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Functions and Their Fourier Transforms with Supports of Finite Measure for Certain Locally Compact Groups

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It is well known that if the supports of a function $f \in L^1(\mathbf{R}^d)$ and its Fourier transform $\hat{f}$ are contained in bounded rectangles, then $f = 0$ almost everywhere. In 1974 Benedicks relaxed the requirements for this conclusion by showing that the supports of f and $\hat{f}$ need only have finite measure. In this paper we extend the validity of this property to a wide variety of locally compact groups. These include $\mathbf{R}^d \times K$, where K is a compact connected Lie group, the motion group, the affine group, the Heisenberg group, $SL(2, \mathbf{R})$, and all noncompact semisimple groups with some additional restrictions on the functions f.

0. INTRODUCTION

Throughout G will denote a locally compact group equipped with left Haar measure dm. (If G is compact, we take $m(G) = 1$, and if $G = \mathbf{R}^d$, dm will be Lebesgue measure.) For simplicity, instead of $\int_G \dots dm(x)$ we will usually write $\int_G \dots dx$. Always $\hat{G}$ will denote the dual of G, that is, a maximal set of pairwise inequivalent unitary irreducible (continuous) representations of G. The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is defined by $\hat{f}(\lambda) = \lambda(f) = \int_G f(x)\lambda(x)\,dx$ (or sometimes $\int_G f(x)\,\lambda(x^{-1})\,dx$) for $\lambda \in \hat{G}$. For such functions we introduce the notation

$$A = A_f = \{x \in G: f(x) \neq 0\}, \qquad B = B_f = \{\lambda \in \hat{G}: \hat{f}(\lambda) \neq 0\}.$$

(To avoid any possible ambiguity, f and $\hat{f}$ are to be taken as specific functions rather than equivalence classes.)

We are interested in establishing that the Fourier transforms on a wide variety of such groups have the following *reciprocal-supports* property:

if $f \in L^1(G)$ satisfies

$$m(A_f) < m(G), \qquad \mu(B_f) < \mu(\hat{G}), \tag{0.1}$$

then $f = 0$ a.e.

(Here μ denotes some type of "measure" on $\hat{G}$, usually closely akin to the Plancherel measure. Also on some occasions it is necessary to slightly tighten the condition on the support of f, and on others it may be substantially relaxed.)

As stated in the abstract, this property was established for the Fourier transform on $\mathbf{R}^d$ by Benedicks [3]. It is a generalization of the classical result that if the supports of $f \in L^1(\mathbf{R}^d)$ and its Fourier transform are contained in rectangles, then $f =$ a.e. [7, 2.9]. Amrein and Berthier [1] later gave a different proof. A related result due to Matolcsi and Szücs [15] is the following: suppose $f \in L^1(G)$, where G is a locally compact abelian group. If $m(A_f)\, \hat{m}(B_f) < 1$, where $\hat{m}$ denotes the Plancherel–Haar measure on $\hat{G}$, then $f = 0$ a.e.

There is a considerable body of results for $\mathbf{R}$ proved by such people as Beurling, Levinson, Malliavin, Paley, F. and M. Riesz, and Wiener which show that $f = 0$ a.e. follows from various conditions on the size and nature of A_f and B_f. For a survey see Benedetto [2]. A recent work which incorporates and generalizes many of the early results is Benedicks [4]. We content ourselves with just one example, a classical result which will be used in Section 3.

0.1. THEOREM. *Suppose $f \in L^2(\mathbf{R})$ satisfies $m(\mathbf{R} \setminus A_f) > 0$ and $B_f \subseteq [\Omega, \infty)$ for some $\Omega \in \mathbf{R}$. Then $f = 0$ a.e.*

Proof. Suppose $f \neq 0$ in $L^2(\mathbf{R})$. The condition $B_f \subseteq [\Omega, \infty)$ implies

$$\int_{\mathbf{R}} \frac{\log |f(x)|}{1 + x^2}\, dx > -\infty$$

(see [7, 3.4]). Since $\log |f(x)| < f(x)$, this is impossible if $\{x \in \mathbf{R} : f(x) = 0\}$ has positive measure.

Benedicks proof in [3] for $\mathbf{R}^d$ is based on the validity of property 0.1 for the torus. As an introduction we show that the same idea used for the torus, namely the analyticity of trigonometric polynomials, establishes the property for all compact connected Lie groups. (A *trigonometric polynomial* on a compact group is a finite linear combination of matrix elements arising from the (finite-dimensional) members of $\hat{G}$.)

0.2. *Compact Groups.* Suppose that G is a compact (nonfinite) group and let μ be the counting measure on $\hat{G}$, that is, $\mu(E) = \# E$, the cardinality of E, for each set $E \subseteq \hat{G}$. The condition that $\# B_f < \# \hat{G} = \infty$ for $f \in L^1(G)$ means precisely that f equals a trigonometric polynomial almost everywhere. If G is also a Lie group, such an f must be analytic. But nonzero analytic functions cannot vanish on a set of positive measure when G is connected. This proves part (i) of:

0.3. Lemma. *Let G be a compact group.*

(i) *Suppose G is also connected and Lie. If a trigonometric polynomial f on G satisfies $m(A_f) < 1$, then $f = 0$.*

(ii) *Suppose the identity component G_0 of G satisfies $0 < m(G_0) < 1$. (For example, G is a disconnected Lie group.) Then there exists a trigonometric polynomial $f \neq 0$ in G with $0 < m(A_f) < 1$.*

Proof of (ii). Since G_0 is normal [12, (7.1)], G/G_0 is a nontrivial compact group and so we may choose $\gamma \in (G/G_0)^\wedge$ with $\gamma \neq 1$. Now γ can be identified with a member of $\hat{G}$ and as such is constant on cosets of G_0 [12, (28.10)]. Define $\chi_\gamma = \mathrm{tr}(\gamma)$: $G \to \mathbf{C}$, the *character of* γ (tr denotes trace), and observe that $\chi_\gamma - \chi_\gamma(e)$ is a nontrivial trigonometric polynomial which is identically zero on G_0.

