A Complete Formulation of Baum-Connes' Conjecture for the Action of Discrete Quantum Groups

by

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Abstract

We formulate a version of Baum-Connes' conjecture for a discrete quantum group, building on our earlier work ([9]). Given such a quantum group $\mathcal{A}$, we construct a directed family $\{\mathcal{E}_F\}$ of C^*-algebras (F varying over some suitable index set), borrowing the ideas of [5], such that there is a natural action of $\mathcal{A}$ on each $\mathcal{E}_F$ satisfying the assumptions of [9], which makes it possible to define the "analytical assembly map", say $\mu_i^{r,F}$, $i = 0, 1$, as in [9], from the $\mathcal{A}$-equivariant K-homolgy groups of $\mathcal{E}_F$ to the K-theory groups of the "reduced" dual $\hat{\mathcal{A}}_r$ (c.f. [9] and the references therein for more details). As a result, we can define the Baum-Connes' maps $\mu_i^r : \overset{\lim}{\longrightarrow} KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \to K_i(\hat{\mathcal{A}}_r)$, and in the classical case, i.e. when $\mathcal{A}$ is $C_0(G)$ for a discrete group, the isomorphism of the above maps for $i = 0, 1$ is equivalent to the Baum-Connes' conjecture. Furthermore, we verify its truth for an arbitrary finite dimensional quantum group and obtain partial results for the dual of $SU_q(2)$.

Key words : Baum-Connes Conjecture, Discrete Quantum Group, Equivariant KK-Theory
AMS Subject classification numbers : 19K35, 46L80, 81R50

1 Introduction

1.1 Classical Baum-Connes' Conjecture

Baum-Connes' conjecture has been occupying the centre-stage of K-Theory and geometry for more than two decades. Given a locally compact group G, and a locally compact Hausdorff space X equipped with a G-action such that X is proper and G-compact, there are canonical maps $\mu_i^r : KK_i^G(C_0(X), \mathbb{C}) \to K_i(C_r^*(G))$, and $\mu_i : KK_i^G(C_0(X), \mathbb{C}) \to K_i(C^*(G))$, for $i = 0, 1$, where $C_0(X)$ is the commutative C^*-algebra of continuous complex-valued functions on X vanishing at infinity, $C_r^*(G)$ and $C^*(G)$ are respectively the reduced and free groups C^*-algebras, and $KK_.^G$ denotes the Kasparov's equivariant KK-functor. In particular, $KK^G(C_0(X), \mathbb{C})$ is identified with the G-equivariant K-homology of X, and thus is essentially something geometric or topological, whereas the object $K_i(.)$ on the right hand side involves the reduced or free group algebras, which are analytic in some sense.

Now, let $\underline{EG}$ be the universal space for proper actions of G. The definition of proper G-actions and explicit constructions in various cases of interest can be found in [18],[19] and the references therein. The equivariant K-homology of $\underline{EG}$, say $RK_i^G(\underline{EG}), i = 0, 1$, can be defined as the inductive limit of $KK_i^G(C_0(X), \mathbb{C})$, over all possible locally compact, G-proper and G-compact subsets X of the universal space $\underline{EG}$. Since the construction of KK_i^G and K_i commute with the procedure of taking an inductive limit, it is possible to define μ_i^r, μ^i on the equivariant K-homology $RK_i^G(\underline{EG})$. The conjecture of Baum-Connes states that $\mu_i^r, i = 0, 1$ are isomorphisms of abelian groups. This conjecture admits certain other generalizations, such as the Baum-Connes conjecture with coefficients (which seems to be false from some recent result announced by M. Gromov, see [18] for references), but we do not want to discuss those here. However, we would like to point out that the Baum-Connes conjecture has already been verified for many classical groups (and for a wide variety of coefficient algebras as well) , using different methods and ideas from many diverse areas of mathematics, and has given birth to many new and interesting tools and techniques in all these areas. In fact, the truth of this conjecture, if established, will prove many other famous conjectures in topology, geometry and K-theory.

1.2 Motivation for a quantum version

During the last two decades, the theory of quantum groups, which is a natural and far-reaching generalization of the concept of topological groups, has become another fast-growing branch of mathematics and mathematical physics, thanks to the works by Woronowicz, Drinfeld, Jimbo and many other mathematicians ([20], [6], [**?**]). On the other hand, with the pioneering efforts of Connes (see [4]), followed by himself and many other mathematicians, a powerful generalization of classical differential and Riemannian geometry has emerged under the name of noncommutative geometry, which has had, since its very beginning, very close connections with K-theory too. Furthermore, Baaj and Skandalis ([3]) have been able to construct an analogue of equivariant KK-theory for the actions of quantum groups, as natural extension of Kasparov's equivariant KK-theory. This motivates one to think of a possibility of generalizing the Baum-Connes construction in the framework of quantum groups, as one hopes for a possible rich interplay between noncommutative geometry and quantum groups in this context. In the present article, we make an attempt towards this generalization, completing our formulation of this conjecture for discrete quantum groups as a follow-up of our earlier paper [9] in this direction.

1.3 Plan of the paper

As remarked in [9], the classical formulation of the Baum-Connes conjecture for the action of locally compact groups or discrete groups G could be achieved in two steps. The first step is to define assembly maps μ_i, μ_i^r for G-compact and G-proper actions. We achieved the analogue of this step for discrete quantum groups in [9]. More precisely, given an action of a discrete quantum group $\mathcal{A}$ on a C^*-algebra $\mathcal{C}$ satisfying certain regularity assumptions (resembling the notion of proper G-compact action of classical discrete groups on some space), we constructed canonical maps μ_i, μ_i^r $(i = 0, 1)$ from the $\mathcal{A}$-equivariant K-homology groups $KK_i^{\mathcal{A}}(\mathcal{C}, \mathbb{C})$ to the K-theory groups $K_i(\hat{\mathcal{A}}), K_i(\hat{\mathcal{A}}_r)$ respectively, where $\hat{\mathcal{A}}, \hat{\mathcal{A}}_r$ denote respectively the quantum analogue of the full and reduced group C^*-algebras. For readers' convenience, we briefly review in section 4 our constructions and results in [9]. We also illustrate these constructions with examples in 4.3.

The second step in the classical formulation of the Baum-Connes conjecture is to define a universal space $\underline{EG}$ for the proper actions of G and build explicit good models for $\underline{EG}$ to show that it can be approximated by its subsets X having G-proper and G-compact actions, thereby defining appropriate maps from the equivariant K-homology groups $RK_i^G(\underline{EG}) = \xrightarrow{\lim} KK_i^G(C_0(X), \mathbb{C})$ to $K_i(\hat{\mathcal{A}}), K_i(\hat{\mathcal{A}}_r)$ by inductive limits. We achieve this step in this paper for the action of discrete quantum groups by borrowing ideas from the work of Cuntz [5], who used the language of noncommutative simplicial complexes to describe this step in the classical situation. More precisely, we construct a directed family $\{\mathcal{E}_F\}$ of C^*-algebras (F varying over some index set) and prove that the natural action of $\mathcal{A}$ on $\mathcal{E}_F$ satisfies the assumptions in [9] which makes it possible to define analytic assembly maps $\mu_i^F, \mu_i^{r,F} (i = 0, 1)$ as in [9] from the $\mathcal{A}$-equivariant K-homology groups of $\mathcal{E}_F$ to the K-theory groups of $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}_r$. Consequently, we are able to define the Baum-Connes maps

$$\mu_i : \xrightarrow{\lim} KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \rightarrow KK_i(\mathbb{C}, \hat{\mathcal{A}}),$$

$$\mu_i^r : \xrightarrow{\lim} KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \rightarrow KK_i(\mathbb{C}, \hat{\mathcal{A}}_r);$$

so that in the classical case when $\mathcal{A} = C_0(G)$, the isomorphism of μ_i^r is equivalent to the Baum-Connes conjecture (see Proposition 5.2.1).

We verify the conjecture for an arbitrary finite dimensional quantum groups (Theorem 5.2.2), and also give some interesting examples and computations of the analytical assembly map (see 4.3 and Remark 5.2.3).

2 Notes on notation

We shall mostly follow the notation used in [9]. However, for readers' convenience, let us briefly mention some of those here.

2.1 Hilbert spaces and operators :

All the Hilbert spaces considered in this paper are over the field of complex numbers.

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and some bounded operator $X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, we denote by X_{12} the operator $X \otimes 1_{\mathcal{H}_2}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$, and denote by X_{13} the operator $(1_{\mathcal{H}_1} \otimes \Sigma)(X \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes \Sigma)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$, where $\Sigma : \mathcal{H}_2 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_2 \otimes \mathcal{H}_2$ flips the two copies of $\mathcal{H}_2$. For two vectors $\xi, \eta \in \mathcal{H}_1$ we define a map $T_{\xi\eta} : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2)$ by setting $T_{\xi\eta}(A \otimes B) :=< \xi, A\eta > B$, where $A \in \mathcal{B}(\mathcal{H}_1), B \in \mathcal{B}(\mathcal{H}_2)$, and extend this definition to the whole of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ in the obvious way. For some Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}_0(\mathcal{H})$ the C^*-algebra of compact operators on $\mathcal{H}$.

2.2 Multiplier algebras

For a Hilbert space $\mathcal{H}$, and a pre-C^*-algebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H})$, we shall denote the multiplier algebra of the norm-closure of $\mathcal{A}_0$ by $\mathcal{M}(\mathcal{A}_0)$. $\mathcal{M}(\mathcal{A}_0)$ has two natural topologies : one is the norm topology in which it becomes a unital (but typically nonseparable) C^*-algebra, and the other is the strict topology, which makes it a locally convex topological $*$-algebra.

The case when $\mathcal{A}_0$ is an algebraic direct sum of finite dimensional matrix algebras is of special interest to us. For such an algebra, say of the form, $\mathcal{A}_0 = \oplus_{\alpha \in I} \mathcal{A}_\alpha$, where I in some index set and $\mathcal{A}_\alpha$ is $n_\alpha \times n_\alpha$ complex matrix algebra (n_α positive integer), the multiplier algebra $\mathcal{M}(\mathcal{A}_0)$ can be described as the set of all collections $(a_\alpha)_{\alpha\in I}$ with $a_\alpha \in \mathcal{A}_\alpha$ for each α, and $\sup_\alpha \|a_\alpha\| < \infty$. The $*$-algebra operations are taken to be the obvious ones; i.e. $(a_\alpha)+(b_\alpha) := (a_\alpha+b_\alpha)$, $(a_\alpha).(b_\alpha) := (a_\alpha b_\alpha)$ and $(a_\alpha)^* := (a_\alpha^*)$. Similarly, $\mathcal{M}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ consists of all collections of the form $(a_\alpha \otimes b_\beta)$ where α, β vary over I.

2.3 Algebraic multiplier

We shall also use the following algebraic analogue of multiplier algebra.

