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A NOTE ON THREE-DECISION ASR PLAN PROVIDING AVERAGE QUALITY PROTECTION IN TERMS OF INFLECTION AVERAGE OUTGOING QUALITY

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SUMMARY. In this paper three-decision accept-screen-reject (ASR) plan providing average quality protection in terms of inflection average outgoing quality (IAOQ) is introduced. Determination of optimal ASR plan providing IAOQ protection is illustrated by numerical example. It is found that IAOQ plans are easier to obtain as compared to AOQL plans and provide comparable protection to consumer.

1. Introduction

Average outgoing quality limit (AOQL) is used to express average quality protection since Dodge and Romig (1959) proposed the sampling plans by attributes. Pandey (1977, 1984) includes, among other three-decision plans, accept-screen-reject (ASR) plans (n, a, r) providing average quality protection in terms of average outgoing quality limit (AOQL) which is operated as follows :

From a lot of N items a random sample of n items is selected and the number of defectives x in the sample is found and the lot is

$$\left.\begin{array}{l} \text{accepted if } 0 \leqslant x \leqslant a \\ \text{screened if } a < x < r \\ \text{rejected if } r \leqslant x \leqslant n \end{array}\right\}. \qquad \ldots \quad (1.1)$$

In view of the criticism of AOQL as a measure of outgoing quality as pointed out in Section 4, any other index for average quality, in particular, IAOQ introduced in the present paper, may serve the purpose of providing average quality protection.

We have chosen only ASR plan for introducing, in this paper, inflection average outgoing quality (IAOQ) to provide average quality protection. It

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has been shown that IAOQ, as expected, is smaller than AOQL for a given three-decision ASR plan. Further, assuming that incoming lot quality $p \leqslant 0.10$ (i.e., Poisson condition, see Hald, 1981), it has been shown that IAOQ has an expression similar to that of AOQL. Moreover, the former is much easier to obtain. Numerical example is provided to illustrate the procedure to obtain optimal three-decision ASR plan providing average quality protection in terms of IAOQ and a set of such plans is also given.

2. Inflection average outgoing quality (IAOQ)

Let a lot of N items of quality p be submitted under a three-decision plan (n, a, r) where n denotes the size of the random sample from the lot, a and r are the two decision numbers. Following the notations of Hald (1981) and Pandey (1984) define

$$P_a(p) = P(x \leqslant a \,|\, n, a, r, p) = B(a\,; n, p) \qquad \dots \quad (2.1)$$

$$P_t(p) = P(a < x < r \,|\, n, a, r, p) = B(r-1\,; n, p) - B(a\,; n, p) \qquad \dots \quad (2.2)$$

$$P_r(p) = P(x \geqslant r \,|\, n, a, r, p) = 1 - B(r-1\,; n, p) \qquad \dots \quad (2.3)$$

where $r-a > 1$ and the subscripts a, t and r indicate acceptance, total inspection (screening) and outright rejection of the lot respectively.

The point of inflection of $P_a(p)$ is obtained by solving $P_a''(p) = 0$ for p. Writing $P_a(p) = \sum_{x=0}^{a} [(np)^x/x!] \exp(-np)$ under Poisson condition and differentiating twice with respect to p gives

$$P_a''(p) = -n^2 \, [((np)^{a-1}/a!)\,(a-np)] \exp(-np). \qquad \dots \quad (2.4)$$

Equating (2.4) to zero we get $p = a/n = p^0$(say) as the point of inflection of $P_a(p)$. We shall define inflection average outgoing quality (IAOQ) denoted by P_A^0 as follows :

Definition : Inflection average outgoing quality (IAOQ) of a three-decision ASR plan (n, a, r) is defined as its average outgoing quality at the point of inflection p^0 of $P_a(p)$.

For incoming lot quality p and lot size N the expression for average outgoing quality (AOQ) of three-decision ASR plan (n, a, r), following Pandey (1977) is given by

$$P_A(p) = [(N-n)/N]\,[pP_a(p) + o.P_t(p)]/[P_a(p) + P_t(p)] \qquad \dots \quad (2.5)$$

Assuming Poisson condition to approximate binomial probabilities in (2.1)–(2.3) and writing gamma cumulative probability for Poisson cumulative probability and substituting $p^0 = a/n$ for p in (2.5) we write inflection average outgoing quality, p_A^0 following the definition as follows :

$$P_A(p^0) = p_A^0 = [(N-n)/N]a\Gamma(r)\int_a^{\infty} e^{-w}w^a dw/[n\Gamma(a+1)\int_a^{\infty} e^{-w}w^{r-1}dw] \quad \ldots \quad (2.6)$$

for $a > 0$.

