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MONOMIALS IN THE JONES PROJECTIONS

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ABSTRACT. It is shown that every monomial $e_I = e_{i_1} e_{i_2} \cdots e_{i_n}$ in the Jones projections (with parameter τ) satisfies $e_I = \tau^{n(I)/2} u_I$ where u_I is a partial isometry and $n(I)$ is an integer for which an explicit formula is given.

Let $\{e_n : n \geq 0\}$ be the sequence of Jones projections associated to a fixed number $\tau \in (0, \frac{1}{4}] \cup \frac{1}{4} \sec^2 \pi/n : n \geq 3\}$; thus each e_i is an orthogonal projection and the following commutation relations hold:

$$(1) \qquad \begin{aligned} e_i e_j &= e_j e_i \quad \text{if } |i - j| > 1, \\ e_i e_{i \pm 1} e_i &= \tau e_i. \end{aligned}$$

For any string $I = i_1 i_2 \cdots i_n$ of nonnegative integers, we shall write $e_I = e_{i_1} e_{i_2} \cdots e_{i_n}$. With I as above, we shall write $I \backslash \{i_r\}$ for the string $i_1 \cdots i_{r-1} i_{r+1} \cdots i_n$. More generally, if a string I is obtained by dropping some integers from a string J, we shall call I a substring of J.

The relations (1) above are seen to bear a striking resemblance to the relations in the presentations of the Braid groups as well as the Hecke algebras. Both these connections have, as is well-known, been very fruitfully exploited by Jones, Ocneanu, Wenzl and several others. As is customary when dealing with generators and relations, it is often convenient to work with reduced words. (For the case of Coxeter groups, this is classical and may be found in several places; see, for instance, [**1**].) For the particular case in hand, we shall find it convenient to use the following description of reduced words, which may be found in [**2**] (Aside 4.1.4): If J is any string, then there exists a string I and an integer l such that $e_J = \tau^l e_I$ and such that I has the 'canonical form'

$$(2) \quad I = j_1(j_1 - 1) \cdots k_1 j_2 (j_2 - 1) \cdots k_2 \cdots j_p (j_p - 1) \cdots k_p, \quad \text{with } k_i < k_{i+1} \quad \text{and} \quad j_i < j_{i+1} \quad \text{for } 1 \leq i < p.$$

When I is as in (2), we shall say that I is in canonical form, with p blocks, the ith block of I being the substring $j_i(j_i - 1) \cdots k_i$, finally, we write $b(I) = \sum_{i=1}^{p} (j_i - k_i)$ and refer to $b(I)$ as the block-length of I.

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LEMMA 1. *Let I be as in (2) above. Define*

$$J = j_1 j_2 \cdots j_p (j_1 - 1)(j_1 - 2) \cdots k_1 (j_2 - 1)(j_2 - 2) \cdots k_2 \cdots (j_p - 1)(j_p - 2) \cdots k_p,$$
$$K = j_1 (j_1 - 1) \cdots (k_1 + 1) j_2 (j_2 - 1) \cdots (k_2 + 1) \cdots j_p (j_p - 1) \cdots (k_p + 1) k_1 k_2 \cdots k_p,$$

it being understood that if some $j_i = k_i$, the corresponding 'empty' string in J and K is said to be omitted; then $e_I = e_J = e_K$.

PROOF. This follows immediately from the commutation relations (1) and the inequalities in the definition of the canonical form (2). □

PROPOSITION 1. *If I is any string, there exist an integer $n(I)$ and a partial isometry u_I such that $e_I = \tau^{n(I)/2} u_I$.*

PROOF. The proof is by induction on $l(I)$, the length of the string I. The statement is obvious when $l(I) = 1$. Suppose the statement is valid for any string J with $l(J) < l(I)$.

We may clearly assume that I is in canonical form and given by (2). If $j_1 > k_1$, then $e_I^* e_I = \tau e_J^* e_J$ where $J = I \backslash \{j_1\}$; since $l(J) = l(I) - 1$, the induction hypothesis settles this case. We may, thus, assume that $j_1 = k_1$.

Next, if $k_1 (= j_1) < k_2 - 1$, it follows from (1) that if $J = I \backslash \{k_1\}$, then e_{k_1} commutes with e_J, so $e_I^* e_I = e_{k_1} e_J^* e_J = \tau^{n(J)} e_{k_1} u_J^* u_J$, and $e_{k_1} u_J^* u_J$ is a projection (being a product of commuting projections). Thus, we may also assume that $k_2 = k_1 + 1$.