Definition 2.3.1 *An "algebraic multiplier" of the $*$-algebra $\mathcal{A}_0$ above is a collection $(a_\alpha)_{\alpha\in I}$, with $a_\alpha \in \mathcal{A}_\alpha \forall \alpha$ (no restriction on norms). The set of all algebraic multipliers of $\mathcal{A}_0$ will be denoted by $\mathcal{M}_{\rm alg}(\mathcal{A}_0)$, and we equip it with the structure of a $*$-algebra by defining the following obvious operations :*

$$\text{For } U = (u_\alpha), V = (v_\alpha) \in \mathcal{M}_{\rm alg}(\mathcal{A}_0),\ \ UV := (u_\alpha v_\alpha)_\alpha,\ \ U^* := (u_\alpha^*).$$

Remark 2.3.2 *Clearly, any element of $\mathcal{A}_0$ can be viewed as an element of $\mathcal{M}_{\rm alg}(\mathcal{A}_0)$, by thinking of $a \in \mathcal{A}_0$ as (a_α), where a_α is the component of a in $\mathcal{A}_\alpha$. It is easy to see that $Ua, aU \in \mathcal{A}_0$ for $a \in \mathcal{A}_0, U \in \mathcal{M}_{\rm alg}(\mathcal{A}_0)$. Similarly, an algebraic multiplier of $\mathcal{A}_0 \otimes \mathcal{A}_0$ is given by any collection of the form $M \equiv (m_{\alpha\beta})_{\alpha,\beta\in I}$, with $m_{\alpha\beta} \in \mathcal{A}_\alpha \otimes \mathcal{A}_\beta$.*

Remark 2.3.3 *Let $\mathcal{K}$ be the smallest Hilbert space containing the algebraic direct sum $\oplus_{\alpha\in I}\mathcal{K}_\alpha \equiv \oplus_\alpha \mathbb{C}^{n_\alpha}$, i.e. $\mathcal{K} = \{(f_\alpha)_{\alpha\in I} : f_\alpha \in \mathcal{K}_\alpha = \mathbb{C}^{n_\alpha}, \sum_\alpha \|f_\alpha\|^2 < \infty\}$, where the possibly uncountable sum $\sum_\alpha$ means the limit over the net consisting of all possible sums over finite subsets of I. Let us consider the canonical imbedding of $\mathcal{A}_0$ in $\mathcal{B}(\mathcal{K})$, with $\mathcal{A}_\alpha$ acting on $\mathbb{C}^{n_\alpha}$. Let us fix some matrix units $e^\alpha_{ij}, i, j = 1, ..., n_\alpha$ for $\mathcal{A}_\alpha = M_{n_\alpha}$, w.r.t. some fixed orthonormal basis $e^\alpha_i, i = 1, ..., n_\alpha$, of $\mathbb{C}^{n_\alpha}$, and thus the norm-closure of $\mathcal{A}_0$, to be denoted by $\mathcal{A}$, is the C^*-algebra generated by e^α_{ij}'s. It is easy to see that any element of $\mathcal{M}_{\rm alg}(\mathcal{A}_0)$ can be viewed as a possibly unbounded operator on $\mathcal{K}$, with the domain containing the algebraic direct sum of $\mathcal{K}_\alpha$'s. Similarly, elements of $\mathcal{M}_{\rm alg}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ can be thought of as possibly unbounded operators on $\mathcal{K} \otimes \mathcal{K}$ with suitable domain.*

2.4 C^* and von Neumann Hilbert modules

For a Hilbert C^*-module E, we denote by $\mathcal{L}(E)$ the C^*-algebra of adjointable linear maps on E. Furthermore, for a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, and some Hilbert space $\mathcal{H}'$, We denote by $\mathcal{H}' \otimes \mathcal{B}$ the Hilbert von Neumann module obtained from the algebraic $\mathcal{B}$-module $\mathcal{H}' \otimes_{\text{alg}} \mathcal{B}$ by completing this algebraic module in the strong operator topology inherited from $\mathcal{B}(\mathcal{H}, \mathcal{H}' \otimes \mathcal{H})$, where we have identified an element of the form $(\xi \otimes b)$, $\xi \in \mathcal{H}', b \in \mathcal{B}$, with the operator which sends a vector $v \in \mathcal{H}$ to $(\xi \otimes bv) \in \mathcal{H}' \otimes \mathcal{H}$.

We also introduce the following notation :

$$\text{For } \eta \in \mathcal{H}', X \in \mathcal{B}(\mathcal{H}') \otimes \mathcal{B} \equiv \mathcal{L}(\mathcal{H}' \otimes \mathcal{B}), \;\; X\eta := X(\eta \otimes 1_{\mathcal{B}}) \in \mathcal{H}' \otimes \mathcal{B}.$$

Similarly, for a possibly nonunital C^*-algebra $\mathcal{A}$, we can complete the algebraic pre-Hilbert $\mathcal{A}$ module $\mathcal{H}' \otimes_{\text{alg}} \mathcal{A}$ in the locally convex topology coming from the strict topology on $\mathcal{M}(\mathcal{A})$, so that the completion becomes in a natural way a locally convex Hilbert $\mathcal{M}(\mathcal{A})$-module, to be denoted by $\mathcal{H}' \otimes \mathcal{M}(\mathcal{A})$. It is also easy to see that if $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}') \otimes \mathcal{A})$, $\eta \in \mathcal{H}'$, then we have $X\eta \equiv X(\eta \otimes 1) \in \mathcal{H}' \otimes \mathcal{M}(\mathcal{A})$.

2.5 Strictly continuous extensions

If $\mathcal{B}_1, \mathcal{B}_2$ are two von Neumann algebras, $\mathcal{H}'$ a Hilbert space, and $\rho : \mathcal{B}_1 \to \mathcal{B}_2$ a normal $*$-homomorphism, then it is easy to show that $(id \otimes \rho) : \mathcal{H}' \otimes_{\text{alg}} \mathcal{B}_1 \to \mathcal{H}' \otimes_{\text{alg}} \mathcal{B}_2$ admits a unique extension (to be denoted again by $(id \otimes \rho)$) from the Hilbert von Neumann module $\mathcal{H}' \otimes \mathcal{B}_1$ to the Hilbert von Neumann module $\mathcal{H}' \otimes \mathcal{B}_2$. Furthermore, one has that $(id \otimes \rho)(X\eta) = (id \otimes \rho)(X)\eta$ for $X \in \mathcal{B}(\mathcal{H}') \otimes \mathcal{B}$, $\eta \in \mathcal{H}'$. By very similar arguments one can also prove that if $\mathcal{A}_1, \mathcal{A}_2$ are two C^*-algebras, and $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ is a nondegenerate $*$-homomorphism (hence extends uniquely as a unital strictly continuous $*$-homomorphism from $\mathcal{M}(\mathcal{A}_1)$ to $\mathcal{M}(\mathcal{A}_2)$), then $(id \otimes \pi) : \mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_1 \to \mathcal{H}' \otimes_{\text{alg}} \mathcal{A}_2$ admits a unique extension (to be denoted by the same notation) from $\mathcal{H}' \otimes \mathcal{M}(\mathcal{A}_1)$ to $\mathcal{H}' \otimes \mathcal{M}(\mathcal{A}_2)$, which is continuous in the locally convex topologies coming from the respective strict topologies. We also have that $(id \otimes \pi)(X\eta) = (id \otimes \pi)(X)\eta$, for $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}') \otimes \mathcal{A}), \eta \in \mathcal{H}'$.

3 Preliminaries on discrete quantum groups

3.1 Definition

We briefly discuss the theory of *discrete quantum groups* as developed in [15], [8],[17], [12] and other relevant references to be found there. As in the previous section, let us fix an index set I (possibly uncountable), and let $\mathcal{A}_0 := \oplus_{\alpha \in I} \mathcal{A}_\alpha$ be the *algebraic* direct sum of $\mathcal{A}_\alpha$'s, where for each α, $\mathcal{A}_\alpha = M_{n_\alpha}$ is the finite dimensional C^*-algebra of $n_\alpha \times n_\alpha$ matrices with complex entries, and n_α is some positive integer. We also take $\mathcal{K} = \oplus_\alpha \mathbb{C}^{n_\alpha}$ as in the previous section.

Definition 3.1.1 Discrete Quantum Group

We say that $\mathcal{A}$ (which is the norm-closure of $\mathcal{A}_0$) is a discrete quantum group if there is a unital C^*-homomorphism $\Delta : \mathcal{M}(\mathcal{A}_0) \to \mathcal{M}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ which satisfies the following :
(i) For $a, b \in \mathcal{A}_0$, we have

$$T_1(a \otimes b) := \Delta(a)(1 \otimes b) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0,$$

and

$$T_2(a \otimes b) := (a \otimes 1)\Delta(b) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0;$$

(ii) $T_1, T_2 : \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0$ are bijections;
(iii) Δ is coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes id)(\Delta(b)(1 \otimes c)) = (id \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c),$$

for $a, b, c \in \mathcal{A}_0$.

Remark 3.1.2 *As explained in the relevant references mentioned above, $(\Delta \otimes id)$, $(id \otimes \Delta)$ admit extensions as C^*-homomorphisms from $\mathcal{M}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ to $\mathcal{M}(\mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \mathcal{A}_0)$ (we denote these extensions by the same notation) and the condition (iii) translates into $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.*

Let us recall (without proof) from [15] and [12] some of the important properties of our discrete quantum group $\mathcal{A}$. It is remarkable that it is possible to deduce from (i) to (iii) the existence of a canonical antipode $S : \mathcal{A}_0 \to \mathcal{A}_0$ satisfying $S(S(a)^*)^* = a$ and other usual properties of the antipode of a Hopf algebra. Furthermore, there exists a counit $\epsilon : \mathcal{A}_0 \to \mathbb{C}$. For details of the constructions of these maps and their properties we refer to [15].

Lemma 3.1.3 *([15])*

(i) There is a bijection of the index set I, say $\alpha \mapsto \alpha'$, such that $S(e_\alpha) = e_{\alpha'}, S(e_{\alpha'}) = e_\alpha$.

(ii) For fixed $\alpha, \beta \in I$, there is a finite number of $\gamma \in I$ such that $\Delta(e_\gamma)(e_\alpha \otimes e_\beta)$ is nonzero.

This allows us to make the following

Definition 3.1.4 *Define* $S(X)$ *for* $X = (x_\alpha)_I \in \mathcal{M}_{\rm alg}(\mathcal{A}_0)$ *by* $S(X) := X' = (x'_\alpha)_I$ *where* $x'_\alpha = S(x_{\alpha'})$. *Similarly, using (ii) in the Lemma 3.1.3,* $\Delta(X)$ *is defined as the element* $Y \in \mathcal{M}_{\rm alg}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ *such that* $Y = (y_{\alpha\beta})$, *where* $y_{\alpha\beta} = \sum_\gamma \Delta(x_\gamma)\Delta(e_\gamma)(e_\alpha \otimes e_\beta)$.

Remark 3.1.5 *For algebraic multipliers* A, B *of* $\mathcal{A}_0$ *and* L *of* $\mathcal{A}_0 \otimes \mathcal{A}_0$, *it is clear that* $\Delta(A) = L$ *if and only if* $\Delta(Aa) = L\Delta(a)$ $\forall a \in \mathcal{A}_0$, *and* $S(A) = B$ *if and only if* $S(Aa) = S(a)B$ *for* $a \in \mathcal{A}_0$.

3.2 Invariant functionals and modular operator

Let us denote by $\mathcal{A}'_0$ the set of all linear functionals on $\mathcal{A}_0$ having "finite support", i.e. they vanish on $\mathcal{A}_\alpha$'s for all but finite many $\alpha \in I$. It is clear that any $f \in \mathcal{A}'_0$ can be identified as a functional on $\mathcal{M}_{\rm alg}(\mathcal{A}_0)$, by defining $f((a_\alpha)_I) := \sum_{\alpha \in I} f(a_\alpha) \equiv \sum_{I_0} f(a_\alpha)$, where I_0 is the finite set of α's such that for α's not belonging to I_0, $f|_{\mathcal{A}_\alpha} = 0$. With this identification, $f(1)$ makes sense for any functional f on $\mathcal{M}_{\rm alg}(\mathcal{A}_0)$. Let us denote by e_α the identity of $\mathcal{A}_\alpha = M_{n_\alpha}$, which is a minimal central projection in $\mathcal{A}_0$. For any subset I_1 of I we denote by e_{I_1} the direct sum of e_α's for $\alpha \in I_1$. It is clear that a functional f on $\mathcal{A}_0$ is in $\mathcal{A}'_0$ if and only if there is some finite I_1 such that $f(a) = f(e_{I_1}a)$ for all $a \in \mathcal{A}_0$.