3. Relationship between AOQL and IAOQ

Differentiating (2.5) with respect to p and equating the derivative to zero, we obtain

$$\frac{dp_A}{dp} = \frac{N-I_1}{N} - \frac{p}{N}\frac{dI_1}{dp} = 0 \quad \ldots \quad (3.1)$$

where

$$I_1 = \frac{[n\, P_a(p)+N\, P_t(p)]}{[P_a(p)+P_t(p)]}. \quad \ldots \quad (3.2)$$

Let the solution of (3.1) be $p = p_m$. Substituting for I_1 in (3.1), assuming Poisson condition and writing the probabilities in terms of cumulative Poisson, and denoting np_m as x and substituting it for np in (3.1) we obtain

$$\frac{g(a, x)}{G(a, x)} = \frac{1}{x} + \frac{g(r-1, x)}{G(r-1, x)} \quad \ldots \quad (3.3)$$

where $g(a, x) = e^{-x}x^a/a!$ and $G(a, x) = \sum_{z=0}^{a} g(z, x)$ (see Pandey, 1977). Average outgoing quality limit (AOQL) denoted by p_L is the maximum value of p_A and can be obtained by substituting x for np in (2.5). Writing probabilities in terms of cumulative gamma and substitutting x for np we get from (2.5)

$$p_L = \frac{[(N-n)/N]x\Gamma(r)\int_x^{\infty} e^{-w}w^a dw}{[n\Gamma(a+1)\int_x^{\infty} e^{-w}w^{r-1}dw]} \quad \ldots \quad (3.4)$$

as AOQL value for a three-decision ASR plan (n, a, r) where x is the solution of (3.3).

Let I denote a set of positive integers and

$$S = \left\{ x : \frac{g(a, x)}{G(a, x)} = \frac{1}{x} + \frac{g(r-1, x)}{G(r-1, x)} \right\}.$$

Theorem 1 : *For $x \in S$, $0 < a \in I$, $r \in I$ and $r-a > 1$ where a and r are the values corresponding to $x \in S$*

$$\text{(i)}\quad a\int_a^\infty e^{-w}w^a dw \Big/ \int_a^\infty e^{-w}w^{r-1}dw < x\int_x^\infty e^{-w}w^a dw \Big/ \int_x^\infty e^{-w}w^{r-1}dw \qquad \ldots (3.5)$$

and

$$\text{(ii)}\quad f(x) = x\int_x^\infty e^{-w}w^a dw \Big/ \int_x^\infty e^{-w}w^{r-1}dw \qquad \ldots (3.6)$$

is a monotonically increasing or decreasing function of x according as $\phi(x)=x\,[u_x(r-1)-u_x(a)]$ is strictly greater or less than $r-a-2$ respectively where

$$u_x(a) = \int_1^\infty e^{-wx}w^{a+1}dw \Big/ \int_1^\infty e^{-wx}w^a dw.$$

Proof : (i) The proof of the first part is trivial. Since $p_L = \max_p p_A(p) > p_L^0$. (2.6) and (3.4) imply the result.

(ii) Substituting wx for w in (3.6) we get

$$f(x) = x^{a+2-r}\int_1^\infty e^{-wx}w^a dw \Big/ \int_1^\infty e^{-wx}w^{r-1}\,dw$$

$$f'(x) = v(x)\,[a+2-r-x[\int_1^\infty e^{-wx}w^{a+1}\,dw \Big/ \int_1^\infty e^{-wx}\,w^a\,dw$$

$$-\int_1^\infty e^{-wx}w^r dw \Big/ \int_1^\infty e^{-wx}w^{r-1}\,dw]] \qquad \ldots (3.7)$$