Before passing to less obvious cases we remark that the reciprocal-supports property is a simple type of uncertainty principle since it restricts the amount to which both a function and its Fourier transform can be concentrated. Its validity for $\mathbf{R}$ is used in [5, Sect. 4] to establish generalizations of the classical Heisenberg uncertainty inequality.

The cases in each of the following sections are all quite different so we have given more introduction in each section than if the paper was aimed at a particular group of specialists.

1. Products with R

As usual take $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$ and $\mathbf{Z}^+ = \{1, 2, ...\}$. In this section $G = \mathbf{R}^d \times K$, where $d \in \mathbf{Z}^+$ and K is a compact group (except for Corollary 1.4, where K is locally compact). Its Haar measure is $dg = dx\, dk$, where dx is Lebesgue measure on $\mathbf{R}^d$ and dk is normalized Haar measure on K. The dual $\hat{G}$ of G is $\mathbf{R}^d \times \hat{K}$, $\hat{K}$ being as usual a maximal set of pairwise inequivalent unitary irreducible representations of K. We will give two versions of a reciprocal-supports property for the Fourier transform on

G, Theorems 1.1 and 1.3, corresponding to two different "measures" on $\hat{G}$. The results of this section are not only of intrinsic interest (since they extend the results of Benedicks) but one of them, Corollary 1.4, is crucial for the analysis on the Heisenberg group given in Section 4.

Each member (y, γ) of $\hat{G}$ is a map $(x, k) \to e^{-2\pi i xy}\gamma(k)$ taking values in $\mathscr{U}(\mathscr{H}_\gamma)$, the Banach space of unitary operators on a Hilbert space $\mathscr{H}_\gamma$ of (finite) dimension $d(\gamma)$. (Of course, $x = (x_1, ..., x_d)$, $y = (y_1, ..., y_d)$, and $xy = x_1 y_1 + \cdots + x_d y_d$.) The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is given by

$$\hat{f}(y, \gamma) = \int_{\mathbf{R}^d} \int_K f(x, k)\, e^{-2\pi i xy} \gamma(k^{-1})\, dk\, dx$$

for $(y, \gamma) \in \hat{G} = \mathbf{R}^d \times \hat{K}$.

Let ϕ be a function on $\hat{G}$ so that $\phi(y, \gamma) \in \mathscr{B}(\mathscr{H}_\gamma)$, the Banach space of bounded linear operators on $\mathscr{H}_\gamma$. Suppose that ϕ is measurable, meaning that it is measurable in the first variable. Extending the usual notions of L^1-norms for $\mathbf{R}^d$ and $\hat{K}$ (see [12, Vol. II, (28.24)]) means that the L^1-norm of ϕ is

$$\|\phi\|_1 = \sum_{\gamma \in \hat{K}} d(\gamma) \int_{\mathbf{R}^d} \operatorname{tr}(|\phi(y, \gamma)|)\, dy,$$

where $|T|$ signifies the absolute value of the operator T, that is, $|T|$ is the unique positive definite operator satisfying $|T|^2 = TT^*$. The corresponding measure μ is defined by

$$\mu(E) = \|1_E\|_1 \qquad \text{for measurable } E \subseteq \hat{G}. \tag{1.1}$$

(Here $1_E(y, \gamma) = I_\gamma$, the identity operator in $\mathscr{B}(\mathscr{H}_\gamma)$ if $(y, \gamma) \in E$ and 0 otherwise.) Notice that

$$\mu(E) = \sum_{\gamma \in \hat{K}} d(\gamma)^2 m(E_\gamma), \tag{1.2}$$

where $E_\gamma = \{y \in \mathbf{R}^d : (y, \gamma) \in E\}$. This is the first measure on $\hat{G}$.

The proof of our first result is based on Benedicks' proof for $\mathbf{R}^d$. We will be frequently referring back to this result or, more precisely, to Corollary 1.2.

1.1. THEOREM. *Let $G = \mathbf{R}^d \times K$, where $d \in \mathbf{Z}^+$ and K is a compact connected Lie group. Suppose that $f \in L^1(G)$ has $m(A_f) < \infty$ and $\mu(B_f) < \infty$, μ being defined by* (1.1). *Then $f = 0$ a.e.*

Proof. In the proof we assume that $K \neq \{e\}$ although it is easily seen that the proof is valid in this case. Suppose $f \in L^1(G)$ satisfies the hypotheses of the theorem. Replacement of f by a suitable dilate in the first variable shows that we may assume $m(A_f) < 1$. Using monotone convergence

$$\int_{\square} \sum_{\gamma \in \hat{K}} \sum_{n \in \mathbf{Z}^d} d(\gamma) \operatorname{tr}(1_B(y+n, \gamma))$$

$$= \sum_{\gamma} d(\gamma) \sum_{n} \int_{\square} \operatorname{tr}(1_B(y+n, \gamma)\, dy = \mu(B) < \infty,$$

where $\square$ is the cube $\{x \in \mathbf{R}^d \colon 0 \leqslant x_j \leqslant 1,\ j = 1, ..., d\}$ and $B = B_f$. Hence there exists a set $E \subseteq \square$ with $m(E) = 1$ so that $y \in E$ implies $\sum_\gamma \sum_n d(\gamma) \operatorname{tr}(1_B(y+n, \gamma))$ is finite, that is, $y \in E$ implies

$$((y + \mathbf{Z}^d) \times \hat{K}) \cap B \qquad \text{is finite.} \tag{1.3}$$

Assume that $\|f\|_1 > 0$. There must exist $\gamma_0 \in \hat{K}$ so that the continuous function $\hat{f}(\cdot, \gamma_0)$ is nonvanishing. But this means that it is nonvanishing on an interval and so we may choose $a \in E$ with the property that

$$\hat{f} \text{ is not identically zero on } (a + \mathbf{Z}^d) \times \hat{K}. \tag{1.4}$$

Using Poisson summation on $f(x, k)\, e^{-2\pi i a x}$, define $\phi \in L^1(\square \times K)$ by

$$\phi(x, k) = \sum_{n \in \mathbf{Z}^d} f(x+n, k)\, e^{-2\pi i a(x+n)}.$$

Its Fourier coefficient at $(m, \gamma) \in \mathbf{Z}^d \times \hat{K}$ is obtained as

$$\hat{\phi}(m, \gamma) = \int_K \gamma(k^{-1})\, dk \int_{\square} f(x+n, k)\, e^{-2\pi i a(x-n)}\, e^{-2\pi i m x}\, dx$$

$$= \int_K \gamma(k^{-1})\, dk \sum_n \int_{\square} f(x+n, k)\, e^{-2\pi i (x-n)(a+m)}\, dx$$

$$= \hat{f}(a+m, \gamma).$$

Now $\hat{f}(a+m, \gamma) \neq 0$ only if $(a+m, \gamma) \in B$ and so, from (1.3) and (1.4), ϕ is a nonzero trigonometric polynomial on the connected Lie group $\square \times K$. (Here we think of $\square$ as the d-dimensional torus.) As such it can only vanish on a set of measure zero by Lemma 0.3(i).