Definition 3.2.1 *We say that a linear functional* ϕ *(not necessarily with finite support) on* $\mathcal{A}_0$ *is left invariant if we have* $(id \otimes \phi)((b \otimes 1)\Delta(a)) = b\phi(a)$ *for all* $a, b \in \mathcal{A}_0$, *or equivalently,* $\phi((\omega \otimes id)(\Delta(a))) = \omega(1)\phi(a)$ *for all* $a \in \mathcal{A}_0$, $\omega \in \mathcal{A}'_0$. *Similarly, a linear functional* ψ *on* $\mathcal{A}_0$ *is called right invariant if* $(\psi \otimes id)((1 \otimes b)\Delta(a)) = \psi(a)b$ *for all* $a, b \in \mathcal{A}_0$.

We now recall some of the main results regarding left and right invariant functionals as proved in [15].

Proposition 3.2.2 *Up to constant multiples, there is a unique left invariant functional, as well as a unique right invariant functional. However, in general (unless* $S^2 = id$*) left and right invariant functionals are not the same.*

Let us summarize some of the important and useful facts here. For details, we refer to [15], [12] and [9].

Proposition 3.2.3 *(a) There exists a positive invertible element* $\theta \in \mathcal{M}_{\rm alg}(\mathcal{A}_0)$, *identified as a possibly unbounded operator on* $\mathcal{K}$, *with its domain containing*

all $\mathcal{K}_\alpha$*'s, such that* $\theta_\alpha = \theta|_{\mathcal{K}_\alpha} \in \mathcal{A}_\alpha$ *satisfies* $\Delta(\theta) = (\theta \otimes \theta)$, $S(\theta) = \theta^{-1}$, *and* $S(\theta^{-1}) = \theta$.
(b) $S^2(a) = \theta^{-1} a \theta$ *for all* $a \in \mathcal{A}_0$.
(c) We can choose a positive faithful left invariant functional (to be referred to as left haar measure later on) ϕ *and a positive faithful right invariant functional (to be referred to as right haar measure)* ψ *such that* $\psi(a) = \phi(a\theta^2) = \phi(\theta^2 a)$ *for* $a \in \mathcal{A}_0$.
(d) $\phi(S^2(a)) = \phi(a), \psi(S^2(a)) = \psi(a)$ *for all* $a \in \mathcal{A}_0$*, where* ϕ, ψ *as in (c).*

Let us now extend the definition of ϕ and ψ on a larger set than $\mathcal{A}_0$ as follows.

Definition 3.2.4 *For a nonnegative element* $a \in \mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{B}(\mathcal{K})$*, we define* $\phi(a)$ *as the limit of* $\phi_J(a)$*, whenever this limit exists as a finite number, and where* J *is any finite subset of* I*,* $\phi_J(.) := \phi(e_J.) = \phi(.e_J)$*, and the limit is taken over the net of finite subsets of* I *partially ordered by inclusion. Similarly, we set* $\psi(a) = \lim_J \psi(e_J a)$ *whenever the limit exists as a finite number. Since a general element* $a \in \mathcal{M}(\mathcal{A}_0)$ *can be canonically written as a linear combination of four nonnegative elements, we can extend the definition of* ϕ *on* $\mathcal{M}(\mathcal{A}_0)$ *by linearity. For any nonnegative* $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ *(where* $\mathcal{H}$ *is some Hilbert space), we define* $(id \otimes \phi)(X)$ *as the limit in the weak-operator topology (if it exists as a bounded operator) of the net* $(id \otimes \phi_J)(X)$ *over finite subsets* $J \subseteq I$*, and extend this definition for a general* $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ *in the usual way. Similar definition is given for* $(id \otimes \psi)$*.*

Lemma 3.2.5 *[9] If we choose* $\mathcal{H} = \mathcal{K}$ *in the above, and take any* $a \in \mathcal{M}(\mathcal{A}_0)$ *such that* $\phi(a)$ *is finite, then* $(id \otimes \phi)(\Delta(a)) = \phi(a) 1_{\mathcal{M}(\mathcal{A})}$.

We remark that an analogous fact is true for ψ.
The following fact, proved in [15], will be useful later on.

Proposition 3.2.6 *There is a one-dimensional component* M_{α_0} *for some* $\alpha_0 \in I$ *such that the identity* h_0*, say, of this component has the property that* $h_0.a = a.h_0 = \epsilon(a) h_0 \forall a \in \mathcal{A}_0$*, and also* $\phi(h_0) = 1$*.*

3.3 Representation and dual of a discrete quantum group

Definition 3.3.1 *We say that a unitary element in* $\mathcal{M}(\mathcal{L}(\mathcal{H} \otimes \mathcal{A})) \equiv \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ *is a unitary representation of the discrete quantum group* $\mathcal{A}$ *if* $(id \otimes \Delta)(U) = U_{12} U_{13}$*, and* $(id \otimes S)(U) = U^*$*.*

Remark 3.3.2 *Note that the second equality in the above definition has to be understood in the sense of the definition of S on the algebraic multiplier, i.e. $(id \otimes S)(U(1 \otimes a)) = (1 \otimes S(a))U^*$ for all $a \in \mathcal{A}_0$. Let us also make the following useful observation : for $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$, and $\xi, \eta \in \mathcal{H}$, we have that $T_{\xi,\eta}(X) \in \mathcal{M}(\mathcal{A})$.*

We shall now define a $*$-algebra structure on $\mathcal{A}_0'$, and identify $\mathcal{A}_0$ with suitable elements of $\mathcal{A}_0'$, thereby equipping $\mathcal{A}_0$ with this new $*$-algebra structure, and finally consider suitable C^*-completions. This will give rise to the analogues of the full and reduced group C^*-algebra in the framework of discrete quantum groups.

Definition 3.3.3 *Define $f*g$ for $f, g \in \mathcal{A}_0$ by $(f*g)(a) := (f \otimes g)(\Delta(a)), a \in \mathcal{A}_0$. We also define an adjoint by $f^*(a) := \bar{f}(S(a)^*), a \in \mathcal{A}_0$.*

Remark 3.3.4 *Note that since f, g have finite supports, there is some finite subset J of I such that $(f \otimes g)(\Delta(a)) = (f \otimes g)((e_J \otimes e_J)\Delta(a))$, and since $(e_J \otimes e_J)\Delta(a) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0$, $f * g$ is well defined.*

We define for each $a \in \mathcal{A}_0$, an element $\psi_a \in \mathcal{A}_0'$ by $\psi_a(b) := \psi(ab)$. It is easy to verify the following by using standard formulae involving Δ and S.

Proposition 3.3.5 *[9]*
For $a, b \in \mathcal{A}_0$, $\psi_a * \psi_b = \psi_{a*b}$, where $a * b := (id \otimes \psi)((1 \otimes b)((id \otimes S^{-1})(\Delta(a)))) = (\phi \otimes id)((a \otimes 1)((S \otimes id)(\Delta(b))))$. Furthermore, $\psi_a^* = \psi_{a^\sharp}$, where $a^\sharp := \theta^{-2} S^{-1}(a^*)$.

We denote by $\hat{\mathcal{A}}_0$ the set $\mathcal{A}_0$ equipped with the $*$-algebra structure given by $(a, b) \mapsto a * b, a \mapsto a^\sharp$ described by the above proposition. There are two different natural ways of making $\hat{\mathcal{A}}_0$ into a C^*-algebra, and thus we obtain the so-called reduced C^*-algebra $\hat{\mathcal{A}}_r$ and the free or full C^*-algebra $\hat{\mathcal{A}}$. This is done in a similar way as in the classical case : one can realize elements of $\hat{\mathcal{A}}_0$ as bounded linear operators on the Hilbert space $L^2(\phi)$ (the GNS-space associated with the positive linear functional ϕ, see [16] and [8] for details) and complete $\hat{\mathcal{A}}_0$ in the norm inherited from the operator-norm of $\mathcal{B}(L^2(\phi))$ to get $\hat{\mathcal{A}}_r$. The definition of $\hat{\mathcal{A}}$ is slightly more complicated and involves the realization of $\hat{\mathcal{A}}_0$ as elements of the Banach $*$-algebra $L^1(\phi)$ (see [8] and other relevant references) and then taking the associated universal C^*-completion. However, it is not important for us how the explicit constructions of these two C^*-algebras are done; we refer to [16], [8] for that; all we need is that $\hat{\mathcal{A}}_0$ is dense in both of them in the respective norm-topologies. It should also be mentioned that exactly as in the classical case, there is a canonical surjective C^*-homomorphism from $\hat{\mathcal{A}}$ to $\hat{\mathcal{A}}_r$.

4 Analytical assembly map for "proper and relatively compact" action of a discrete quantum group

4.1 An Analogue of proper and relatively compact action in the quantum case

Let us first recall from [9], without proof, how one can construct an analogue of the Baum-Connes analytic assembly map for the action of the discrete quantum group $\mathcal{A}$ on some C^*-algebra, under some additional assumptions on the action, which may be called "properness and $\mathcal{A}$-compactness", since these assumptions actually hold for a proper and G-compact action by a discrete group G. Our construction is analogous to that described in, for example, [18],[19], for the discrete group (see also [13] for some analogous constructions in the classical context). We essentially translate that into our noncommutative framework step by step, and verify that it really goes through. However, in case S^2 is not identity, it is somewhat tricky to give the correct definition of $\hat{\mathcal{A}}_0$-valued inner product, and prove the required properties, as one has to suitably incorporate the modular operator θ.

Let $\mathcal{C}$ be a C^*-algebra (possibly nonunital). Assume furthermore that there is an action of the quantum group $\mathcal{A}$ on it, given by $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}(\mathcal{C} \otimes \mathcal{A})$, which is a coassociative C^*-homomorphism, and assume also that there is a dense $*$-subalgebra $\mathcal{C}_0$ of $\mathcal{C}$ such that the following conditions are satisfied :

A1 $\Delta_{\mathcal{C}}(c)(c' \otimes 1) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$ for all $c, c' \in \mathcal{C}_0$;
A2 $\Delta_{\mathcal{C}}(c)(1 \otimes a) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$ for all $c \in \mathcal{C}_0$, $a \in \mathcal{A}_0$;
A3 There is a positive element $h \in \mathcal{C}_0$ such that

$$(id \otimes \phi)(\Delta_{\mathcal{C}}(h^2)) = 1,$$

or equivalently $(id \otimes \phi)(\Delta_{\mathcal{C}}(h^2)(c \otimes 1)) = c, \forall c \in \mathcal{C}_0$.

Remark 4.1.1 *Assume that X is a locally compact Hausdorff space equipped with an action of a discrete group Γ such that X is Γ-proper and Γ-compact. It is then easy to verify that A1,A2 and A3 hold if we take $\mathcal{C} = C_0(X)$, $\mathcal{A} = C_0(\Gamma)$ and $\mathcal{C}_0 = C_c(X)$. The choice of a positive element h as in A3 can be found in [18].*

Remark 4.1.2 *In the quantum case, some interesting examples where A1,A2, A3 are valid can be obtained by taking $\mathcal{C} = \mathcal{A}$, with the obvious action of $\mathcal{A}$ on itself, and $\mathcal{C}_0 = \mathcal{A}_0$.*

4.2 The construction of analytical assembly map

Now, our aim is to construct maps $\mu_i : KK_i^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}}) \equiv K_i(\hat{\mathcal{A}})$, and $\mu_i^r : KK_i^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}}_r) \equiv K_i(\hat{\mathcal{A}}_r)$, for $i = 0, 1$, i.e. even and odd cases. For simplicity let us do it for $i = 1$ only, the other case can be taken care of by obvious modifications. We have chosen the convention of [18] to treat separately odd and even cases, instead of treating both of them on the same footing as in the original work of Kasparov or in [10]. This is merely a matter of notational simplicity. For the definition and properties of equivariant KK groups $KK_{\cdot}^{\mathcal{A}}(.,.)$, we refer to the paper by Baaj and Skandalis ([3]) (with the easy modifications of their definitions to treat odd and even cases separately).