Hence there exist indices i ($i = 1$ works) such that $k_{i+1} = k_i + 1$. Let r be the largest such index. Then $k_{r+1} < k_i - 1$ for all $i > r + 1$. It follows from Lemma 1 and the relations (1) that $e_I = e_L$ where

$$L = j_1 \cdots k_1 \cdots j_{r-1} \cdots k_{r-1} j_r \cdots (k_r + 1) j_{r+1} \cdots (k_{r+1} + 1) j_{r+2} \cdots k_{r+2} \cdots j_p \cdots k_p k_r k_{r+1};$$

then, $e_I e_I^* = e_L e_L^* = \tau e_J e_J^*$ where $J = L \backslash \{k_{r+1}\}$. By induction hypothesis, $e_J e_J^* = \tau^{n(J)} q_J$ where $q_J = u_J u_J^*$ is a projection; hence $e_I e_I^* = \tau^{n(J)+1} q_J$, so that the proposition is valid for I, with $n(I) = n(J) + 1$. □

We turn now to the determination of $n(I)$ (as in Proposition 1). Notice that $\tau^{n(I)} = \|e_I\|^2$.

PROPOSITION 2. *Let I be in canonical form and given by (2). Inductively define the integers $l_1, l_2, \ldots, l_p$ thus: $l_1 = k_1$; if $1 < i \leq p$ and l_{i-1} has been defined, let*

$$l_i = \begin{cases} \min\{j_i, l_{i-1} + 2\}, & \text{if } j_i > k_i \leq l_{i-1} + 1, \\ k_i, & \text{otherwise.} \end{cases}$$

Then $n(I) = b(I) + \#\{i \colon 1 \leq i < p, l_{i+1} = l_i + 1\}$.

PROOF. The proof is by induction of $b(I)$. (Recall that $b(I) = \sum_{i=1}^{p} (j_i - k_i)$.)

If $b(I) = 0$, then for each i, we have $j_i = k_i = l_i$. Thus $e_I = e_{l_1} e_{l_2} \cdots e_{l_p}$, and since $l_1 < \cdots < l_p$, it is clear from (1) that $n(I) = \#\{i \colon 1 \leq i < p, l_{i+1} = l_i + 1\}$.

Assume now that the proposition is valid for any string J which is in canonical form and satisfies $b(J) < b(I)$, and that $b(I) > 0$. The induction step will be complete once we can find a substring J of I such that

(i) J is in canonical form;

(ii) $\|e_I\|^2 = \tau\|e_J\|^2$ (so that $n(I) = n(J) + 1$);

(iii) $b(I) = b(J) + 1$;

(iv) both I and J have the same associated l_i's.

We shall obtain J by dropping from I some member of some nontrivial block. Thus, suppose $1 \leq r \leq p$, $j_r > k_r$ and $j_r \geq m \geq k_r$ and $J = I\backslash\{m\}$. Since the l_i's are defined inductively, it is clear that $l_i(I) = l_i(J)$ for $i < r$. Further, if it is the case that $l_r(I) = l_r(J)$, it would then follow (from the fact that l_i depends only on the ith block and l_{i-1}) that also $l_i(I) = l_i(J)$ for $i > r$. So if J is constructed in the manner described, we would only have to verify $l_r(I) = l_r(J)$ in order to establish (iv).

Let s be the index of the first nontrivial block in I; i.e., $s = \min\{i\colon 1 \leq i \leq p, j_i > k_i\}$.

Case (i). $s = 1$, or $s > 1$ and $k_s - 1 > k_{s-1}$ $(= j_{s-1})$.

In this case, we let $J = I\backslash\{j_s\}$, and observe that (i) follows at once, as does (iii), while the commutation relations (1) ensure that $e_I^* e_I = \tau e_J^* e_J$ and hence (ii) follows. As for (iv), notice that $l_i(I) = l_i(J) = j_i$ for $i < s$; since $k_s > l_{s-1} + 1$, we have $l_s(I) = l_s(J) = k_s$, and hence (iv) follows from the above remarks.

Case (ii). $s > 1$ and $k_s - 1 = k_{s-1}$ $(= j_{s-1})$.

In this case, there exist indices i (for instance, $i = s$) which correspond to nontrivial blocks and satisfy $k_i = k_{i-1} + 1$. Let r be the largest such index, and put $J = I\backslash\{k_r\}$.