Suppose for the moment that $\|f\|_\infty < \infty$. Then

$$|\phi(x, k)| \leqslant \|f\|_\infty \sum_n 1_A(x+n, k),$$

where $A = A_f$. Also

$$\int_K \int_{\square} \sum_n 1_A(x+n, k)\, dx\, dk = m(A) < 1.$$

Hence $\sum_n 1_A(x+n, k)$, which only takes values in $\{0\} \cup \mathbf{Z}^+$, vanishes on a set of positive measure in $\square \times K$. Thus so does ϕ, which contradicts the conclusion of the previous paragraph.

It only remains to prove that $\|f\|_\infty$ is finite. First of all $\|\hat{f}(y, \gamma)\| \leqslant \|f\|_1$, where the first norm is the operator norm. Since $f \in L^1$ and $\mu(B_f) < \infty$, from the inversion formula for Fourier transforms and (1.2),

$$\begin{aligned}
|f(x, k)| &= \left| \int_{\mathbf{R}^d} \sum_{\gamma \in \hat{K}} d(\gamma) \operatorname{tr}(\hat{f}(y, \gamma)\, \gamma(x))\, e^{2\pi i x y}\, dy \right| \\
&\leqslant \int_{\mathbf{R}^d} \sum_\gamma d(\gamma)^2 \, \|\hat{f}(y, \gamma)\| \, dy \\
&\leqslant \|f\|_1 \sum_\gamma d(\gamma)^2 \int_{\mathbf{R}^d} 1_{B_\gamma} \, dy \\
&= \|f\|_1 \sum_\gamma d(\gamma)^2 \, m(B_\gamma) \\
&= \|f\|_1 \, \mu(B) < \infty,
\end{aligned}$$

which completes the proof.

Because we will be continually referring to it, we isolate the following (slight generalization of a) special case of Theorem 1.1. The proof reduces to that of Theorem 1.1 (with $K = \{e\}$) by first establishing that $f \in L^1(\mathbf{R}^d)$.

1.2. Corollary. *Suppose $1 \leqslant p < \infty$ and let $f \in L^p(\mathbf{R}^d)$ or $M(\mathbf{R}^d)$, the space of bounded measures on $\mathbf{R}^d$. Assume $m(A_f) < \infty$ in the first case or $m(\operatorname{supp} f) < \infty$ in the second. In both cases $\hat{f}$ is defined. If $m(B_f) < \infty$, then $f = 0$.*

We now show that the same conclusion holds for $G = \mathbf{R}^d \times K$ if we tighten the restriction on the support of f but relax it on the support of its Fourier transform. If $E \subseteq G$, let $EK = \{(x, hk) : (x, h) \in E, k \in K\}$.

1.3. Theorem. *Suppose $f \in L^1(G)$ with $G = \mathbf{R}^d \times K$, where $d \in \mathbf{Z}^+$ and K is a compact group. For each $k \in K$, $\sigma \in \hat{K}$ define*

$$(AK)_k = \{x \in \mathbf{R}^d : (x, k) \in AK\}, \qquad B_\sigma = \{y \in \mathbf{R}^d : (y, \sigma) \in B\},$$

where $A = A_f$ and $B = B_f$. If $m((AK)_k) < \infty$ and $m(B_\sigma) < \infty$ for all $k \in K$, $\sigma \in \hat{K}$, then $f = 0$ a.e.

Proof. Assume that $K \neq \{e\}$, otherwise the result collapses to

Corollary 1.2. If necessary, redefine f on a set of measure zero so that $x \to f(x, k) \in L^1(\mathbf{R}^d)$ for all $k \in K$. Given $k \in K$, $\sigma \in \hat{K}$, define $f_{k,\sigma} \in L^1(\mathbf{R}^d)$ by

$$f_{k,\sigma}(x) = f * \chi_\sigma(x, k) = \int_K f(x, kh^{-1})\, \chi_\sigma(h)\, dh,$$

where $\chi_\sigma = \operatorname{tr}(\sigma(\cdot))$. Notice that $f * \chi_\sigma(x, k) \neq 0$ implies $(x, k) \in AK$ and so

$$\{x \in \mathbf{R}^d : f_{k,\sigma}(x) \neq 0\} \subseteq (AK)_k. \tag{1.5}$$

The Euclidean Fourier transform $\hat{f}_{k,\sigma}$ of $f_{k,\sigma}$ is given by

$$\begin{aligned}
\hat{f}_{k,\sigma}(y) &= \int_{\mathbf{R}^d} \int_K f(x, kh^{-1})\, e^{-2\pi i x y}\, \chi_\sigma(h)\, dh\, dx \\
&= \operatorname{tr} \int_{\mathbf{R}^d} \int_K f(x, kh^{-1})\, \sigma(h)\, dh\, e^{-2\pi i x y}\, dx \\
&= \operatorname{tr} \int_{\mathbf{R}^d} \int_K f(x, h)\, \sigma(h^{-1})\, dh\, e^{-2\pi i x y}\, dx\, \sigma(k) \\
&= \operatorname{tr}(\hat{f}(y, \sigma)\, \sigma(k)),
\end{aligned}$$

where the penultimate equality was obtained via the transformation $h \to h^{-1}k$. Now $\hat{f}(y, \sigma) = 0$ implies $\hat{f}_{k,\sigma}(y) = 0$ for all $k \in K$ and so

$$\{y \in \mathbf{R}^d : \hat{f}_{k,\sigma}(y) \neq 0\} \subseteq B_\sigma. \tag{1.6}$$

From (1.5), (1.6), the hypotheses of the theorem, and Corollary 1.2, $f_{k,\sigma} = 0$ a.e. for all $k \in K$, $\sigma \in \hat{K}$. Now for each $x \in \mathbf{R}^d$, the Fourier series of $k \to f(x, k)$ is

$$f(x, k) \sim \sum_{\sigma \in \hat{K}} d(\sigma)\, f_{k,\sigma}(x).$$

Thus $f = 0$ a.e. by the uniqueness of Fourier series on compact groups.