Let $(U, \pi, , F)$ be a cycle (following [18]) in $KK_1^{\mathcal{A}}(\mathcal{C}, \mathbb{C})$, i.e.
(i) $U \in \mathcal{L}(\mathcal{H} \otimes \mathcal{A})) \cong \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ is a unitary representation of $\mathcal{A}$, where $\mathcal{H}$ is a separable Hilbert space, i.e. U is unitary and $(id \otimes \Delta)(U) = U_{12}U_{13}$, $(id \otimes S)(U) = U^*$;
(ii) $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ is a nondegenerate $*$-homomorphism such that $(\pi \otimes id)(\Delta_{\mathcal{C}}(a)) = U(\pi(a) \otimes 1)U^*, \forall a \in \mathcal{C}$;
(iii) $F \in \mathcal{B}(\mathcal{H})$ is self-adjoint, $[F, \pi(c)], \pi(c)(F^2 - 1) \in \mathcal{B}_0(\mathcal{H}) \forall c \in \mathcal{C}$, and $(F \otimes 1) - U(F \otimes 1)U^* \in \mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}$.

Definition 4.2.1 *We say that a cycle (U, π, F) is equivariant (or F is equivariant) if $U(F \otimes 1)U^* = F \otimes 1$. We say that F is properly supported if for any $c \in \mathcal{C}_0$, there are* **finitely many** *$c_1, ...c_k, b_1, ..., b_k \in \mathcal{C}_0$ and $A_1, ...A_k \in \mathcal{B}(\mathcal{H})$ (all depending on c) such that $F\pi(c) = \sum_i \pi(c_i)A_i\pi(b_i)$.*

Before we proceed further, let us make the following convention : for any element $A \in \mathcal{B}(\mathcal{H})$, we shall denote by $\tilde{A}$ the element $A \otimes 1_{\mathcal{K}}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Theorem 4.2.2 *[9] Given a cycle (U, π, F), we can find a homotopy-equivalent cycle (U, π, F') such that (U, π, F') is equivariant and F' is properly supported.*

Sketch of Proof :
Since π is nondegenerate, we can choose a net e_ν of elements from $\mathcal{C}_0$ such that $\pi(e_\nu)$ converges to the identity of $\mathcal{B}(\mathcal{H})$ in the strict topology, i.e. in the strong $*$-topology. Now, let $X_\nu := \tilde{\pi(e_\nu)}^* U \tilde{\pi(h)} \tilde{F} \tilde{\pi(h)} U^* \tilde{\pi(e_\nu)} = \tilde{\pi(e_\nu)}^* (\pi \otimes id)(\Delta_{\mathcal{C}}(h)) U \tilde{F} U^* (\pi \otimes id)(\Delta_{\mathcal{C}}(h)) \tilde{\pi(e_\nu)}$. Since by our assumption $\tilde{e_\nu^*} \Delta_{\mathcal{C}}(h) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$, and similar thing is true for $\Delta_{\mathcal{C}}(h)\tilde{e_\nu}$, it is easy to see that X_ν is of the form $X_\nu = \sum_j (\pi(c_j) \otimes a_j)(U\tilde{F}U^*)(\pi(c_j') \otimes a_j')$, for some finitely many $c_j, c_j' \in \mathcal{C}_0$ and $a_j, a_j' \in \mathcal{A}_0$. Choosing a suitably large enough finite subset I_1

of I, we can assume that all the a_j, a_j''s are in the support of e_{I_1}, and hence it is easy to see that $X_\nu \in \mathcal{B}(\mathcal{H}) \otimes_{\mathrm{alg}} (e_{I_1}\mathcal{A}_0 e_{I_1})$, so $(id \otimes \phi)(X_\nu)$ is finite. Similarly, $(id \otimes \phi)(\pi(\tilde{e_\nu})^* U \pi(\tilde{h}^2) U^* \pi(\tilde{e_\nu}))$ is finite, and by assumption **A3**, is equal to $(\pi(e_\nu^* e_\nu) \otimes 1)$. Now, from the operator inequality $-\|F\|1 \leq F \leq \|F\|1$, we get the operator inequality

$$-\pi(\tilde{e_\nu})^* U \pi(\tilde{h}^2) U^* \pi(\tilde{e_\nu}) \|F\| \leq X_\nu \leq \pi(\tilde{e_\nu})^* U \pi(\tilde{h}^2) U^* \pi(\tilde{e_\nu}) \|F\|;$$

from which it follows after applying $(id \otimes \phi)$ that

$$-\pi(e_\nu^* e_\nu)\|F\| \leq (id \otimes \phi) X_\nu \leq \pi(e_\nu^* e_\nu)\|F\|.$$

Since $\pi(e_\nu^* e_\nu) \to 1_{\mathcal{B}(\mathcal{H})}$ in the strong operator topology, one can easily prove by the arguments similar to those in [18] that $(\mathrm{id} \otimes \phi)(X_\nu)$ converges in the strong operator topology of $\mathcal{B}(\mathcal{H})$, and let us denote this limit by F'. It is also easy to see that in fact $F' = (id \otimes \phi)(U(\pi(h) F \pi(h) \otimes 1) U^*)$, where we have used the extended definition of $(id \otimes \phi)$ on $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$ as discussed in the previous section.

Fix some $c \in \mathcal{C}_0$. Clearly we have $F'\pi(c) = (id \otimes \phi)(U \pi(\tilde{h}) \tilde{F} \pi(\tilde{h}) U^* \pi(\tilde{c}))$. Now, note that $U \pi(\tilde{h}) \tilde{F} \pi(\tilde{h}) U^* \pi(\tilde{c}) = (\pi \otimes id)(\Delta_{\mathcal{C}}(h)) U \tilde{F} U^* (\pi \otimes id)((\Delta_{\mathcal{C}}(h)(c \otimes 1))$. Since $\Delta_{\mathcal{C}}(h)(c \otimes 1) \in \mathcal{C}_0 \otimes_{\mathrm{alg}} \mathcal{A}_0$, we can write it as a finite sum of the form $\sum_{ij,\alpha} x_{ij}^\alpha \otimes e_{ij}^\alpha$, with $x_{ij}^\alpha \in \mathcal{C}_0$, and where e_{ij}^α 's are the matrix units of $\mathcal{A}_\alpha$, as described in the previous section, and α in the above sum varies over some finite set T, say, with $i, j = 1, ..., n_\alpha$. Thus, $U \pi(\tilde{h}) \tilde{F} \pi(\tilde{h}) U^* \pi(\tilde{c}) = \sum_{\alpha,i,j} (\pi \otimes id)(\Delta_{\mathcal{C}}(h))(1 \otimes e_{ij}^\alpha)(F x_{ij}^\alpha \otimes 1)$. Since for each α, i, j, $\Delta_{\mathcal{C}}(h)(1 \otimes e_{ij}^\alpha) \in \mathcal{C}_0 \otimes_{\mathrm{alg}} \mathcal{A}_0$, we can write $\Delta_{\mathcal{C}}(h)(1 \otimes e_{ij}^\alpha)$ as a finite sum of the form $\sum x_p \otimes a_p$ with $x_p \in \mathcal{C}_0, a_p \in \mathcal{A}_0$, and hence $U \pi(\tilde{h}) \tilde{F} \pi(\tilde{h}) U^* \pi(\tilde{c})$ is clearly a finite sum of the form $\sum_k \pi(c_k) A_k \pi(c_k') \otimes a_k$, with $c_k, c_k' \in \mathcal{C}_0$, $A_k \in \mathcal{B}(\mathcal{H})$ and $a_k \in \mathcal{A}_0$. From this it follows that F' is properly supported.

It is easy to show the equivariance of F'. Indeed, $U(F' \otimes 1)U^* = (id \otimes id \otimes \phi)((id \otimes \Delta)(U \pi(\tilde{h}) F \pi(\tilde{h}) U^*))$ by using the fact that $(id \otimes \Delta)(U) = U_{12} U_{13}$ and Δ is a $*$-homomorphism. Now, since it is easy to see using what we have proved in the earlier section that $(id \otimes id \otimes \phi)((id \otimes \Delta)(X)) = (id \otimes \phi)(X) \otimes 1$, for $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$, from which the equivariance of F' follows.

Finally, we can verify that $\pi(c)(F - F')$ is compact for $c \in \mathcal{C}_0$, hence for all $c \in \mathcal{C}$, by very similar arguments as in [18], adapted to our framework in a suitable way. We omit this part of the proof, which is anyway straightforward.

Let $\mathcal{H}_0 := \pi(\mathcal{C})\mathcal{H}$. By the fact that F' is properly supported, it is clear that $F'\mathcal{H}_0 \subseteq \mathcal{H}_0$. We now equip $\mathcal{H}_0$ with a right $\hat{\mathcal{A}}_0$-module structure. Define

$$(\xi.a) := (id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U)\xi,$$

for $\xi \in \mathcal{H}_0, a \in \mathcal{A}_0$. It is useful to note that for $c \in \mathcal{B}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{A}_0$, $(id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(c) = (id \otimes (\psi_{\theta a} \circ S^{-1}))(c)$ by simple calculation using the properties of ψ and θ described in the previous section. By taking suitable limit, it is easy to extend this for $c \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$, in particular for U. So we also have that $\xi.a = (id \otimes \psi_{\theta a} \circ S^{-1})(U)\xi$.

Proposition 4.2.3 *[9] $(\xi.a).b = \xi.(a * b)$ for $a, b \in \mathcal{A}_0, \xi \in \mathcal{H}_0$. That is, $(\xi, a) \mapsto \xi.a$ is indeed a right $\hat{\mathcal{A}}_0$-module action.*

Proof :-
Choosing finite subsets J, K of I such that $\theta^{-1}S(a)\theta^{-2} \in supp(e_K), \theta^{-1}S(b)\theta^{-2} \in supp(e_J)$, we have that

$$\begin{aligned}
&(\xi.a).b \\
&= \sum_{\alpha \in J; i,j=1,..,n_\alpha} U^\alpha_{ij}(\xi.a)\psi_{\theta^{-1}S(b)\theta^{-2}}(e^\alpha_{ij}) \\
&= \sum_{\alpha \in J; i,j=1,...,n_\alpha} \sum_{\beta \in K; k,l=1,...,n_\beta} U^\alpha_{ij} U^\beta_{kl} \xi \psi_{\theta^{-1}S(b)\theta^{-2}}(e^\alpha_{ij})\psi_{\theta^{-1}S(a)\theta^{-2}}(e^\beta_{kl}) \\
&= (id \otimes \psi_{\theta^{-1}S(b)\theta^{-2}} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U_{12}U_{13})\xi \\
&= (id \otimes \psi_{\theta^{-1}S(b)\theta^{-2}} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})((id \otimes \Delta)(U))\xi \\
&= (id \otimes (\psi_{\theta^{-1}S(b)\theta^{-2}} * \psi_{\theta^{-1}S(a)\theta^{-2}}))(U)\xi \\
&= (id \otimes \psi_{(\theta^{-1}S(b)\theta^{-2})*(\theta^{-1}S(a)\theta^{-2})})(U)\xi.
\end{aligned}$$

Now, by a straightforward calculation using the properties of ψ, S and θ one can verify that $(\theta^{-1}S(b)\theta^{-2}) * (\theta^{-1}S(a)\theta^{-2}) = \theta^{-1}S(a * b)\theta^{-2}$, which completes the proof.

For $\xi, \eta \in \mathcal{H}_0$, say of the form $\xi = \pi(c_1)\xi', \eta = \pi(c_2)\eta'$, it is clear that $T_{\xi\eta}(U)$ is an element of $\mathcal{A}_0$, since $(\pi(\tilde{c}_1^*)U\pi(\tilde{c}_2) = (\pi \otimes id)((c_1^* \otimes 1)\Delta_{\mathcal{C}}(c_2))U$, which belongs to $(\pi(\mathcal{C}_0) \otimes_{\mathrm{alg}} \mathcal{A}_0)\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{A}_0$. We define

$$< \xi, \eta >_{\hat{\mathcal{A}}_0} := \theta^{-1}T_{\xi\eta}(U) \in \hat{\mathcal{A}}_0,$$

identifying $\mathcal{A}_0$ as the $*$-algebra $\hat{\mathcal{A}}_0$ described earlier.