Rewrite (3.7) as

$$f'(x) = v(x)\,[a+2-r-x\,[u_x(a)-u_x(r-1)]] \qquad \ldots (3.8)$$

where

$$0 < v(x) = x^{a+1-r}\int_1^\infty e^{-wx}w^{r-1}\,dw\int_1^\infty e^{-wx}w^a dw \Big/ \Big(\int_1^\infty e^{-wx}w^{r-1}dw\Big)^2.$$

For real $g(w)$ and $h(w)$ and $w \in A$ Schwarz's inequality states

$$[\int_A g(w)h(w)dw]^2 \leqslant [\int_A g^2(w)dw]\,[\int_A h^2(w)dw]. \qquad \ldots (3.9)$$

Let $\qquad g(w) = e^{-wx/2}\,w^{(a+1)/2}\,;\; h(w) = e^{-wx/2}\,w^{(a-1)/2},$

$$A = \{w : 1 \leqslant w \leqslant \infty\}$$

then from (3.9) we obtain $u_x(a) > u_x(a-1)$. Since $0 < a \in I$ is an arbitrary number and $r-a > 1$ the result $u_x(a) < u_x(r-1)$ follows in general. Since $r(x) > 0$, $x \in S$ and $u_x(a) < u_x(r-1)$ we get from (3.8)

$$f'(x) > 0 \text{ if } \phi(x) > r-a-2 \quad \ldots \quad (3.10)$$

$$f'(x) < 0 \text{ if } \phi(x) < r-a-2 \quad \ldots \quad (3.11)$$

proving the result.

The first part of Theorem 1 shows that for a given x there exists an "a" such that $f(a) < f(x)$ is always true. The second part of the theorem shows that $f(x)$ is a monotonically increasing function of x if $\phi(x) > r-a-2$ for a and r corresponding to $x \in S$. Suppose $\phi(x) > r-a-2$ holds, as for example is the case for the tabulated values in the Table 1. For the first part of the theorem to be true we must have $a < x$ in the Table 1 for each x and the corresponding value of a. Since $0 < a \in I$ and $x \in S$ the inequality $a < x$ is violated mainly due to the fact that "a" takes only integer values whereas x may take any fractional values under (3.3). For example we have $a > x$ for $a = 15(1)20$ in the Table 1 but it does not imply that $f(a) > f(x)$ otherwise the theorem will be self contradictory under its two parts.

For $a = 1(1)20$ Table 1 provides the solution x of the equation (3.3) and the value of the function $\phi(x)$.

TABLE 1. THE VALUES OF x AND $\phi(x)$ FOR $a = 1(1)20$

a	r	x	$\phi(x)$	a	r	x	$\phi(x)$
1	3	2.236375	0.43480	11	13	11.215362	0.48082
2	4	3.002727	0.43607	12	14	12.140244	0.47604
3	5	3.965123	0.43984	13	15	13.068409	0.47106
4	6	4.849036	0.453852	14	16	14.000797	0.46793
5	7	5.742012	0.45110	15	17	14.930630	0.40451
6	8	6.641530	0.44510	16	18	15.864714	0.40128
7	9	7.547888	0.46182	17	19	16.800401	0.45797
8	10	8.450209	0.45637	18	20	17.737806	0.48594
9	11	9.374142	0.47155	19	21	18.677091	0.58284
10	12	10.202835	0.44702	20	22	19.617890	0.47997

To obtain the solution x, in the Table 1, we have chosen systematically the values of "a" and the corresponding smallest feasible values of r in equation (3.3). The solutions were obtained by a computer programme using Newton-Raphson method. The values of $\phi(x)$ are obtained by computing.

$$[r\,G(r, x)/G(r-1, x)]-[(a+1)\,G(a+1, x)/G(a, x)].$$

As it can be seen from the Table 1 $\phi(x) > r-a-2 = 0$ for all $a = 1(1)20$ and hence $f(x)$ is increasing function of x. Figure 1 gives the curve of $f(x)$ against x for the chosen values of "a" in the Table 1.

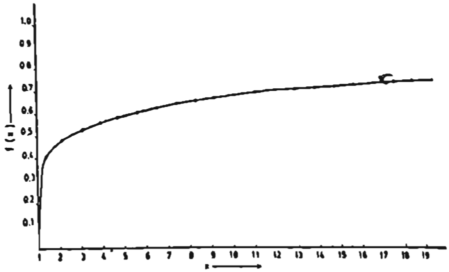

Fig. 1

For a given ASR plan (n, a, r) computation of p_A^0 does not require the value of x as it is clear from (2.6).