Further results are available for $G = \mathbf{R} \times H$, where H is a noncompact locally compact group, which don't require the supports of the function nor its transform to have finite measure. They are achieved by reducing the problem to $\mathbf{R}$. One form is used in Section 4.

As before, $\hat{G} = \mathbf{R} \times \hat{H}$. Variables in G and $\hat{G}$ will be written as (s, x) and (t, γ), respectively, where $s, t \in \mathbf{R}$, $x \in H$, and $\gamma \in \hat{H}$. Given $f \in L^1(G)$, define

$$\hat{f}(t, \gamma) \int_{\mathbf{R}} \int_H f(s, x)\, \gamma(x^{-1})\, e^{-2\pi i s t}\, dx\, ds.$$

For $f \in L^2(G)$ with G abelian, $\hat{f}$ is defined using the usual completion argument so that the Fourier transform provides an isometric isomorphism between $L^2(G)$ and $L^2(\hat{G})$ by Plancherel's theorem.

1.4. COROLLARY. *Let $G = \mathbf{R} \times H$, where H is a locally compact topological group. Suppose that the following conditions on $f: G \to \mathbf{C}$ are satisfied:*

(i) *$f \in L^1(G)$ OR $f \in L^2(G)$ and H is abelian,*

(ii) *there exists $E \subseteq \mathbf{R}$ with $m(E) < \infty$ such that*

$$A_f \subseteq E \times H,$$

(iii) *for each $\gamma \in \hat{H}$,*

$$m\{t \in \mathbf{R}: \hat{f}(t, \gamma) \neq 0\} < \infty.$$

(In the case when H is abelian, this condition can be relaxed to

(iii)′ *for a.a. $\gamma \in \hat{H}$, $m\{t \in \mathbf{R}: \hat{f}(t, \gamma) \neq 0\} < \infty$.)*

Then $f = 0$ a.e.

Proof. Assume that $f \in L^1(G)$ satisfies conditions (ii) and (iii). Also modify f on a set of measure zero so that all its sections in both directions are integrable. For each $\gamma \in \hat{H}$ define ϕ_γ on $\mathbf{R}$ by

$$\phi_\gamma(s) = (\mathscr{F}_2 f)(s, \gamma). \tag{1.7}$$

(In this proof $\mathscr{F}_1$ and $\mathscr{F}_2$ will denote the operations of taking Fourier transforms, in the L^1 or L^2 sense, in the first and second variables, respectively.) Evidently $\{s: \phi_\gamma(s) \neq 0\} \subseteq E$ from (ii) and so, once again from (ii), we have

$$m\{s \in \mathbf{R}: \phi_\gamma(s) \neq 0\} < \infty \qquad \text{for} \quad \gamma \in \hat{H}. \tag{1.8}$$

By writing out the relevant integrals and by applying Fubini's theorem for functions with values in a Banach space, we see

$$(\phi_\gamma)^\wedge(t) = (\mathscr{F}_1(\mathscr{F}_2 f))(t, \gamma) = \hat{f}(t, \gamma). \tag{1.9}$$

Since $\{t: (\phi_\gamma)^\wedge(t) \neq 0\} = \{t: \hat{f}(t, \gamma) \neq 0\}$, the measure of the first set is finite by (iii). Combining this fact with (1.8) and applying Corollary 1.2 shows that $\phi_\gamma = 0$ a.e. for each $\gamma \in \hat{H}$. Thus $\hat{f} = 0$ on $\hat{G}$ implying $f = 0$ a.e. [6, 18.2.4].

Now let $f \in L^2(G)$ with H abelian and suppose that (ii) and (iii) are satisfied. The proof mimics that above except that Plancherel's theorem is invoked in several places. Redefine f and $\hat{f}$ on sets of measure zero so that all these sections are square integrable. As before, define ϕ_y on $\mathbf{R}$ by (1.7) (all Fourier transforms are now in the L^2-sense). Since

$$\iint_{\mathbf{R}\times\hat{H}} |\mathscr{F}_2 f(s, y)|^2 \, ds \, dy = \int_{\mathbf{R}} \left(\int_{\hat{H}} |\mathscr{F}_2 f(s, y)|^2 \, dy \right) ds$$

$$= \int_{\mathbf{R}} \left(\int_H |f(t, y)|^2 \, dt \right) dy = \|f\|_{L^2(G)},$$

$\mathscr{F}_2 f \in L^2(\mathbf{R} \times \hat{H})$. Hence $\mathscr{F}_1 \mathscr{F}_2 f$ is well-defined almost everywhere. Suppose for the moment that

$$\mathscr{F}_1 \mathscr{F}_2 f = \hat{f} \qquad \text{a.e.} \tag{1.10}$$

so that $(\phi_y)^\wedge(t) = \hat{f}(t, y)$ for a.a. $(t, y) \in \hat{G}$ (cf. (1.9)). The proof is completed as in the previous case by using (ii) and (iii) to show that for a.a. $y \in \hat{H}$ the sets on which ϕ_y and its transform are nonzero have finite measure.

It only remains to establish (1.10). It is valid when $f \in L^1(G)$ so its validity for $L^2(G)$ follows by approximating $f \in L^2(G)$ by functions in $L^1(G) \cap L^2(G)$.

