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# A NOTE ON IYENGAR'S METHOD OF ESTIMATING ENGEL ELASTICITIES FROM GROUPED DATA

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*SUMMARY*. This note develops formulae for Engel elasticity based on the generalised non-linear Engel function at any given income level in terms of two types of concentration curves or two types of concentration-ratios. It also outlines a procedure for consistently estimating the Engel elasticities and the underlying Engel function from grouped data.

## 1. Introduction

For estimating Engel elasticities one needs to specify certain forms of Engel function and then estimate them from family budget data which are usually available in the form of grouped means. The method of least squares applied to the grouped arithematic mean data, however, introduces bias and inconsistency in the estimation of parameters of an Engel function in which the original variables appear in transformed form. In order to meet the problem of inconsistency, Iyengar (1960, 1964) developed two alternative methods, viz., M-I and M-II, for estimating Engel elasticities from grouped data, M-I based on the use of two types of concentration curves, and M-II on the use of two types of concentration-ratios. Both methods rest on the basic assumption that the size distribution of income (or total expenditure) follows log-normal law. Following Method-I Iyengar (1964) obtained formulae for consistently estimating elasticities at median income level based on the log-linear Engel function and at both median and mean income levels when based on the semi-log Engel function. Using the estimates of income elasticity at median level, he further obtained consistent estimates of the parameters of the underlying Engel function. Method-II was suggested for estimating constant Engel elasticities.

Iyengar's attempt is, however, restrictive as it provides for the estimation of only two particular forms of the Engel function and estimates of elasticities only at one or two particular average income levels based on the two Engel functions. Secondly, the estimation procedure suggested by Iyengar (1964, 1967) for estimating the Engel elasticity and the parameters of the underlying Engel function is based on linear approximations resulting in an over-estimation of the points on the two concentration curves and an under-estimation of the two concentration-ratios.

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This note attempts to extend Iyengar's approach to a family of non-linear Engel functions for estimating the Engel elasticity and the underlying Engel function. Following both methods of Iyengar we develop in Section 2 formulae for Engel elasticity based on the generalized Engel function involving Box-Cox (1964) type transformation which reduces to conventional non linear Engel functions, viz., semi-log and hyperbolic, on assigning particu. values to the transformation parameter. These elasticity formulae are derived at any income level in general and at alternative (i.e. arithmetic, geometric and harmonic) mean levels in particular. Section 3 presents an estimation procedure for consistently estimating Engel elasticities and the parameters of the underlying Engel function by applying both methods of Iyenger. This procedure is definitely better and expected to yield more efficient estimates than the one suggested by Iyengar (1967). Section 4 presents consistent estimation of semi-log and hyperbolic Engel functions from grouped data.

## 2. Formulation of Engel elasticities

An Engel curve is defined by $E(y|x) = \Psi(x)$ where $y$ and $x$ represent respectively per capita household expenditure on the specific commodity and per capita income (or total expenditure). $x$ is here assumed to be two-parameter log-normally distributed with its distribution function denoted as $\Lambda(x\ \theta, \lambda)$ such that $\theta$ and $\lambda^2$ are, respectively, the mean and the variance of the normally distributed variable log $x$. For this distribution, the arithemetic, the geometric (which also happens to be the median) and the harmonic means are respectively, given by $\bar{x}_a = e^{\theta+\lambda^2/2}$, $\bar{x}_g = e^{\theta}$, and $\bar{x}_h = e^{\theta-\lambda^2/2}$

The following notations shall be used throughout the paper :

$p(x)$ = proportion of persons earning a given income $x$ or less ;

$q(x)$ = proportion of aggregate income earned by the above stratum of persons ;

$Q(x)$ = proportion of aggregate consumption expenditure on a specific commodity accruing to the persons with income $x$ or less ;

$\eta_a, \eta_g, \eta_h$ = Engel elasticities at $\bar{x}_a$, $\bar{x}_g$, $\bar{x}_h$ respectively ;

$\phi(k)$ = ordinate of the standard normal curve for value $k$ of the normal deviate ;

$t_k$ = standard normal deviate corresponding to the cumulative area $k$ of the standard normal curve ; and $\Phi(k)$ denotes cumulative area of the standard normal curve for value $k$ of the standard normal deviate.