We recall here the proof in [9] of the fact that $\mathcal{H}_0$ with the above right $\hat{\mathcal{A}}_0$-action and the $\hat{\mathcal{A}}_0$-valued bilinear form $< .,. >_{\hat{\mathcal{A}}_0}$ is indeed a pre-Hilbert $\hat{\mathcal{A}}_0$-module.

Define $\Sigma : \mathcal{H}_0 \to \mathcal{F}_0$ by

$$\Sigma(\xi) := ((\pi(h) \otimes \theta^{-1})U)\xi,$$

for $\xi \in \mathcal{H}_0$. Note that by writing $\xi = \pi(c)\xi'$ for some $\xi' \in \mathcal{H}, c \in \mathcal{C}_0$, we have that $((\pi(h) \otimes 1)U)\xi = ((\pi \otimes id)((h \otimes 1)\Delta_{\mathcal{C}}(c))U)\xi'$, and since $(\pi \otimes id)((h \otimes 1)\Delta_{\mathcal{C}}(c))U \in \pi(\mathcal{C}_0) \otimes_{\rm alg} \mathcal{A}_0$, the range of Σ is clearly in $\mathcal{H} \otimes_{\rm alg} \mathcal{A}_0$. It follows that Σ is in fact a module map and preserves the bilinear form $< .,. >_{\hat{\mathcal{A}}_0}$ on $\mathcal{H}_0$.

Proposition 4.2.4 *[9] For $\xi, \eta \in \mathcal{H}_0, a \in \mathcal{A}_0$, we have that*
(i) $\Sigma(\xi.a) = \Sigma(\xi)a$.
(ii) $< \Sigma(\xi), \Sigma(\eta) > = < \xi, \eta >_{\hat{\mathcal{A}}_0}$.

Proof :-
(i) Choose suitable finite set indexed by p such that $(\tilde{\pi(h)U})\xi = \sum_p \pi(h)U_1^{(p)}\xi \otimes U_2^{(p)}$, where $U_1^{(p)} \in \mathcal{B}(\mathcal{H}), U_2^{(p)} \in \mathcal{A}_0$, and also $\sum_p U_1^{(p)^*} \otimes U_2^{(p)^*} = \sum_p U_1^{(p)} \otimes S(U_2^{(p)})$. Using the facts that $\Delta(\theta) = \theta \otimes \theta$, $S^{-1}(\theta) = \theta^{-1}$ and that $\psi(b\theta) = \psi(\theta b) \forall b \in \mathcal{A}_0$, and also the easily verifiable relation $\psi_a \circ S^{-1} = \psi_{\theta^{-1}S(a)\theta^{-1}}$ for $a \in \mathcal{A}_0$, we have that

$$\begin{aligned}
&\Sigma(\xi)a \\
&= \sum_p \pi(h)U_1^{(p)} \otimes (id \otimes \psi_{\theta^{-1}S(a)\theta^{-1}})(\Delta(\theta^{-1}U_2^{(p)})) \\
&= \sum_p \pi(h)U_1^{(p)} \otimes (id \otimes \psi_{\theta^{-1}S(a)\theta^{-1}})((\theta^{-1} \otimes \theta^{-1})\Delta(U_2^{(p)})) \\
&= (\pi(h) \otimes \theta^{-1} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})((\mathrm{id} \otimes \Delta)(U\xi)) \\
&= (\pi(h) \otimes \theta^{-1} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})((id \otimes \Delta)(U)\xi) \\
&= (\pi(h) \otimes \theta^{-1} \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U_{12}U_{13})\xi \\
&= (\pi(h) \otimes \theta^{-1})(U)((id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U)\xi) \\
&= (\pi(h) \otimes \theta^{-1})(U)(\xi.a) \\
&= \Sigma(\xi a).
\end{aligned}$$

(ii) Choosing suitable finite index sets as explained before, such that $(\pi(h) \otimes 1)U\xi = \sum_p U_1^{(p)} \otimes U_2^{(p)}$, with $\sum_p U_1^{(p)} \otimes S(U_2^{(p)}) = \sum_p U_1^{(p)^*} \otimes U_2^{(p)^*}$, and similarly for $(\pi(h)\otimes 1)U\eta$ with the index p replaced by say q, we can write

$$
\begin{aligned}
&< \Sigma(\xi), \Sigma(\eta) > \\
&= \sum_{p,q} < U_1^{(p)}\xi, \pi(h^2)U_1^{(q)}\eta > (\theta^{-1}U_2^{(p)})^\sharp * (\theta^{-1}U_2^{(q)}) \\
&= \sum_{p,q} < \xi, U_1^{(p)^*}\pi(h^2)U_1^{(q)}\eta > (\theta^{-2}S^{-1}(\theta^{-1})S^{-1}(U_2^{(p)^*})) * (\theta^{-1}U_2^{(q)}) \\
&= \sum_{p,q} < \xi, U_1^{(p)^*}\pi(h^2)U_1^{(q)}\eta > (\theta^{-1}S^{-1}(U_2^{(p)^*})) * (\theta^{-1}U_2^{(q)}) \\
&= \sum_{p,q} < \xi, U_1^{(p)}\pi(h^2)U_1^{(q)}\eta > \theta^{-1}(U_2^{(p)} * U_2^{(q)}) \\
&= \sum_{p,q} < \xi, U_1^{(p)}\pi(h^2)U_1^{(q)}\eta > (\phi\otimes\theta^{-1})((U_2^{(p)}\otimes 1)(S\otimes id)(\Delta(U_2^{(q)})))....(1),
\end{aligned}
$$

using the fact that $\sum_p U_1^{(p)^*}\otimes U_2^{(p)^*} = \sum_p U_1^{(p)}\otimes S(U_2^{(p)})$ and the simple observation that $(\theta^{-1}x)*(\theta^{-1}y) = \theta^{-1}(x*y)$. Now,

$$
\begin{aligned}
&\sum_q \pi(h^2)U_1^{(q)}\eta\otimes((S\otimes id)(\Delta(U_2^{(q)}))) \\
&= (\pi(h^2)\otimes S\otimes id)((id\otimes\Delta)(U\eta)) \\
&= (\pi(h^2)\otimes S\otimes id)((id\otimes\Delta)(U)\eta) \\
&= (\pi(h^2)\otimes id\otimes id)((U^*)_{12}U_{13})\eta....(2).
\end{aligned}
$$

Thus, from (1) and (2), $< \Sigma(\xi), \Sigma(\eta) >= (T_{\xi\eta}\otimes\phi\otimes\theta^{-1})((U(\pi(h^2)\otimes 1)U^*\otimes 1)U_{13}) = \theta^{-1}T_{\xi\eta}(U)$, since $(id\otimes\phi)(U\pi(h^2)U^*) = 1$. This completes the proof.

Note that from the above proposition it follows in particular that $< \xi, \eta a >_{\hat{\mathcal{A}}_0} =< \Sigma(\xi), \Sigma(\eta a) > =< \Sigma(\xi), \Sigma(\eta)a > =< \Sigma(\xi), \Sigma(\eta) > *a =< \xi, \eta >_{\hat{\mathcal{A}}_0} *a$. Similarly, $< \xi, \eta >^\sharp_{\hat{\mathcal{A}}_0} =< \eta, \xi >_{\hat{\mathcal{A}}_0}$, and $< \xi, \xi >$ is a nonnegative element in the $*$-algebra $\hat{\mathcal{A}}_0$, since $< .,. >$ on $\mathcal{F}_0$ is a nonnegative definite form.

Given any C^*-algebra which contains $\mathcal{A}_0$ as a dense $*$-subalgebra, we can complete $\mathcal{F}_0$ w.r.t. the corresponding norm to get a Hilbert C^*-module in which $\mathcal{F}_0$ sits as a dense submodule. Let us denote by $\mathcal{F}$ and $\mathcal{F}_r$ the Hilbert $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}_r$-modules respectively obtained in the above mentioned procedure, by considering $\mathcal{A}_0$ as dense $*$-subalgebra of $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}_r$ respectively. The corresponding completions of $\mathcal{H}_0$ will be denoted by $\mathcal{E}$ and $\mathcal{E}_r$ respectively.

By construction, Σ extends to an isometry from $\mathcal{E}$ to $\mathcal{F}$ and also from $\mathcal{E}_r$ to $\mathcal{F}_r$. We denote both these extensions by the same notation Σ, as long as no confusion arises. Clearly, $\mathcal{E} \cong \Sigma\mathcal{E} \subseteq \mathcal{F}$ as closed submodule, and similar statement will be true for $\mathcal{E}_r$ and $\mathcal{F}_r$.

Theorem 4.2.5 *[9] Let $T \in \mathcal{B}(\mathcal{H})$ be equivariant, i.e. $U(T \otimes 1)U^* = T \otimes 1$, and also assume that it satisfies the following condition which is slightly weaker than being properly supported :*
For $c \in \mathcal{C}_0$, one can find $c_1, ..., c_m \in \mathcal{C}_0, A_1, ..., A_m \in \mathcal{B}(\mathcal{H})$ (for some integer m) such that $T\pi(c) = \sum_k \pi(c_k)A_k$.
Then we have the following :
(i) $T(\xi a) = (T\xi)a \ \forall a \in \mathcal{A}_0$, and thus T is a module map on the $\hat{\mathcal{A}}_0$-module $\mathcal{H}_0$. Furthermore, if T is self-adjoint in the sense of Hilbert space, then $< \xi, T\eta >_{\hat{\mathcal{A}}_0} =< T\xi, \eta >_{\hat{\mathcal{A}}_0}$ for $\xi, \eta \in \mathcal{H}_0$.
(ii) T is continuous in the norms of $\mathcal{E}$ as well as $\mathcal{E}_r$, thus admits continuous extensions on both $\mathcal{E}$ and $\mathcal{E}_r$. We shall denote these extensions by $\mathcal{T}$ and $\mathcal{T}_r$ respectively.
(iii) If $T\pi(h)$ is compact in the Hilbert space sense, i.e. in $\mathcal{B}_0(\mathcal{H})$, then $\mathcal{T}$ and $\mathcal{T}_r$ are compact in the Hilbert module sense.