2. The Motion Group $M(2)$

Let $SO(2)$ act on $\mathbf{R}^2$ in the usual manner. The group $G = M(2)$ of rigid transformations of $\mathbf{R}^2$

$$g = (x, k) \colon y \to ky + x$$

for $x \in \mathbf{R}^2$, $k \in SO(2)$ is called the *motion group of* $\mathbf{R}^2$. It is the semidirect product of $SO(2)$ and $\mathbf{R}^2$. Haar measure on G is $dg = dx\, dk$, where dx is Lebesgue measure on $\mathbf{R}^2$ and dk is normalized Haar measure on $SO(2)$. Details of the following summary may be found in [17, Chap. IV].

To each $r \in \mathbf{R}^+ = (0, \infty)$ assign the unitary irreducible representation U^r of G as operators in $\mathscr{U}(L^2(SO(2)))$ defined by

$$(\mathscr{U}_g^r \phi)(t) = e^{-2\pi i r \langle t e_2, x \rangle} \phi(k^{-1} t),$$

where $g = (x, k) \in M(2)$, $\phi \in L^2(SO(2))$, $t \in SO(2)$, and $e_2 = (0, 1)$. This family of representations makes up $\hat{G}_r$, a subset of $\hat{G}$ which supports the Plancherel measure, that is, $\mu(\hat{G} \setminus \hat{G}_r) = 0$, where μ is the Plancherel measure. The Plancherel measure μ on $\mathbf{R}^+$ $(\leftrightarrow \hat{G}_r)$ is defined by

$$\mu(E) = \int_E r\, dr \qquad \text{for measurable } E \subseteq \mathbf{R}^+, \tag{2.1}$$

where dr is Lebesgue measure on $\mathbf{R}$.

The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is a function $\mathbf{R}^+ \to \mathscr{B}(L^2(SO(2)))$ defined by

$$(\hat{f}(r)\phi)(t) = \iint_{\mathbf{R}^2 \times SO(2)} f(x, k)\, e^{-2\pi i r\langle te_2, x\rangle}\, \phi(k^{-1}t)\, dk\, dx$$

for $r > 0$, $\phi \in L^2(SO(2))$, and $t \in SO(2)$. Let B_f be the set $\{r \in \mathbf{R}^+ : \hat{f}(r) \neq 0\}$.

2.1. THEOREM. *Let $f \in L^1 \cap L^2(M(2))$ and suppose that for a.a. $k \in SO(2)$ the Lebesgue measure of $\{x \in \mathbf{R}^2 : (x, k) \in A_f\}$ and the Plancherel measure (2.1) of B_f are finite. (From the continuity of the Fourier transform it follows that B_f is indeed a measurable set.) Then $f = 0$ a.e.*

Proof. Let $\mathscr{F}_1$ denote the usual Fourier transform of f in the first variable. Arguing as in the proof of the second case of Corollary 1.4, $k \to \mathscr{F}_1 f(y, k) \in L^2(SO(2))$ for a.a. $y \in \mathbf{R}^2$.

Suppose $r \notin B_f$. From the definition of $\hat{f}$, after the transformation $k \to tk^{-1}$ we see that

$$\int_{SO(2)} \mathscr{F}_1 f(rte_2, tk^{-1})\phi(k)\, dk = 0$$

for $\phi \in L^2(SO(2))$ and a.a. $t \in SO(2)$. Hence $\mathscr{F}_1 f(rte_2, tk^{-1}) = 0$ for a.a. $k, t \in SO(2)$. By choosing k, t appropriately we conclude that for a.a. $u \in SO(2)$

$$\mathscr{F}_1 f(\xi, u) \neq 0 \qquad \text{only if} \quad \xi \in SO(2)\, Be_2 \cap E,$$

where $E \subseteq \mathbf{R}^2$ has measure zero. Note further that the Lebesgue measure of the generalized annulus $SO(2)Be_2$ is finite by hypothesis.

On the other hand, the measure of the set $\{x \in \mathbf{R}^2 : f(x, u) \neq 0\}$ is finite for a.a. $u \in SO(2)$. Applying Corollary 1.2 completes the proof.

3. THE AFFINE GROUP

In this section G denotes the "$ax + b$" group, that is, the group of affine transformations of $\mathbf{R}$. It consists of the subset $\{(a, b) : a, b \in \mathbf{R}^2, a > 0\}$ of $\mathbf{R}^2$ with the product

$$(a, b)(a', b') = (aa', ab' + b). \tag{3.1}$$

Left Haar measure on G is $dg = a^{-2}\, da\, db$, where da and db are Lebesgue measures on $\mathbf{R}$. The following facts about G are proved and developed in [10, 13].

Let $L^2(\mathbf{R}^+)$ denote the Hilbert space of square integrable functions on the multiplicative group $\mathbf{R}^+ = \{t \in \mathbf{R}: t > 0\}$ equipped with Haar measure $t^{-1}\,dt$. There is a direct analogue of Plancherel's theorem for unimodular groups valid for G. It is based on just two unitary irreducible representations π_+, π_- acting on $L^2(\mathbf{R}^+)$. They are defined by

$$\pi_\pm(a, b)\,\phi(t) = e^{\pm 2\pi i b t}\phi(at),$$

where $\phi \in L^2(\mathbf{R}^+)$, $t \in \mathbf{R}^+$. Denote the set $\{\pi_+, \pi_-\}$ by $\hat{G}_r$.

Given $f \in L^1(G)$, its Fourier transform $\hat{f}$ on $\hat{G}_r$ is defined by

$$(\hat{f}(\pi_\pm)\phi)(t) = \int_{\mathbf{R}}\int_{\mathbf{R}^+} f(a, b)\,e^{\mp 2\pi i b t}\,\phi(at)\,a^{-2}\,da\,db$$

for $\phi \in L^2(\mathbf{R}^+)$ and $t \in \mathbf{R}^+$. Our result for the affine group only places a very minor restriction on A_f.

3.1. THEOREM. *Let $f \in L^1(G)$ also satisfy $a^{-1/2} f \in L^2(G)$. If $m\{b \in \mathbf{R}: f(a, b) = 0\} > 0$ for a.a. $a \in \mathbf{R}^+$ and $\hat{f}(\pi_+) = 0$ (or $\hat{f}(\pi_-) = 0$), then $f = 0$ a.e.*

Proof. Suppose f satisfies the above hypotheses (with $\hat{f}(\pi_-) = 0$). Let $\mathcal{F}_2 f$ denote the usual Fourier transform of f with respect to the second variable. Using an L^2 argument similar to that in the proofs of 1.4 and 2.1 we conclude that $a \to a^{-1}\mathcal{F}_2(a, b) \in L^2(\mathbf{R}^+)$ for a.a. $b \in \mathbf{R}$.