We consider the generalised Engel function involving Box-Cox type transformation :

(G.1) $E(y|x) = \alpha+\beta(x^{\mu}-1)/\mu$, where the transformation parameter $\mu$ determines the degree and the type of non-linearity. Notice that

for $\mu = 1$, (G.1) = linear : $E(y|x) = a+bx$ where $a = \alpha-\beta$ and $b = \beta$;

for $\mu \to 0$, (G.1) $\to$ semi-log : $E(y|x) = a+b \log x$ where $a = \alpha$ and $b=\beta$;

for $\mu= -1$, (G.1) = hyperbolic : $E(y|x)=a+b/x$ where $a=\alpha+\beta$ and $b=-\beta$;

Following both methods of Iyengar, we, therefore, develop the formulae for Engel elasticity based on the generalized function and derive from them the formulae based on semi-log and hyperbolic Engel functions by assigning the relevent values of parameter $\mu$.

Notice that one can also easily obtain the formulation of Engel elasticities for several non-conventional and non-linear Engel functions which follow from the generalised function on assigning the parameter $\mu$ values other than $-1$, 1 and 0.

2.1. *Method* I. Income (expenditure) elasticity for the generalised Engel function is, by definition, given by

$$\eta(x) = \mu\beta x^{\mu}/[\mu\alpha+\beta(x^{\mu}-1)].$$

As $\log x = \theta+\lambda l_{p(x)}$ at an income level $x$ under the log-normality hypothesis, elasticity may, therefore, be rewritten as

$$\eta(x) = \mu\beta e^{\mu\theta+\mu\lambda l}p(x)/[(\alpha\mu-\beta)+\beta e^{\mu\theta+\mu\lambda l}p(x)].$$

It can be easily proved that the specific concentration curve in the present case is given by

$$Q(x) = [(\alpha\mu-\beta)p(x)+\beta e^{\mu\theta+\mu^2\lambda^2/2}\Phi(l_{p(x)}-\lambda\mu)]/[(\alpha\mu-\beta)+\beta e^{\mu\theta+\mu^2\lambda^2/2}],$$

and $\lambda$ is given by the equation of the Lorenz curve as $\lambda = l_{p(x)}-l_{Q(x)}$. Using the equations of the two concentration curves with the view to eliminate parameters $\alpha$, $\beta$, $\theta$ and $\lambda$ from $\eta(x)$, we obtain

$$\eta(x) = \mu[p(x)-Q(x)]/[p(x)-Q(x)+\{Q(x)-\Phi(l_{p(x)}-\lambda\mu)\}e^{\mu^2\lambda^2/2-\mu\lambda l_{p(x)}}], \quad \ldots \quad (2.1)$$

where $\lambda = l_{p(x)}-l_{q(x)}$. This provides a general formula for Engel elasticity at any income level $x$. In particular, notice that $l_{p(\bar{x}_a)} = \lambda/2$, $p(\bar{x}_a)+q(\bar{x}_a) = 1$, $l_{p(\bar{x}_g)} = 0$ and $l_{p(\bar{x}_h)} = -\lambda/2$. Therefore, the formulae for elasticities at the over-all arithmetic, geometric and harmonic mean income levels are, respectively, obtained as

$$\eta_a = -\mu[Q(\bar{x}_a)+q(\bar{x}_a)-1]/[e^{(\mu^2-\mu)\lambda^2/2}\{Q(\bar{x}_a)-\Phi(\lambda(0.5-\mu))\}-Q(\bar{x}_a)-q(\bar{x}_a)+1],$$
$$\text{where } \lambda = -2t_{q(\bar{x}_a)}$$

$$\eta_g = -\mu[Q(\bar{x}_g)-0.5]/[e^{\mu^2\lambda^2/2}\{Q(\bar{x}_g)-\Phi(\lambda\mu)-1\}-Q(\bar{x}_g)+0.5],$$
$$\text{where } \lambda = -t_s(\bar{x}_g);$$

$$\eta_h = -\mu[Q(\bar{x}_h)-\Phi(-\lambda/2)]/[e^{(\mu^2+\mu)\lambda^2/2}\{Q(\bar{x}_h)-\Phi(-\lambda(0.5+\mu))\}-Q(\bar{x}_h)+\Phi(-\lambda/2)]$$
$$\text{where } \lambda = -\frac{2}{3}\, t_{q(\bar{x}_h)}.$$