Proof :
(i) is obvious from the defintion of the right $\hat{\mathcal{A}}_0$ action, the definition of $< .,. >_{\hat{\mathcal{A}}_0}$, and the equivariance of T. Let us prove (ii) and (iii) only for $\mathcal{T}$, as the proof for $\mathcal{T}_r$ will be exactly the same. In fact, it is enough to show that $\Sigma\mathcal{T}\Sigma^*$ is continuous on $\mathcal{F}$, and is compact if $T\pi(h)$ is compact in the Hilbert space sense. Let us introduce the following notation : for $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A}), a \in \mathcal{A}_0, \eta \in \mathcal{H}$, define $X * b := (id \otimes id \otimes \psi_b \circ S^{-1})((id \otimes \Delta)(X))$, and $X * (\eta \otimes a) := (X * a)\eta$. Note that clearly $X * a \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{A})$, so $(X * a)\eta$ makes sense. Now, we observe using the equivariance of T and the explicit formula for Σ^* derived earlier that for $\eta \in \mathcal{H}, a \in \mathcal{A}_0$,

$$\begin{aligned} &\Sigma\mathcal{T}\Sigma^*(\eta \otimes a) \\ &= (\pi(h) \otimes 1)U\beta, \end{aligned}$$

where $\beta \in \mathcal{H}$ is given by $\beta = (id \otimes \psi_{\theta^{-1}S(a)\theta^{-2}})(U)(T\pi(h)\eta)$. Now, by using the fact that $(id \otimes \Delta)(U) = U_{12}U_{13}$, it follows by a straightforward computation that

$$\begin{aligned} &(1 \otimes \theta^{-1})U\beta \\ &= (id \otimes \theta^{-1} \otimes (\psi_{\theta a} \circ S^{-1}))((id \otimes \Delta)(U)(T\pi(h)\eta). \end{aligned}$$

But $\psi_{\theta a}(S^{-1}(b)) = \psi(\theta a S^{-1}(b)) = \psi(aS^{-1}(b)\theta) = \psi(aS^{-1}(\theta^{-1}b)) = (\psi_a \circ S^{-1})(\theta^{-1}b)$, and hence $(id \otimes \theta^{-1} \otimes (\psi_{\theta a} \circ S^{-1}))((id \otimes \Delta)(U)) = (id \otimes id \otimes (\psi_a \circ S^{-1}))((id \otimes \Delta)((1 \otimes \theta^{-1})U)) = ((1 \otimes \theta^{-1})U) * a$. From this, it is clear that

$$(\Sigma \mathcal{T} \Sigma^*)(\eta \otimes a) = ((\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1)) * (\eta \otimes a).$$

Now, note that $T\pi(h) = \sum_{k=1}^m \pi(c_k)A_k$, for some $c_1, ..., c_m \in \mathcal{C}_0, A_1, ..., A_m \in \mathcal{B}(\mathcal{H})$, and so we have $(\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1) = \sum_k (1 \otimes \theta^{-1})(\pi \otimes id)((h \otimes 1)\Delta_{\mathcal{C}}(c_k))U(A_k \otimes 1)$. But $(h \otimes 1)\Delta_{\mathcal{C}}(c_k)$ is in $\mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$ for each $k = 1, ..m$, and thus $(\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1) \in \mathcal{B}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{A}_0$ clearly. Choosing some large enough finite subset J of I such that $(\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1) = \sum_{\alpha \in J, ij=1,..,n_\alpha} B_{ij}^\alpha \otimes e_{ij}^\alpha$, (with $B_{ij}^\alpha \in \mathcal{B}(\mathcal{H})$), it is clear that $\Sigma \mathcal{T} \Sigma^* = \sum_{\alpha \in J, ij=1,...,n_\alpha} B_{ij}^\alpha \otimes L_{e_{ij}^\alpha}$, where for $x \in \mathcal{A}_0$, $L_x : \mathcal{A}_0 \to \mathcal{A}_0$ with $L_x(a) = x*a$. As L_x is a norm-continuous map on $\hat{\mathcal{A}}$, the above finite sum shows that $\Sigma \mathcal{T} \Sigma^*$ indeed admits a continuous extension on the Hilbert $\hat{\mathcal{A}}$-module $\mathcal{F}$. This proves (ii).

Furthermore, since $\mathcal{K}(\mathcal{H} \otimes \hat{\mathcal{A}}) \cong \mathcal{B}_0(\mathcal{H}) \otimes \hat{\mathcal{A}}$, where $\mathcal{K}(E)$ means the set of compact (in the Hilbert module sense) opeartors on the Hilbert module E, it is easy to see that $\Sigma \mathcal{T} \Sigma^*$ is compact on $\mathcal{F}$ if B_{ij}^α's are compact on the Hilbert space $\mathcal{H}$. Now, $B_{ij}^\alpha = (id \otimes \phi_{ij}^\alpha)((\pi(h) \otimes \theta^{-1})U(T\pi(h) \otimes 1)) = \pi(h).(id \otimes \phi_{ij}^\alpha)((1 \otimes \theta^{-1})U)T\pi(h)$, where ϕ_{ij}^α is the functional on $\mathcal{A}_0$ which is 0 on all e_{kl}^β except $\beta = \alpha, (kl) = (ij)$, with $\phi_{ij}^\alpha(e_{ij}^\alpha) = 1$. It follows that B_{ij}^α 's are all compact if $T\pi(h)$ is so, which completes the proof.

Now, let us come to the construction of the Baum-Connes maps $\mu_1 : KK_1^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_1(\mathbb{C}, \hat{\mathcal{A}})$ and $\mu_1^r : KK_1^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_1(\mathbb{C}, \hat{\mathcal{A}}_r)$. Let us do it only for μ_1, as the case of μ_1^r is similar, and in fact μ_1^r will be the compositon of μ_1 and the canonical map from $KK_1(\mathbb{C}, \hat{\mathcal{A}})$ to $KK_1(\mathbb{C}, \hat{\mathcal{A}}_r)$ induced by the canonical surjective C^*-homomorphism from $\hat{\mathcal{A}}$ to $\hat{\mathcal{A}}_r$. Note that an element of $KK_1(\mathbb{C}, \hat{\mathcal{A}}) \cong K_1(\hat{\mathcal{A}})$ is given by the suitable homotopy class $[E, L]$ of a pair of the form (E, L), where E is a Hilbert $\hat{\mathcal{A}}$-module and $L \in \mathcal{L}(E)$ (the set of adjointable $\hat{\mathcal{A}}$-linear maps on E) such that $L^* = L$, $L^2 - 1$ is compact in the sense of Hilbert module. For more details, see for example [10].

Theorem 4.2.6 *[9] Given a cycle $(U, \pi, F) \in KK_1^{\mathcal{A}}(\mathcal{C}, \mathbb{C})$, let $F' \equiv F_h'$ be the equivariant and properly supported operator as constructed in 4.2.2, with a given choice of h as in that theorem. Then the continuous extension of F_h' on the Hilbert module $\mathcal{E}$ (as described by the Theorem 4.2.5), to be denoted by say $\mathcal{F}_h'$, satisfies the conditions that $(\mathcal{F}_h')^* = \mathcal{F}_h'$ (as module map), and*

$(\mathcal{F}'_h)^2 - I$ is compact on $\mathcal{E}$. Define

$$\mu_1((U, \pi, F)) := [\mathcal{E}, \mathcal{F}'_h] \in KK_1(\mathbb{C}, \hat{\mathcal{A}}) \cong K_1(\hat{\mathcal{A}}).$$

In fact, $[\mathcal{E}, \mathcal{F}'_h]$ is independent (upto operatorial homotopy) of the choice of h.

Proof :-
Since F'_h is equivariant and properly supported, it is clear that $T_h := (F'_h)^2 - 1$ is equivariant and for any $c \in \mathcal{C}_0$, there are finitely many $c_1, ..., c_m \in \mathcal{C}_0, A_1, ..., A_m \in \mathcal{B}(\mathcal{H})$ such that $T_h \pi(c) = \sum_k \pi(c_k) A_k$. Furthermore, by the Theorem 4.2.2, we have that $\pi(c)T_h$, and hence $T_h \pi(c)$ is compact operator on $\mathcal{H}$ for every $c \in \mathcal{C}$. So, in particular, $T_h \pi(h)$ is compact. By Theorem 4.2.5, it follows that the continuous extension of T_h on $\mathcal{E}$ is compact in the sense of Hilbert modules. Furthermore, the fact that $(\mathcal{F}'_h)^* = \mathcal{F}'_h$ is clear from (i) of the Theorem 4.2.5. So, $[\mathcal{E}, \mathcal{F}'_h] \in KK_1(\mathbb{C}, \hat{\mathcal{A}})$. Furthermore, as we can see from the proof of the Theorem 4.2.2, $(F'_h - F)\pi(c) \in \mathcal{B}_0(\mathcal{H}) \; \forall c \in \mathcal{C}_0$, and so for h, h' satisfying **A3**, we have $(F'_h - F'_{h'})\pi(c) \in \mathcal{B}_0(\mathcal{H})$, and hence by Theorem 4.2.5, $\mathcal{F}'_h - \mathcal{F}'_{h'}$ is compact in the Hilbert module sense. Thus, for each $t \in [0, 1]$, setting $\mathcal{F}(t) := t\mathcal{F}'_{h'} + (1 - t)\mathcal{F}'_h$, we have that $\mathcal{F}(t)^2 - I$ is compact on $\mathcal{E}$, and this gives a homotopy in $KK_1(\mathbb{C}, \hat{\mathcal{A}})$ between $[\mathcal{E}, \mathcal{F}'_h]$ and $[\mathcal{E}, \mathcal{F}'_{h'}]$.

4.3 Examples and computations

Example 1 : Finite Dimensional Quantum Groups

Assume that $\mathcal{A}$ is a finite dimensional quantum group (hence both discrete and compact). Note that in this case $\hat{\mathcal{A}}$ coincides with $\hat{\mathcal{A}}_r$, $\phi = \psi$ is a tracial state, $S^2 = id$, $h = 1$ and $\theta = 1$. We would now like to prove the following

Proposition 4.3.1 *The analytical assembly map gives an isomorphism between the abelian groups $KK_i^{\mathcal{A}}(\mathbb{C}, \mathbb{C})$ and $KK_i(\mathbb{C}, \hat{\mathcal{A}})$ for $i = 0, 1$.*

Proof :-
It is enough to do the case $i = 0$, since for $i = 1$, both $KK_1^{\mathcal{A}}(\mathbb{C}, \mathbb{C})$ and $KK_1(\mathbb{C}, \hat{\mathcal{A}})$ are easily seen to be $\{0\}$. Let us assume that the set of irreducible representation of the quantum group $\mathcal{A}$ has cardinality N, and let us index them by say $\sigma_1, ...\sigma_N$. Then, $\hat{\mathcal{A}} \cong \oplus_{i=1}^N M_{d_i}$, where d_i denotes the dimension of the representation σ_i. Consider a cycle $[U, \mathcal{H}, F, \gamma]$

in $KK_0^{\mathcal{A}}(\mathbb{C},\mathbb{C})$, where γ is the grading operator on $\mathcal{H}$, and without loss of generality assume that F and γ are equivariant, F is self-adjoint and $F^2-1 \in \mathcal{B}_0(\mathcal{H})$. We can decompose (by applying the Peter-Weyl Theorem for compact quantum groups to $\mathcal{A}$) $\mathcal{H}$ as $\mathcal{H}=\oplus_i \mathbb{C}^{d_i} \otimes \mathcal{H}_i$, so that $U=\oplus_i U^{(\sigma_i)} \otimes I_{\mathcal{H}_i}$, $F=\oplus_i I \otimes F_i$, $\gamma=\oplus_i I \otimes \gamma_i$, where $U^{(\sigma)} \equiv ((U^{\sigma}_{kl}))^{d_\sigma}_{k,l=1}$ denotes the $d_\sigma \times d_\sigma$ unitary matrix corresponding to the irreducible representation of type σ and $\mathcal{H}_i$ is separable Hilbert space, $F_i \in \mathcal{B}(\mathcal{H}_i)$ is a self-adjoint operator acting on $\mathcal{H}_i$ such that $F_i^2-1_{\mathcal{H}_i} \in \mathcal{B}_0(\mathcal{H}_i)$, γ_i is a grading operator acting on $\mathcal{H}_i$ with $F_i\gamma_i=-\gamma_i F_i$. In other words, $\beta_i \equiv (\mathcal{H}_i, F_i, \gamma_i)$ can be considered as a cycle in $KK_0(\mathbb{C},\mathbb{C})=K_0(\mathbb{C}) \cong \boldsymbol{Z}$, and it is clear that the map $[U,\mathcal{H},F,\gamma] \mapsto (\beta_1,...,\beta_N) \in \oplus_i KK_0(\mathbb{C},\mathbb{C}) \cong KK_0(\hat{\mathcal{A}}) \cong \boldsymbol{Z}^N$ is an isomorphism. We claim that the analytical assembly map $\mu_0^r=\mu_0$ is nothing but the above association.

