_0^T X_t\, dW_t.$$

Clearly the quadratic variation process of the martingale $\{Y_T\}$ is

(4.4) $$\langle Y_T \rangle = \int_0^T X_t^2\, dt.$$

Since the parameter $\theta_0 > 0$, the process $\{X_t\}$ is stationary and ergodic and, by the ergodic theorem, it follows that

(4.5) $$\frac{1}{T}\int_0^T X_t^2\, dt \xrightarrow{\text{a.s.}} \mathrm{E}(X_0^2)$$

as $T \to \infty$, where E denotes the expectation with respect to the ergodic measure under θ_0. We assume that $0 < \mathrm{E}(X_0^2) < \infty$. Hence $\langle Y_T \rangle \to \infty$ a.s $[P_{\theta_0}]$ as $T \to \infty$. By the strong law of large numbers for continuous time square integrable martingales (cf. Baswa and Prakasa Rao [1, p. 394]), it follows that

(4.6) $$\frac{Y_T}{\langle Y_T \rangle} \to 0 \quad \text{a.s.}\,[P_{\theta_0}] \quad \text{as } T \to \infty.$$

Hence

(4.7) $$\hat{\theta}_T - \theta_0 \to 0 \quad \text{a.s.}\,[P_{\theta_0}] \quad \text{as } T \to \infty.$$

This shows that condition (A_3) holds. Relation (4.7) also follows from Lemma 17.4 in Liptser and Shiryaev [5, p. 210]. Let $Q(T) = T^{1/2}$. Clearly,

$$\beta_T = \frac{1}{T}\int_0^T X_t^2\, dt \xrightarrow{\text{a.s.}} \beta = \mathrm{E}(X_0^2) > 0 \quad \text{as } T \to \infty$$

by (4.5.). This relation shows that (A_2) holds. Hence Theorems 2.1–2.3 hold for the Ornstein–Uhlenbeck process defined by (4.1) under condition (A_1).

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