The fact that $\hat{f}(\pi_+) = 0$ leads to

$$\int_{\mathbf{R}^+} a^{-1}\mathcal{F}_2 f(a/t, t)\,\phi(a)\,a^{-1}\,da = 0 \qquad \text{for} \quad \phi \in L^2(\mathbf{R}^+) \text{ and a.a. } t \in \mathbf{R}^+.$$

Hence for a.a. $a \in \mathbf{R}^+$

$$\mathcal{F}_2 f(a, t) = 0 \qquad \text{for} \quad \text{a.a. } t \in \mathbf{R}^+.$$

But by hypothesis, for a.a. $a \in \mathbf{R}^+$ the function $b \to f(a, b)$ vanishes on a set of positive measure. The L^2 condition on f ensures that almost all of these functions belong to $L^2(\mathbf{R})$. Application of Theorem 0.1 completes the proof. If $\hat{f}(\pi_-) = 0$, replace $f(a, b)$ with $f(a, -b)$.

3.2. *Full Affine Group.* Let G denote the *full affine group*, that is, the group of pairs $\{(a, b): a, b \in \mathbf{R}, a \neq 0\}$ with the product (3.1). It is amusing to notice that the reciprocal-supports property holds for G in the simplest possible manner. In this case the corresponding $\hat{G}_r$ consists of just one element π and a multiple of π is equivalent to the left regular representation. Hence if $f \in L^1(G)$, then $\hat{f}(\pi) = 0$ implies $f = 0$ a.e.

4. The Heisenberg Group

The Heisenberg group G is the subgroup of $GL(3, \mathbf{R})$ consisting of the matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad x, y, z \in \mathbf{R}.$$

For simplicity we write these elements as $g = (x, y, z)$. Haar measure on G is $dg = dx\, dy\, dz$.

Corresponding to each $\lambda \in \mathbf{R}^\times = \mathbf{R} \setminus \{0\}$ there is an irreducible unitary representation T^λ of G consisting of unitary operators in $\mathscr{U}(L^2(\mathbf{R}))$. It is defined by

$$(T_g^\lambda \phi)(t) = e^{-2\pi i \lambda((t-y)x+z)} \phi(t-y).$$

These representations make up the "reduced" dual $\hat{G}_r$ of G [18, 2.12]. Plancherel measure on $\mathbf{R}^\times$ $(\leftrightarrow \hat{G}_r)$ is $|\lambda|\, d\lambda$ although in our case the appropriate measure is simply Lebesgue measure $d\lambda$. The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is defined by

$$(\hat{f}(\lambda)\phi)(t) = \iiint_{\mathbf{R}^3} f(x, y, z)\, e^{-2\pi i \lambda((t-y)x+z)} \phi(t-y)\, dx\, dy\, dz,$$

where $\lambda \in \mathbf{R}^\times$, $\phi \in L^2(\mathbf{R})$, and $t \in \mathbf{R}$.

4.1. Theorem. *Let $f \in L^1(G) \cap L^2(G)$, where G is the Heisenberg group. Suppose that $m\{z \in \mathbf{R}: f(x, y, z) \neq 0\} < \infty$ for a.a. $x, y \in \mathbf{R}$ and $m\{\lambda \in \mathbf{R}^\times : \hat{f}(\lambda) \neq 0\} < \infty$. Then $f = 0$ a.e. (In both cases m stands for ordinary Lebesgue measure and the corresponding sets are measurable.)*

Proof. Let $\mathscr{F}_{13} f$ denote the Euclidean Fourier transform of f in the first and third variables. Then $\mathscr{F}_{13} f \in L^2(\mathbf{R}^3)$ so that, using a linear change of variables,

$$\int_{\mathbf{R}} \int_{\mathbf{R}} |\mathscr{F}_{13} f(\lambda y, t-y, \zeta)|^2\, dy\, dt = \int_{\mathbf{R}} \int_{\mathbf{R}} |\mathscr{F}_{13} f(\zeta, y, \xi)|^2\, d\xi\, dy < \infty$$

for $\lambda \neq 0$ and a.a. ζ. Hence $y \to \mathscr{F}_{13} f(\lambda y, t-y, \zeta) \in L^2(\mathbf{R})$ for $\lambda \neq 0$ and a.a. $t, \zeta \in \mathbf{R}$.

From the assumption $\hat{f}(\lambda)=0$ we deduce that

$$0=\iiint_{\mathbf{R}^3} f(x, y, z)\, e^{-2\pi i\lambda((t-y)x+z)}\phi(t-y)\,dx\,dy\,dz$$

$$=\int_{\mathbf{R}} \mathscr{F}_{13} f(\lambda(t-y), y, \lambda)\,\phi(t-y)\,dy$$

$$=\int_{\mathbf{R}} \mathscr{F}_{13} f(\lambda y, t-y, \lambda)\,\phi(y)\,dy$$

for all $\phi \in L^2(\mathbf{R})$ and a.a. $t \in \mathbf{R}$. Hence $\mathscr{F}_{13} f(\lambda y, t-y, \lambda)=0$ for a.a. $t, y \in \mathbf{R}$. This implies that for a.a. $y \in \mathbf{R}$

$$\{(\xi, \zeta) \in \mathbf{R}^2 : \mathscr{F}_{13} f(\xi, y, \zeta) \neq 0\} \subseteq_{\text{a.e.}} \mathbf{R} \times B, \tag{4.1}$$

where $B=B_f=\{\lambda \in \mathbf{R}^\times : \hat{f}(\lambda) \neq 0\}$. Recall that $m(B)<\infty$.

On the other hand, for a.a. $x, y \in \mathbf{R}$, $m\{z : f(x, y, z) \neq 0\}<\infty$. Now $(x, z) \rightarrow f(x, y, z) \in L^2$ for a.a. $y \in \mathbf{R}$ so that the preceding fact couples with (4.1) to give $f=0$ a.e. using Corollary 1.4.