Recall from earlier sections that as a C^*-algebra $\hat{\mathcal{A}}$ is nothing but $\mathcal{A}$ (here the dimension is finite, so no further completion is needed) with the product given by $(a,b) \mapsto a*b$. It is not difficult to verify that $U^{(\sigma_i)}_{kl} * U^{(\sigma_j)}_{pq} = \delta_{ij}\delta_{lp}U^{(\sigma_i)}_{kq}$, upto some constant multiple, (as finite dimensional quantum groups are unimodular, the matrix elements $U^{(\sigma)}_{kl}$ are orthogonal w.r.t. the inner product coming from the Haar state) and we can identify the $*$-algebra generated by $U^{(\sigma_i)}_{kl}$ with the component M_{d_i} in $\hat{\mathcal{A}}=\oplus_j M_{d_j}$, identifying (upto some constant) $U^{(\sigma_i)}_{kl}$ with $|e^i_k><e^i_l|$, where $\{e^i_k, k=1,2,...\}$ is an orthonormal basis of $\mathbb{C}^{d_i}$. Let us now look at the action of μ_0 on the cycle $[U^{(\sigma_i)} \otimes I, \mathbb{C}^{d_i} \otimes \mathcal{H}_i, I \otimes F_i, I \otimes \gamma_i]$. For $\xi,\eta \in \mathcal{H}_i$, and $k,l \in \{1,2,...,d_i\}$, it is easy to see that $<e^i_k \otimes \xi, e^i_l \otimes \eta>_{\hat{\mathcal{A}}}=<\xi,\eta>|e^i_k><e^i_l|$, and the $\hat{\mathcal{A}}$-module obtained by the map μ_0 is $E_i \otimes \mathcal{H}_i$, where E_i denotes the $\mathbb{C}^{d_i}$ with the canonical M_{d_i}-module given by $\xi.a := a'\xi$, $<\xi,\eta>:=\sum_{kl}\bar{\xi}_k\eta_l|e^i_k><e^i_l|$, for $\xi=\sum_k \xi_k e^i_k, \eta=\sum_l \eta_l e^i_l \in \mathbb{C}^{d_i}$. Here a' denotes the transpose of the matrix a. Clearly, E_i can be viewed as $\hat{\mathcal{A}}$-module with the same inner product and the right action extended by $\xi.b=0$ for b in any component other than M_{d_i}. Thus, μ_0 takes $[U^{(\sigma_i)} \otimes I, \mathbb{C}^{d_i} \otimes \mathcal{H}_i, I \otimes F_i, I \otimes \gamma_i]$ to $(E_i \otimes \mathcal{H}_i, I \otimes F_i, I \otimes \gamma_i)$, which can be identified with the cycle $(\mathcal{H}_i, F_i, \gamma_i)$ of $KK_0(\mathbb{C}, M_{d_i}) = KK_0(\mathbb{C},\mathbb{C})$. This completes the proof.

Example 2 :-

Let us consider a general (possibly infinite dimensional) separable discrete quantum group $\mathcal{A}$. For simplicity, assume that $\hat{\mathcal{A}}_r$ coincides with $\hat{\mathcal{A}}$ (there are many interesting quantum groups satisfying this, e.g. the dual of $SU_q(n)$, $n=2,3,...$, q positive). Consider $\mathcal{C}=\mathcal{A}$, with its canonical action on itself, denoted by Δ. Take $\mathcal{C}_0=\mathcal{A}_0$, as mentioned in the Remark

4.1.2. Let us recall the notation and discussion on general discrete quantum groups in Section 3. The dual $\hat{\mathcal{A}}$ is a compact quantum group, and from the general theory it is known that the GNS space for $(\mathcal{A}, \phi)$ is isomorphic as a Hilbert space with the GNS space of the Haar state on $\hat{\mathcal{A}}$ via a quantum-analogue of the classical Fourier transform. Let $\mathcal{H}$ be the GNS space of $(\mathcal{A}, \phi)$. It is known that both $\mathcal{A}$ and $\hat{\mathcal{A}}$ can be represented as subalgebras of $\mathcal{B}(\mathcal{H})$. Furthermore, there is a unitary U such that $\Delta(a) = U(a \otimes I)U^*$ for $a \in \mathcal{A}$, and in fact this U is given by $U(a \otimes b) = \Delta(a)(b \otimes 1)$ for $a, b \in \mathcal{A}_0$. It can be verified by direct computation, following step by step the construction of $\mu_i, i = 0, 1$ discussed in the previous subsection that the Hilbert $\hat{\mathcal{A}}$-module arising out of this construction can be identified with the trivial module $\hat{\mathcal{A}}$ itself, with its own right action and the canonical inner product $< x, y >= x^* y$. Let us see this for the case when $S^2 = id$, just for sake of computational simplicity, keeping in mind that the general case can be treated essentially in a similar manner. Note that in this case $\theta = 1, \psi = \phi$, and we have for $\xi \in \mathcal{A}_0$, viewed as an element in $L^2(\mathcal{A}, \phi)$, $a \in \mathcal{A}_0$, $\xi.a = (id \otimes \psi_a \circ S^{-1})(U\xi) = (id \otimes \phi_a \circ S)(\Delta(\xi)) = \xi * a$. Similarly, $< \xi, \eta >= \xi^\sharp * \eta$ can be easily seen. This proves our claim.

As a particular example of the above, we can consider $\mathcal{A} = \hat{SU_q}(2)$, q positive. Fix a generator of $K_0(\mathbb{C}) = \boldsymbol{Z}$, realized as $(\mathcal{H}_0, F, \gamma)$, say, and then consider $\mathcal{H} = L^2(\mathcal{A}, \phi) \otimes \mathcal{H}_0$, and a cycle $[U \otimes I, \mathcal{H}, I \otimes F, I \otimes \gamma] \in KK_0^{\mathcal{A}}(\mathcal{A}, \mathbb{C})$. Here $\mathcal{A}$ is represented as $a \mapsto (a \otimes I_{\mathcal{H}_0})$ in $\mathcal{B}(\mathcal{H})$. The analytical assembly map μ_0 will take this cycle to the canonical generator of $K_0(SU_q(2)) = \boldsymbol{Z}$, since the inclusion of $1 \in SU_q(2)$ is an isomorphism of K_0-groups.

5 Formulation of the conjecture

5.1 Formulation

Now we shall formulate the Baum-Connes' conjecture for $\mathcal{A}$, following the approach (for a classical discrete group) taken by Cuntz in [5]. We shall use the notation introduced and used in Section 2.

Fix any finite subset F of I. Define a C^*-algebra $\mathcal{E}_F$ as follows (note that for $n = 1, 2, ..$ we denote by $\Delta^n : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A}^{\otimes^{n+1}})$ the map $...(\Delta \otimes id \otimes id)(\Delta \otimes id)\Delta.)$:

Definition 5.1.1 *Let $\mathcal{E}_F$ be the universal C^*-algebra generated by elements $\lambda(a), a \in \mathcal{A}_0$ satisfying the following :*
(i) $\mathcal{A}_0 \ni a \mapsto \lambda(a) \in \mathcal{E}_F$ is completely positive,
(ii) $(\lambda \otimes id)(\Delta^n(a))(\lambda \otimes id)(\Delta^n(x_1))...(\lambda \otimes id)(\Delta^n(x_k))(\lambda \otimes id)(\Delta^n(b)) =$

$0 \forall x_1, ..., x_k \in \mathcal{A}_0$ *whenever* a *and* b *are such that* $(1 \otimes e_F)(\sigma \circ \Delta(a))(b \otimes 1) = 0$, *(where* σ *is the flip automorphism on* $\mathcal{M}(\mathcal{A} \otimes \mathcal{A})$*),*
(iii) $\sum_\alpha \lambda(e_\alpha)\lambda(a) = \lambda(a)$ *for all* $a \in \mathcal{A}_0$.
Note that the sum in (iii) above is actually a finite sum, as a consequence of (ii).

We shall define a coassociative coaction $\Delta_F : \mathcal{E}_F \to \mathcal{M}(\mathcal{E}_F \otimes \mathcal{A})$. We need the following result for doing this.

Proposition 5.1.2 *Define* $\tilde{\lambda} : \mathcal{A}_0 \to \mathcal{M}(\mathcal{E}_F \otimes \mathcal{A})$ *by* $\tilde{\lambda} = (\lambda \otimes id) \circ \Delta$. *Then (i)-(iii) in the definition of* $\mathcal{E}_F$ *are valid if we replace* λ *by* $\tilde{\lambda}$*; so by the universal property of* $\mathcal{E}_F$*, there is a well-defined* C^**-homomorphism* $\Delta_F : \mathcal{E}_F \to \mathcal{M}(\mathcal{E}_F \otimes \mathcal{A})$ *satisfying* $\Delta_F(\lambda(a)) = \tilde{\lambda}(a)$ *for all* $a \in \mathcal{A}_0$.

Proof :-
Since λ is clearly a nondegenerate, bounded completely positive (to be abbreviated as CP) map (nondegeneracy follows from (iii) of the definition, which says that $\sum_\alpha \lambda(e_\alpha) = 1$ in the strict topology of $\mathcal{M}(\mathcal{E}_F)$), $(\lambda \otimes id)$ makes sense on $\mathcal{M}(\mathcal{A} \otimes \mathcal{A})$. Clearly, $\tilde{\lambda}$ is completely positive on $\mathcal{A}_0$. Now, choose $a, b \in \mathcal{A}_0$ such that $(1 \otimes e_F)(\sigma \circ \Delta(a))(b \otimes 1) = 0$. We have, $(\tilde{\lambda} \otimes id)(\Delta^n(a))(\tilde{\lambda} \otimes id)(\Delta^n(x_1))...(\tilde{\lambda} \otimes id)(\Delta^n(b)) = (\lambda \otimes id)(\Delta^{n+1}(a))(\lambda \otimes id)(\Delta^{n+1}(x_1))...(\lambda \otimes id)(\Delta^{n+1}(b)) = 0$ for all $x_1, ..., x_k \in \mathcal{A}_0$. Finally, it is easy to see that the condition (iii) holds for λ replaced by $\tilde{\lambda}$.

It is easy to show that Δ_F is indeed coassociative, since Δ is coassociative and we have, $(\Delta_F \otimes id)(\lambda \otimes id) = (\lambda \otimes id \otimes id)(\Delta \otimes id)$, from which it is immediate that $(\Delta_F \otimes id)(\Delta_F(\lambda(a))) = (id \otimes \Delta_F)(\Delta_F(\lambda(a)))$. Now we shall show that the $\mathcal{A}$-coaction on $\mathcal{E}_F$ satisfies the conditions A1,A2, A3 mentioned earlier by us (see 3.1), if we choose large enough F such that it contains α_0 mentioned in the earlier section.

Theorem 5.1.3 *A1,A2,A3 are valid for the coaction* $\Delta_F : \mathcal{E}_F \to \mathcal{M}(\mathcal{E}_F \otimes \mathcal{A})$.