4. Justifications for use of IAOQ in place of AOQL

The concept of AOQL acquired an accepted physical meaning and gained usage for specifying consumer's protection over the years. But the physical meaning of AOQL conveying the idea of a limit to average outgoing quality came under heavy criticism by several authors. It was mainly because AOQL does not provide a sharp upper bound for average outgoing quality and hence its role as a measure of average outgoing quality is substantially reduced. Anscombe (1958) characterised AOQL as being only a statistician's guarantee and remarked that in practice it is not the consumer's requirement.

On the other hand, for a given plan (n, a, r) the point of inflection of $P_a(p)$ is given by the ratio a/n and IAOQ is the value of average outgoing quality for incoming quality equal to the ratio a/n. This connection of IAOQ with the ratio a/n provides a clear physical meaning to it in terms of average outgoing quality.

IAOQ is the value of AOQ at incoming quality a/n whereas AOQL is the value of AOQ at p_m which is the solution of a complicated equation (3.3). Thus, the concept of IAOQ is easier to grasp for a given plan.

For specified values of IAOQ or AOQL one can obtain a large number of plans satisfying the stated values. The unique plan from amongst these plans is obtained by picking up a plan minimising the average amount of inspection at specified process average quality. In this context of determining the optimal plan, AOQL is no way superior to IAOQ.

5. Optimal three-decision ASR plan with IAOQ protection

Given lot size N, process average $\bar{p}$ and IAOQ $= p_A^0$ the problem is to determine (n, a, r) minimising average amount of inspection $\bar{I}$ given by

$$\bar{I} = n+(N-n)\, P_i(\bar{p}) \qquad \dots \quad (5.1)$$

and satisfying the specified value of p_A^0.

The procedure to obtain ASR plan providing IAOQ protection and minimizing average amount of inspection at process average quality is as follows :

Obtain the boundary condition using the optimality condition

$$\Delta\, \bar{I}\,(a-1) \leqslant 0 < \Delta\, \bar{I}\,(a) \qquad \dots \quad (5.2)$$

where $$\bar{I}(a) = n+(N-n)\, P(a+1 \leqslant x \leqslant r-1 \,|\, p = \bar{p}). \qquad \dots \quad (5.3)$$

Let $\bar{M} = N\bar{p}$ and $k = \bar{p}/p_A^0$. Since "a" takes only discrete values minimum value of $\bar{I}$ will be found for many pairs $(\bar{M}, k)$ on the $(\bar{M}, k)$-plane for the same value of "a". From this it is evident that on $(\bar{M}, k)$-plane there exist zones in which the values of decision number "a" remains the same. To find the boundary lines it can be noted that for certain pairs $(\bar{M}, k)$ two pairs (a, b) exist giving the same minimum value of $\bar{I}$. These values of "a" differ by 1 in all such cases. We use the notation $b = n\bar{p}$ and b_a as the value of "b" associated with "a". Thus, from (5.1) setting $\bar{I}(a) = \bar{I}(a+1)$ the boundary condition can be written as

$$\bar{M} = \frac{(b_{a+1}-b_a)+b_a \sum\limits_{a+1}^{(r-1)^{(a)}} g(x, b_a) - b_{a+1} \sum\limits_{a+2}^{(r-1)^{(a+1)}} g(x, b_{a+1})}{\sum\limits_{a+1}^{(r-1)^{(a)}} g(x, b_a) - \sum\limits_{a+2}^{(r-1)^{(a+1)}} g(x, b_{a+1})} \qquad \dots \quad (5.4)$$

The chart on the $(\bar{M}, k)$-plane gives the values of "a" minimizing $\bar{I}$ for specified value of $\bar{p}/p_A^0$ and lot size (N).