2.2. *Method* II. For the generalised Engel function, the specific concentration ratio is given by

$$L_s = 1-2\int_0^1 Q\,dp.$$

Let us write

$$K_1 = \alpha\mu-\beta,\ K_2 = e^{\mu\theta+\mu^2\lambda^2/2},\ \text{and } K = K_1+K_2.$$

Then we have

$$\begin{aligned} L_s &= 1-2\int_0^1 [(K_1/K)p+(K_2/K)\Phi(t_p-\lambda\mu)]dp \\ &= 1-K_1/K-2(K_2/K)\int_0^1 \Phi(z-\lambda\mu)d\Phi(z), \\ &= (-K_2/K)[1-2\int_0^1 \Phi(-z+\lambda\mu)d\Phi(z)] = (K_2/K)[2\Phi(\lambda\mu/\sqrt{2})-1] \end{aligned}$$

by convolution theorem.

Since

$$K_2/K = \eta(x)/[\eta(x)+\{\mu-\eta(x)\}e^{\lambda\mu t_{p(x)}-\mu^2\lambda^2/2}],$$

the Engel elasticity is given by

$$\eta(x) = -\mu L_s e^{\lambda\mu t_{p(x)}-\mu^2\lambda^2/2}[L_s(1-e^{\mu\lambda t_{p(x)}-\mu^2\lambda^2/2})-(2\Phi(\lambda\mu/\sqrt{2})-1)] \quad \dots \quad (2.2)$$

where $\lambda$ in terms of the Lorenz-ratio $L_0$ is given by $\lambda = \sqrt{2}t_{(1+L_0)/2}$

Elasticities $\eta_a$, $\eta_g$ and $\eta_h$ at the well-known mean income levels can be easily obtained from this general formula as is done under Method I.

We may now obtain elasticity formulations based on semi-log and hyperbolic Engel functions and by the two methods by taking the value of $\mu$ as 0 and 1, respectively, in (2.1) and (2.2). These are listed in Appendix Table A.1.

Notice that our formula for $\eta_a$ based on semi-log form and using Method I is quite different from $\eta_g/[1+0.5\ \eta_g t^2_{q(\bar{x}_g)}]$ the one obtained by Iyengar (1964). Our formulation is properly developed by considering the points on the two concentration curves corresponding to $\bar{x}_a$, where as that of Iyengar is derived indirectly being based on the use of formula for $\eta_g$.

## 3. Estimation procedure for Engel elasticities

To estimate elasticities at various income levels using grouped data and following the two alternative Methods I and II, it is necessary to calculate $(\hat{p}_i, \hat{q}_i, \hat{Q}_i)$'s corresponding to various income classes as basic datum. The given grouped data on consumer expenditure, for instance, by the National Sample Survey of India, usually provide estimates of: (i) $w_i$, proportion of persons belonging to the $i$-th income[1] class; (ii) $\bar{x}_i$ and $\bar{y}_i$, the mean income and the mean specific expenditure for the $i$-th income class; for $i = 1, \ldots, k$, $k$ being the number of income classes. From this data one may derive for each $i = 1, \ldots, k$,

$$\hat{p}_i = \sum_{j=1}^{i} \hat{w}_j,\ \hat{\bar{x}} = \sum_{j=1}^{k} \hat{w}_j\hat{\bar{x}}_j,\ \hat{\bar{y}} = \sum_{j=1}^{k} \hat{w}_j\hat{\bar{y}}_j,\ \ \hat{q}_i = \sum_{j=1}^{i} \hat{w}_j\hat{\bar{x}}_j/\hat{\bar{x}}$$

and

$$\hat{Q}_i = \sum_{j=1}^{i} \hat{w}_j\hat{\bar{y}}_j/\hat{\bar{y}}.$$

These values evidently provide consistent estimates of the corresponding population parameters.