Proof :-
Let us take $\mathcal{C} = \mathcal{E}_F$ and let $\mathcal{C}_0$ be the $*$-algebra (no completion) generated by $\lambda(a), a \in \mathcal{A}_0$. Take $a, b \in \mathcal{A}_0$, and let $c = \lambda(a) \in \mathcal{C}_0$. From the general theory of discrete quantum groups, we know that $\Delta(a)(1 \otimes b) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0$. Thus, $\Delta_F(\lambda(a))(1 \otimes b) = (\lambda \otimes id)(\Delta(a))(1 \otimes b) = (\lambda \otimes id)((\Delta(a)(1 \otimes b)) \in (\lambda \otimes id)(\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0) = \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$. This proves A2. To prove A1, we note from [15] that given $\beta, \gamma \in I$, there are only finite many $\alpha \in I$ such that $\Delta(e_\alpha)(e_\beta \otimes e_\gamma) \neq 0$. Thus, depending on b and F, we can choose a finite

subset J of I such that $\forall \alpha$ not in J, one has $(1 \otimes e_F)(\Delta(e_\alpha)(b \otimes 1)) = 0$. Hence for any $x \in \mathcal{A}_0$, we have that $(1 \otimes e_F)(\Delta(x e_\alpha)(b \otimes 1)) = \Delta(x)\Delta(e_\alpha)(b \otimes e_F) = 0$, for any α not belonging to J (as e_F is a central element in $\mathcal{M}(\mathcal{A} \otimes \mathcal{A})$.) From this, it follows that $\lambda(x e_\alpha)\lambda(b) = 0$ for α not in J, and for any $x \in \mathcal{A}_0$, and hence for any $x \in \mathcal{M}(\mathcal{A})$, since λ, being a nondegenerate CP map on $\mathcal{A}_0$, has a strictly continuous extension to $\mathcal{M}(\mathcal{A})$. Thus, $(\lambda \otimes id)(X)(\lambda(b) \otimes 1) = (\lambda \otimes id)(X(e_J \otimes 1))(\lambda(b) \otimes 1)$ for all $X \in \mathcal{M}(\mathcal{A} \otimes \mathcal{A})$. In particular, for $a, b \in \mathcal{A}_0$, $(\lambda \otimes id)(\Delta(a))(\lambda(b) \otimes 1) = (\lambda \otimes id)(\Delta(a)(e_J \otimes 1))(\lambda(b) \otimes 1) \in \mathcal{C}_0 \otimes_{\text{alg}} \mathcal{A}_0$, since $\Delta(a)(e_J \otimes 1) \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{A}_0$. This completes the proof of A1. Finally, by taking $h = \lambda(h_0)$, where h_0 is as in the earlier section, we see that A3 is satisfied, as $(id \otimes \phi)(\Delta(h_0)) = \phi(h_0)1 = 1$.

Applying the results of [9], as recalled already, we have canonical homomorphisms $\mu_i^F : KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}})$ and $\mu_i^{r,F} : KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}}_r)$, $i = 0, 1$, and for every F as above. Then, taking the inverse limit w.r.t. a family of F increasing to I, we get maps $\mu_i : lim_F KK_i^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}})$ and $\mu_i^r : lim_F KK_i^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_i(\mathbb{C}, \hat{\mathcal{A}}_r)$.

5.1.4. conjecture : μ_i^r is an isomorphism for $i = 0, 1$.

5.2 Consistency with the classical formulation and verification for some quantum groups

We first point out that in the classical case, this is equivalent to Baum-Connes' conjecture, using the results in [5].

Proposition 5.2.1 *If we take $\mathcal{A} = C_0(G)$, where G is a discrete group, then the conjecture 5.1.4 above is equivalent to the classical Baum-Connes Conjecture for the group G.*

Proof :-
Suppose that F is some finite subset of G containing the identity element e, say, and $\mathcal{A} = C_0(G), \mathcal{A}_0 = C_c(G)$. Since $\mathcal{A}$ is commutative, complete positivity coincides with positivity, and hence any CP map λ from $\mathcal{A}$ to some other C^* algebra $\mathcal{B}$ is determined by a map from G to the positive cone of $\mathcal{B}$ given by $g \mapsto \lambda_g \equiv \lambda(\delta_g) \geq 0$, where δ_g is the indicator function at the point $g \in G$. Thus, in this case, $\lambda(f) = \sum_g f(g)\lambda_g$ for $f \in C_c(G)$. Furthermore, the condition $(1 \otimes e_F)(\sigma \circ \Delta(f))(\psi \otimes 1) = 0$ for some pair (f, ψ) is equivalent to $f(hg)\psi(g) = 0 \forall h \in F$, i.e. $f(s)\psi(t) = 0$ whenever $st^{-1} \in F$. In particular, given two fixed $s, t \in G$, such that st^{-1} is not in F, we see that $(1 \otimes e_F)(\sigma \circ \Delta(\delta_s))(\delta_t \otimes 1) = 0$, so $\lambda_s \lambda_{t_1} ... \lambda_{t_k} \lambda_t = 0$ for any $t_1, ..., t_k \in G$.

Furthermore, the condition that $\lambda_s\lambda_{t_1}...\lambda_{t_k}\lambda_t = 0$ for any $t_1, ..., t_k$ and t, s with st^{-1} not in F, actually implies the apparently stronger condition that $(\lambda \otimes id)(\Delta^n(a))(\lambda \otimes id)(\Delta^n(x_1))...(\lambda \otimes id)(\Delta^n(x_k))(\lambda \otimes id)(\Delta^n(b)) = 0 \forall x_1, ..., x_k \in \mathcal{A}_0$ whenever a and b are such that $(1 \otimes e_F)(\sigma \circ \Delta(a))(b \otimes 1) = 0$. For example, for a, b as above, we have that for any $h \in G$, $(\lambda \otimes id)(\Delta(a))(h).(\lambda \otimes id)(\Delta(b))(h) = \sum_{s,t} a(sh)b(th)\lambda_s\lambda_t = 0$, since the above sum is over (s, t) such that $sh.(th)^{-1} = st^{-1}$ not in F (for otherwise $a(sh)b(th) = 0$), and $\lambda_s\lambda_t = 0$ for such s, t.

Thus, the C^*-algebra $\mathcal{E}_F$ in this case turns out to be the universal C^*-algebra generated by $\{\lambda_g, g \in G\}$ such that each λ_g is a nonnegative element, $\lambda_s\lambda_{t_1}...\lambda_{t_k}\lambda_t = 0$ for any $t_1, ..., t_k$ and s, t with st^{-1} not in F, and the condition that $\sum_g \lambda_g\lambda_h = \lambda_h$ for all fixed $h \in G$. Then a careful look at [5] tells us that the isomorphism conjecture stated by us in the general context of discrete quantum groups coincides with the classical Baum-Connes conjecture for a discrete group.

We now verify the conjecture for any finite dimensional quantum group.

Theorem 5.2.2 *The conjecture 5.1.4 is true for any finite dimensional quantum group.*

Proof :-
Let $\mathcal{A}$ be a finite dimensional quantum group. We have to prove that the analytical assembly maps are isomorphism when applied on the KK-groups corresponding to $\mathcal{E}_F$ for $F = I$, as I itself is a finite set in this case. Now, $\mathcal{E} \equiv \mathcal{E}_I$ is nothing but the universal C^*-algebra generated by $\{\lambda(a), a \in \mathcal{A}\}$, where $\lambda : \mathcal{A} \to \mathcal{E}$ is CP and unital. To see this, it is enough to note that the condition (ii) in the definition of $\mathcal{E}_I$ is automatic in this case, as $e_I = 1$, and $(a \otimes b) \mapsto \Delta(a)(b \otimes 1)$ is a bijection, so that $\Delta(a)(b \otimes 1) = 0$ if and only if $a = b = 0$. Furthermore, $\sum_{\alpha \in I} \lambda(a)$ is a finite sum in this case, so this sum actually belongs to $\mathcal{E}$, which shows that $\mathcal{E}$ is unital.

Now, with the above identification of $\mathcal{E}$, we claim that $\mathcal{E}$ is $\mathcal{A}$-equivariantly homotopic to $\mathbb{C}$, and hence $KK_i^{\mathcal{A}}(\mathcal{E}, \mathbb{C}) \cong KK_i^{\mathcal{A}}(\mathbb{C}, \mathbb{C}), i = 0, 1$, which will complete the proof of the theorem, since we have already proven (4.3.1) that the analytical assembly maps are isomorphisms between $KK_i^{\mathcal{A}}(\mathbb{C}, \mathbb{C})$ and $KK_i(\mathbb{C}, \hat{\mathcal{A}})$. Let us denote the comultiplication on $\mathcal{E}$ by $\Delta_{\mathcal{E}}$. To establish our claim, we first note that since $\mathcal{A}$ is a finite dimensional quantum group, there is a unique faithful Haar state on it, say τ, and this gives a unital CP map from $\mathcal{A}$ to $\mathbb{C}$ given by $a \mapsto \tau(a)$. From the definition of $\mathcal{E}$, there must exist a unital C^*-homomorphsim $\tilde{\tau} : \mathcal{E} \to \mathbb{C}$ such that $\tilde{\tau}(\lambda(a)) = \tau(a)$. Let us denote by π_1 the homomorphism from $\mathcal{E}$ to itself given by $\pi_1(a) = \tilde{\tau}(a)1_{\mathcal{E}}$, and by π_0 the identity map from $\mathcal{E}$ to itself. To verify our claim, we need to give an equivariant homotopy connecting π_0 and π_1. Define for $t \in [0, 1]$ the unital CP map $\lambda_t : \mathcal{A} \to \mathcal{E}$ given by $\lambda_t(a) : (1-t)\lambda(a) + t\tau(a)1_{\mathcal{E}}$. By the universality of $\mathcal{E}$, there exist unital C^*-homomorphisms $\pi_t : \mathcal{E} \to \mathcal{E}$, satisfying $\pi_t \circ \lambda = \lambda_t$. Clearly, $t \mapsto \pi_t(x)$ is continuous for x belonging to the dense $*$-algebra spanned by the elements in the range of λ, and hence for all x. It remains to check that π_t are equivariant. To this end, it is enough to show that $(\pi_t \otimes id)(\Delta_{\mathcal{E}}(\lambda(a))) = \Delta_{\mathcal{E}}(\pi_t(\lambda(a)))$ for all $a \in \mathcal{A}$. Indeed, we have,

$$\begin{aligned}
&(\pi_t \otimes id)(\Delta_{\mathcal{E}}(\lambda(a)) \\
&= \Delta_{\mathcal{E}}(\lambda_t(a)) \\
&= (1-t)\Delta_{\mathcal{E}}(\lambda(a)) + t\tau(a)1 \\
&= (1-t)(\lambda \otimes id)(\Delta(a)) + t(\tau \otimes id)(\Delta(a)) \\
&= (\lambda_t \otimes id)(\Delta(a)) \\
&= (\pi_t \otimes id)((\lambda \otimes id)(\Delta(a))) \\
&= (\pi_t \otimes id)(\Delta_{\mathcal{E}}(\lambda(a))),
\end{aligned}$$

which completes the proof. Note that we have used in the above the fact that $\tau(a)1 = (\tau \otimes id)(\Delta(a))$, which follows from the definition of the Haar state.

Remark 5.2.3 *We have seen in Example 2 of section 4 that μ_0 is surjective from $KK_0^{\mathcal{A}}(\mathcal{A}, \mathbb{C})$ to $KK_0(\mathbb{C}, \hat{\mathcal{A}})$ for $\mathcal{A} = \widehat{SU_q(2)}$. From this, it is possible to argue that $\mu_0^r : lim_F KK_0^{\mathcal{A}}(\mathcal{C}, \mathbb{C}) \to KK_0(\mathbb{C}, \hat{\mathcal{A}}_r)$ is surjective. Indeed, the limit algebra $\mathcal{E} \equiv lim_F \mathcal{E}_F$ is nothing but the universal C^* algebra generated by $\lambda(a), a \in \mathcal{A}$ with λ CP and unital from $\mathcal{M}(\mathcal{A})$ to $\mathcal{M}(\mathcal{E})$. From the universality, one can get an equivariant C^*-homomprhism from $\mathcal{E}$ to $\mathcal{A}$, which in turn gives a homomorphism from the equivariant KK-group of $\mathcal{A}$ to $lim_F KK_0^{\mathcal{A}}(\mathcal{E}_F, \mathbb{C})$ and the composition of the analytical assembly map with this will coincide with the analytical assembly map from $KK_0^{\mathcal{A}}(\mathcal{A}, \mathbb{C})$ to $KK_0(\mathbb{C}, \hat{\mathcal{A}})$. From this, surjectivity will follow.*

We postpone the attempts to verify this conjecture for some non-classical quantum groups. e.g. duals of $SU_q(n)$'s, for future works.

Acknowledgement :
D. Goswami would like to express his gratitude to I.C.T.P. for a visting research fellowship during January-August 2002, and to A.O. Kuku and the other organisers of the "School and Conference on Algebraic K Theory and Its Applications" at I.C.T.P. (Trieste) in July 2002. He would also like to thank T. Schick for sending some relevant preprint, and A. Valette and I. Chatterjee for giving useful information regarding some manuscript (yet to be published) by A. Valette.

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