4.2. *Remarks.* (i) Since the Plancherel measure on $\mathbf{R}^\times$ is given by $|\lambda|\,d\lambda$, the Plancherel measure of a set being finite implies that the Lebesgue measure is finite. Thus we are actually proving a slightly stronger version of 0.1.

(ii) Roughly speaking, the above result says that in the case of the Heisenberg group we can't have nonzero functions $f \in L^2$ which are concentrated in the z direction and which have $\hat{f}$ concentrated. This interpretation is given a quantitative formulation in a forthcoming paper on local uncertainty inequalities for groups.

(iii) Actually Theorem 4.1 is valid for all the Heisenberg groups $\mathbf{H}_n$. The Heisenberg group $\mathbf{H}_n$ is just $\mathbf{R}^{2n+1}$ with the following multiplication,

$$(p, q, t)(p', q', t')=(p+p', q+q', t+t'+(p \cdot q'-p' \cdot q)/2),$$

where $p, q, p', q' \in \mathbf{R}^n$, $t, t' \in \mathbf{R}$, and $\cdot$ denotes the usual inner product for $\mathbf{R}^n$. (When $n=1$, it is easy to prove that $\mathbf{H}_n$ is isomorphic to G.) $\mathbf{H}_n$ is a simply connected two step nilpotent Lie group with Haar measure *dpdqdt*. It has a family of inequivalent irreducible unitary representations $\{\pi_h\}_{h \in \mathbf{R}^\times}$, all realized on $L^2(\mathbf{R}^n)$. As for G, one can write explicit formulae for the π_h. In the general case the Plancherel measure is $|h|^n\,dh$ and one can prove exactly as before the following: Let $f \in L^1(\mathbf{H}_n) \cap L^2(\mathbf{H}_n)$. Suppose $m\{t \in \mathbf{R} : f(p, q, t) \neq 0\}<\infty$ for almost all $p, q \in \mathbf{R}^n$ and $m\{\lambda \in \mathbf{R}^\times : \hat{f}(\lambda) \neq 0\}<\infty$. Then $f=0$ a.e. (In both cases m stands for ordinary Lebesgue measure.)

5. Semisimple Lie Groups

In view of our previous results, in particular Theorem 2.1 for the motion group, it is natural to ask whether an analogue of property (0.1) is valid for any connected noncompact semisimple Lie group. In this section we first prove a version for $G = SL(2, \mathbf{R})$ and then for general semisimple Lie groups with certain restrictions on the functions involved. A good reference for the following background is Lang [14]. See also Ehrenpreis and Mautner [8].

Let $K = SO(2)$ and let π be an irreducible unitary representation of G on the Hilbert space $\mathscr{H}_\pi$. By a *matrix element* of π we mean a function of G of the form $x \to \langle \pi(x)v, w \rangle$, where v, w are K-finite elements in $\mathscr{H}_\pi$. Following Lang [13], let $\{T_\lambda^+\}$ and $\{T_\lambda^-\}$, $\lambda \in \mathbf{R}^+$, be the principal series representations of G, $\{D_n^+\}$ and $\{D_n^-\}$, $n \in \mathbf{Z}^+$, the discrete series representations, and μ the Plancherel measure. These representations form the set $\hat{G}_r$, a subset of the dual of $SL(2, \mathbf{R})$ which supports the Plancherel measure. Also μ restricted to $\{T_\lambda^+\}$ [resp. $\{T_\lambda^-\}$] is $c\lambda \tanh(\pi/2)\lambda \, d\lambda$ [resp. $c\lambda \coth(\pi/2)\lambda \, d\lambda$] and on $\{D_n^+\}$ and $\{D_n^-\}$ it is integer valued. Define $\hat{f}$ on the above representations for $f \in L^1(G)$ in the usual manner by $\hat{f}(\pi) = \pi(f)$.

5.1. Theorem. *Given $f \in L^1(G)$, suppose that*

$$m(A_f K A_f K) < \infty \qquad \textit{and} \qquad \mu(B) < \infty,$$

where m is Haar measure on $SL(2, \mathbf{R})$ (see [14, p. 166]), μ is the Plancherel measure, $A_f = \{x \in G : f(x) \neq 0\}$, and $B = B_f = \{\pi \in \hat{G}_r : \hat{f}(\pi) \neq 0\}$. Then $f = 0$ a.e. (Again B_f is a measurable subset of $\hat{G}_r$.)

Proof. We assume that the reader is somewhat familiar with the representation theory of and harmonic analysis on the group $G = SL(2, \mathbf{R})$. Each $f \in L^1(G)$ can be written as $f = \sum_{m,n \in \mathbf{Z}} f_{mn}$ (in the sense, at least, of distributions), where $f_{mn} = \chi_m * f * \chi_n$, the χ_l, $l \in \mathbf{Z}$, being the characters of $SO(2)$. Hence it suffices to show that each $f_{mn} = 0$ a.e.

The hypothesis on A_f shows that $\{x : f_{mn}(x) \neq 0\}$ always has finite measure. Also, if $\hat{f}(\pi) = 0$ for some $\pi \in \hat{G}_r$, then $\hat{f}_{mn}(\pi) = 0$. Let h be one of the functions f_{mn}. Depending on m, n, $T_\lambda^+(h) = 0$ for all $\lambda \in \mathbf{R}^+$ *or* $T_\lambda^-(h) = 0$ for all $\lambda \in \mathbf{R}^+$. Without loss of generality assume that $\hat{h}$ vanishes on $\{T_\lambda^- : \lambda \in \mathbf{R}^+\}$. One knows that $T_\lambda^+(h)$ is essentially a "scalar-valued function" on $\mathbf{R}^+$ (see [14] for details) and is given by a holomorphic function in a strip containing the real axis. Thus the hypothesis on B forces $T_\lambda^+(h) = 0$ for $\lambda \in \mathbf{R}^+$

Since h is a function such that $T_\lambda^\pm(h) = 0$ for all λ and since $h(k_1 g k_2) = \chi_m(k_1) h(g) \chi_n(k_2)$, $k_1, k_2 \in K$, $g \in SL(2, \mathbf{R})$, for some characters χ_m, χ_n of k, one can show that h must be necessarily a finite linear combination of matrix elements based on members π of the discrete series for which

$\pi(h) \neq 0$. However, these are real analytic functions on G and so the fact that $\{x : h(x) \neq 0\}$ has only finite measure implies $h = 0$.