Estimation of $\eta(x)$ by Method I and based on the generalised Engel function involves finding out the estimates of the corresponding $q(x)$ and $Q(x)$. As these estimates in general, cannot be directly obtained from the given grouped data, it calls for the use of some interpolation technique. The use of linear interpolation, as suggested by Iyengar (1967), is a crude approximation device resulting in over-estimation of both $q(x)$ and $Q(x)$. Moreover, by this device the over-estimation of $Q(x)$ in comparison to that of $q(x)$ is more for the luxuries and less for the necessities. A far better approximating method would be the non-linear interpolation method recently developed by Kakwani (1976) which suggests fitting third degree polynomial concentration curve within each income class except the first and the last open-ended classes where Pareto-type concentration curve is fitted as a further

[1]As reliable data are more easily available for the variable total expenditure, and not for income, total expenditure is regarded proxy for income as a matter of practice. However, for the sake of brevity, we name 'total expenditure' as 'income'.

refinement. This fitted Lorenz curve within the $t$-th income class ($t = 2, \ldots, k-1$) is given by the relation :

$$q = \alpha_{0t}+\alpha_{1t}(p-p_{t-1})+\alpha_{2t}(p-p_{t-1})^2+\alpha_{3t}(p-p_{t-1})^3, \quad \ldots \quad (3.1)$$

where parameters $\alpha_{0t}$, $\alpha_{1t}$, $\alpha_{2t}$ and $\alpha_{3t}$ are obtained as

$$\alpha_{0t} = q_{t-1},\ \alpha_{1t} = x_{t-1}/\bar{x},\ \alpha_{2t} = (3\delta_t-1)\Delta x_t/w_t\bar{x},\ \alpha_{3t} = (1-2\delta_t)\Delta x_t/\bar{x}w_t^2,$$

$$\delta_t = (\bar{x}_t-x_{t-1})/\Delta x_t, \text{ and } \Delta x_t = x_t-x_{t-1}.$$

Based on (3.1) $p$ at a given $x(x_{t-1} \leqslant x \leqslant x_t)$ is given by

$$p = p_{t-1}+[-\alpha_{2t}\pm\sqrt{\alpha_{2t}^2-3\alpha_{1t}\alpha_{3t}+3\alpha_{3t}x/\bar{x}}]3\alpha_{3t}, \quad \ldots \quad (3.2)$$

where sign is to be so chosen that $p_{t-1} \leqslant p \leqslant p_t$. Here the condition $1/3 < \delta_t < 2/3$ is required to ensure that the polynomial (3.1) will be always convex in the $t$-th income class. The corresponding fitted third degree polynomial specific concentration curve, yielding $Q$ at a given $x(x_{t-1} \leqslant x \leqslant x_t)$, is given by the relation similar to (3.1) except that the respective $q$, $\bar{x}_t$ and $x_t$ are replaced by $Q$, $\bar{y}_t$ and $E(y\,|\,x_t)$, the expected value of $y$ corresponding to the class limit $x_t$.

Consistent estimates of $p(x)$[2] and $q(x)$ corresponding to the given income level $x$ can be obtained from (3.2) and (3.1). The estimation of $Q(x)$ requires the knowledge of $E(y\,|\,x_t)$ for each $t = 1, 2, \ldots, k$, which can be approximated through some interpolation technique. Here the use of Lagrange's formula will give us

$$E(y\,|\,x_t) \simeq \sum_{i=1}^{k} \bar{y}_i L_i(x_t),$$

where

$$L_i(x_t) = \prod_{j=1}^{k} (x_t-\bar{x}_j) \Big/ \prod_{\substack{j=1\\ j\neq i}}^{k} (\bar{x}_i-\bar{x}_j)$$

which provides consistent estimate of $E(y\,|\,x_t)$ on replacing the population means $\bar{x}_i$'s and $\bar{y}_i$'s by their consistent estimates available in the corresponding sample means.