If $r = p$, (i) is immediate; if $r < p$, then either $k_{r+1} = j_{r+1} > j_r \geq k_r + 1$, or $j_{r+1} > k_{r+1} > k_r + 1$; so, in any case $k_{r-1} + 1 = k_r < k_i - 1$ for $i > r$. It follows at once that (i) (and clearly, also (iii)) is valid; also, the above inequalities ensure that ($e_{k_{r-1}} e_{k_r}$ can be pulled to the extreme right in e_I and that) $e_I e_I^* = \tau e_J e_J^*$ and hence (ii) holds. It remains only to establish $l_r(I) = l_r(J)$, and we do this by considering two cases.

Case (iia). $l_{r-1} = k_{r-1}$. (Note that $l_{r-1}(I) = l_{r-1}(J)$.)

Here, $k_r = l_{r-1} + 1$ and also $j_r > k_r$ (by the definition of r) and hence,

$$l_r(I) = \min\{j_r, l_{r-1} + 2\} = k_r + 1.$$

On the other hand, $k_r(J) = k_r + 1 > l_{r-1}(J) + 1$ and hence $l_r(J) = k_r(J) = k_r + 1$, as desired.

Case (iib). $l_{r-1} > k_{r-1}$. (Notice that $j_i \geq l_i \geq k_i \forall i$.)

Here, $j_r > k_r = k_{r-1} + 1 \leq l_{r-1}$ and so $l_r(I) = \min\{j_r, l_{r-1} + 2\}$.

On the other hand, $k_r(J) = k_r + 1 \leq l_{r-1} + 1$, and we must distinguish between the cases $j_r = k_r + 1$ and $j_r > k_r + 1$. If $j_r = k_r + 1$, then $l_r(J) = k_r(J) = k_r + 1$, while $j_r \leq l_{r-1} + 1$ implies that

$$l_r(I) = \min\{j_r, l_{r-1} + 2\} = j_r = k_r + 1 = l_r(J).$$

If $j_r > k_r + 1$, then

$$l_r(J) = \min\{j_r, l_{r-1} + 2\} = l_r(I),$$

and the proof is complete. □

EXAMPLES. (1) If $I = 0\ 21\ 432\ 6543\ 765$, then $b(I) = 0 + 1 + 2 + 3 + 2 = 8$, and $l_1 = 0$, $l_2 = 2$, $l_3 = 4$, $l_4 = 6$, $l_5 = 7$; since there is only one pair of successive integers in the l_i sequence, we find that $\|e_I\|^2 = \tau^9$.

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(2) Let $n \geq 1$ and $I = n(n-1)\cdots 0(n+1)n\cdots 1\cdots(2n)(2n-1)\cdots n$. In our notation, $p = n+1$, $j_i = n+i-1$, $k_i = i-1$ and $b(I) = (n+1)n$. It is easy to see, inductively, that $l_i = 2(i-1)$ for each i, and hence $\|e_I\|^2 = \tau^{n(n+1)}$. Also, a repeated application of Lemma 1 shows that

$$\begin{aligned} e_I &= e_n \cdots e_1 e_{n+1} \cdots e_2 \cdots e_{2n} \cdots e_{n+1} e_0 e_1 \cdots e_n \\ &= e_n \cdots e_2 e_{n+1} \cdots e_3 \cdots e_{2n} \cdots e_{n+2} e_1 e_2 \cdots e_{n+1} e_0 e_1 \cdots e_n \\ &= \cdots \\ &= e_n e_{n+1} \cdots e_{2n} e_{n+1} e_{n+2} \cdots e_{2n-1} \cdots e_1 e_2 \cdots e_{n+1} e_0 e_1 \cdots e_n \\ &= e_I^*. \end{aligned}$$

It follows that $\tau^{-n(n+1)/2} e_I$ is a projection. It was shown in [**3** and **4**] that if $M_{-1} \subseteq M_0$ is a pair of II_1 factors with $[M_0 : M_{-1}] = \tau^{-1}$, and if $M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ is the tower of II_1 factors obtained by iterating the basic construction, so that $M_{n+1} = (M_n \cup \{e_n\})''$, then for any $n \geq 1$, $M_{-1} \subseteq M_n \subset M_{2n+1}$ 'is also a basic construction' and the projection in M_{2n+1} which implements the conditional expectation of M_n onto M_{-1} is precisely the '$\tau^{-n(n+1)/2} e_I$' of this example. See [**4**] for another proof of the fact that this is a projection. In fact, it was an examination of this 'example' that resulted in this short note.

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