5.2. *Remark*. An examination of the above proof shows that we have actually proved the following stronger version: Let $f \in L^1(G)$ have the properties that the measure of the complement of KA_fK is positive and the μ-measures of $\{T_\lambda^+\} \setminus (B_f \cap \{T_\lambda^+\})$ and $\{T_\lambda^-\} \setminus (B_f \cap \{T_\lambda^-\})$ are positive. Then $f = 0$ a.e.

5.3. *Semisimple Lie Groups*. Michael Cowling has pointed out that results of Harish–Chandra on representations of semisimple Lie groups allow the above method to go through for arbitrary connected noncompact semisimple Lie groups with finite centres. However, the details require considerable technical knowledge of representations of semisimple Lie groups and will be given in a forthcoming paper [19]. Here we take up a special case—we assume that f is a function on G such that $f(xk) = f(x)$ for all $x \in G$ and $k \in K$ (where K is a fixed maximal compact subgroup of G), that is, f is right K-invariant.

A good reference for this section is the excellent survey article of R. Gangolli [9]. For any unexplained terminology and notation in this section we refer the reader to [9].

Let G be a connected noncompact semisimple Lie group with finite centre and K a fixed maximal compact subgroup of G. Let $G = KAN$ be an Iwasawa decomposition, $\mathfrak{a}$ the Lie algebra of A, $\mathfrak{a}^*$ its dual, and W the Weyl group of the pair (G, A). Let $\{\pi_\lambda\}_{\lambda \in \mathfrak{a}^*}$ be the spherical-principal series representations of G.

All these representations can be realized on the space $H = L^2(K/M)$ (where M is the centralizer of A in K). Then (π_λ, H) is an irreducible unitary representation of G and π_λ and π_ν are unitarily equivalent if and only if $\nu = s\lambda$ for some $s \in W$. Let μ be the (Harish–Chandra) Plancherel measure restricted to $\mathfrak{a}^*/W$. In view of what we said above we can "lift" μ to a W-invariant measure on $\mathfrak{a}^*$. We will denote this measure also by μ. We will need the fact that if a measurable subset U of $\mathfrak{a}^*$ has positive μ-measure, then it has positive Lebesgue measure. This follows from the fact that μ is given by a smooth density which is in fact analytic on $\mathfrak{a}^*$.

As before, let $A_f = \{x \in G : f(x) \neq 0\}$ and $B_f = \{\lambda \in \mathfrak{a}^* : \pi_\lambda(f) \neq 0\}$; both sets are measurable.

5.4. THEOREM. *Let G, K, and μ be as above. Suppose $f \in L^1(G)$ has the following properties*:

(i) *f is right K-invariant*,

(ii) $\mu(\mathfrak{a}^* \setminus B_f) > 0$.

Then $f = 0$ a.e.

Before we give the proof we remark that Theorem 5.4 easily implies the following (weaker) statement, which is an analogue of proper 0.1.

Suppose f is a right K-invariant integrable function on G such that A_f has finite Haar measure and the Plancherel measure of $\{\pi \in \hat{G}: \pi(f) \neq 0\}$ is finite. Then $f=0$ a.e.

Proof of 5.3. The function f^* defined by $f^*(x)=\overline{f(x^{-1})}$ is left K-invariant and hence $g=f^* * f$ is K-biinvariant. Now $\pi_\lambda(f)=0$ if and only if $\pi_\lambda(g)=0$ (because $\pi_\lambda(g)=\pi_\lambda(f)^*\pi_\lambda(f)$). Since g is K-biinvariant, $\pi_\lambda(g)=0$ if and only if $\tilde{g}(\lambda)=0$, where $\tilde{g}$ is the spherical Fourier transform of g (see [9]). However, it is well known that $\tilde{g}$ has a holomorphic extension in a tube containing $\mathfrak{a}^*$ [8] and hence $\tilde{g}(\lambda)=0$ on a set of positive μ-measure implies $\tilde{g}\equiv 0$. Hence, since $h \to \tilde{h}$ is one-to-one on K-biinvariant L^1-functions, $g=0$ a.e. Since $g=f^* * f$, it follows that $f=0$ a.e.

6. Postscript

In the above we have presented a qualitative uncertainty principle for certain groups and families of groups. However, there are versions valid for more general objects. For instance, if X is a compact connected analytic manifold, consider the decomposition

$$L^2(X)=\sum_\lambda^{\oplus} H_\lambda,$$

where the sum is over the eigenvalues λ of the Laplace–Beltrami operator on X and H_λ is the (finite-dimensional) eigenspace corresponding to λ. (There are only countable many such λ's.) Given $f \in L^2(X)$ such that f vanishes on a set of positive measure, then *either* $f=0$ a.e. *or* infinitely many of the projections f_λ on H_λ must be nontrivial. (If f is almost everywhere a finite linear combination of eigenfunctions of the Laplacian, it is equal almost everywhere to a real analytic function. Consequently if f vanishes on a set of positive measure we must have $f=0$ a.e.) A quantitative version of this result will appear in a forthcoming paper [20].

There are limits on how widespread is the phenomenon discussed in this paper. For example, Mautner [16, Theorem 9.1] shows that there exist members π of $\hat{G}$ for $G=PGL(2, \Omega)$, where Ω is a $\mathfrak{P}$-adic field, such that in a suitable orthonormal basis the matrix coefficients of π have compact support. Representations of this type are referred to as supercuspidal and are known to exist for all reductive $\mathfrak{P}$-adic groups [11]. The question arises as to just how general is the principle described in this paper.

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Note added in proof. Let G be a locally compact abelian group. Recently J. A. Hogan has shown that $m(A_f), m(B_f) < \infty$ implies $f = 0$ a.e. for $f \in L^2(G)$ if and only if the identity component of G is noncompact.

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