A systematic iterative procedure to obtain the boundary point on the $(\bar{M}, k)$-plane can be stated on the lines of Pandey (1977) with the above modified notations. To avoid repetition the steps are not reproduced here. When lot size N, a and r are known the sample size n can be obtained from

$$n = NU/(N\,p_\lambda^0+U) \qquad \dots \quad (5.5)$$

where $$U = a[\Gamma(r)/\Gamma(a+1)] \int_a^\infty e^{-w}w^a\,dw \Big/ \int_a^\infty e^{-w}w^{r-1}\,dw.$$

Numerical example : Let $\bar{p} = 0.005$, $p_\lambda^0 = 0.05$. For $k = \bar{p}/p_\lambda^0 = 0.100$ and $a = 1, 2, 3$ and 4 and using the corresponding values of b_a the boundary points $\bar{M}$ for adajcent zones were obtained as given in Table 2

TABLE 2. THE BOUNDARY POINTS $\bar{M}$ FOR ASR PLANS

a	$a+1$	$\bar{M}$
1	2	16.389329
2	3	136.381060
3	4	978.760790
4	5	6430.144500

The value of $\bar{M}$ corresponding to $k = 0.100$ from Table 2 is 16.389329 for $a = 1$ and $a+1 = 2$. The value of lot size at the boundary of the zone for $a = 1$ is obtained by dividing the value of $\bar{M}$ by the value of p i.e.,

$$N = \frac{16.389320}{0.005} = 3277.8658.$$

Using (5.5) we get the value of sample size as 14.95. The average lot size for $a = 1$ is taken as 229 and the lot range is as 16—327[7]. For $a = 2$ and $a+1 = 3$ the value of lot size at boundary of the zone for $a = 2$ is computed in a similar manner and is equal to 27276.212. To compute the value of sample size (n) we require the values of U defined by (5.5). For computation of AOQL for the different plans we shall use the formula

$$p_L = \left(\frac{1}{n}-\frac{1}{N}\right) y \qquad \dots \quad (5.6)$$

where y denotes the coefficient of $(N-n)/nN$ in the expression for p_L given by (3.4). The values of U, y and also n are provided in Table 3 for $a = 1(1)4$.

TABLE 3. THE VALUES OF U AND y FOR $a = 1(1)4$

a	r	x	U	y	n	a/n
1	3	2.230376	0.800000	0.952182	14.06	.0068
2	4	3.002727	1.448274	1.558027	28.91	.0091
3	5	3.965121	2.381670	2.564303	47.01	.0030
4	6	4.849036	3.203608	3.351019	64.07	.0024

A set of illustrative ASR plans with average quality protection in terms of inflection average outgoing quality is given in the Table 4 where $\overline{N}$ is geometric mean of lot range limits.

TABLE 4. SOME ILLUSTRATIVE OPTIMAL ASR SINGLE SAMPLING PLANS FOR $\bar{p} = 0.005$ AND $p_L^* = 0.05$.

lot size	$\overline{N}$	a	r	n	AOQL	IAOQ
16–3277	229	1	3	15	0.0593	0.0498
3278–27276	15277	2	4	29	0.0536	0.0498
27277–195750	111514	3	5	48	0.0534	0.0495
195751–1286028	740890	4	6	64	0.0523	0.0500

As we know from the expression (2.6) inflection average outgoing quality (IAOQ) is defined only for $a > 0$. We do not have ASR plan providing average quality protection in terms of IAOQ for $a = 0$. We note from Table 3 that, for the numerical example, the sample size (n) and decision (a) vary in such a way that the ratio a/n remains constant around 1/15 for the chosen values of $a = 1(1)4$.

For the plan (15, 1, 3), $n = 15$, $\overline{N} = 229$ and $y = 0.952182$. Substituting these values in (5.6) we obtain $p_L = 0.0593$. Similarly, the values of p_L for other illustrative plans are computed. These values along with actual values of IAOQ for different illustrative plans are given in Table 4. It can be noted that the values of IAOQ are close to the specified value of 0.05 and the corresponding values of AOQL, as expected, are not exceeded by the values of IAOQ.

6. Concluding remarks

Three-decision ASR plan providing average quality protection in terms of IAOQ does not require solving the cumbersome AOQL equations using Newton-Raphson method. Under Poisson condition inflection point is uniquely determined as a ratio of acceptance decision number and the sample size which is found quite convenient in practice. It may be interesting to develop procedure to obtain optimal three-decision plan from a family of plans having a fixed value of the ratio a/n instead of a fixed value of IAOQ.

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