Estimation of $\eta(x)$ by Method II involves the use of the estimates of concentration ratios $L_0$ and $L_s$. For estimating $L_0$ and $L_s$ Iyengar (1964) used the linear formulae :

$$\hat{L}_0 = 1-\sum_{t=1}^{k} \hat{w}_t(\hat{q}_t+\hat{q}_{t-1})$$

and

$$\hat{L}_s = 1-\sum_{t=1}^{k} \hat{w}_t(\hat{Q}_t+\hat{Q}_{t-1}),$$

[2]Notice that for log-normally distributed variable $x$, $p(x_g) = 0.5$ and $p(x_h) = 1 - p(x_a)$.

which obviously under-estimate the true concentration-ratios. The exact formulae for $L_0$ and $L_s$ may be expressed as

$$L_0 = 1-2 \sum_{i=1}^{k} A_i$$

and

$$L_s = 1-2 \sum_{i=1}^{k} A_{si},$$

where $A_i$ and $A_{si}$ are the areas under the Lorenz and the specific concentration curves for the $i$-th income class. Fitting third degree polynomial Lorenz and specific concentration curves within each income class except the first and the last open-ended classes where Pareto-type curves are fitted, the above formulae for $L_0$ and $L_s$ give rise to the following formulae [See Kakwani (1976)] for the consistent estimates of $L_0$ and $L_s$, where $L_0$ and $L_s$ correspond to the fitted concentration curves over the entire income range :

$$\hat{L}_0 = 1- \sum_{i=1}^{k} \hat{w}_i(\hat{q}_i+\hat{q}_{i-1})+ \sum_{i=2}^{k-1} \frac{\hat{w}_i^2(x_i-x_{i-1})}{6\hat{\bar{x}}}+\frac{\hat{w}_1^2\hat{\bar{x}}_1(x_1-\hat{\bar{x}}_1)}{\hat{\bar{x}}(x_1+\hat{\bar{x}}_1)}+\frac{\hat{w}_k^2\hat{\bar{x}}_k(\hat{\bar{x}}_k-x_{k-1})}{\hat{\bar{x}}(\hat{\bar{x}}_k+x_{k-1})} \quad \ldots \quad (3.3)$$

$$\hat{L}_s = 1-\sum_{i=1}^{k} \hat{w}_i(\hat{Q}_i+\hat{Q}_{i-1})+\sum_{i=2}^{k-1} \hat{w}_i^2[\hat{E}(y\,|\,x_i)-\hat{E}(y\,|\,x_{i-1})]/(6\hat{\bar{y}})$$

$$+\hat{w}_1^2\hat{\bar{y}}_1[\hat{E}(y\,|\,x_1)-\hat{\bar{y}}_1]/[\hat{\bar{y}}\{\hat{E}(y\,|\,x_1)+\hat{\bar{y}}\}]$$

$$+\hat{w}_k^2\hat{\bar{y}}_k[\hat{\bar{y}}_k-\hat{E}(y\,|\,x_{k-1})]/[\hat{\bar{y}}\{\hat{\bar{y}}_k+\hat{E}(y\,|\,x_{k-1})\}] \quad \ldots \quad (3.4)$$

where $E(y\,|\,x_i)$'s may be estimated the same way as done under Method I.

It may be noted that as $q(x)$ and $q(x)$, and $Q(x)$ and $Q(x)$ as well as $L_0$ and $L_0$, and $L_s$ and $L_s$ are expected to be almost equal, both Methods I and II, using the present estimation procedure, yield almost consistent estimate of Engel elasticity $\eta(x)$.

## 4. Estimation of Engel functions

We have earlier noted that when grouped data with unequal class interval are given, both Methods I and II yield almost consistent (a.c.) estimates of Engel elasticity at any given income level and based on semi-log and hyperbolic Engel functions. Therefore, using these estimated elasticities, a.c. estimates of the parameters of these two Engel functions can be obtained by both Methods as outlined below.

4.1. *Semi-log Engel functions* $E(y|x) = a+b \log x$ : Based on this Engel Function, elasticity at geometric mean level $\bar{x}_g$ is given by $\eta_g = b/(a+b \log \bar{x}_g)$. Since $\bar{y}_a = a+b \log \bar{x}_g$ and $\bar{x}_g = \bar{x}_a \exp(-\lambda^2/2)$, parameters a and $b$ may be estimated almost consistently as

$$\hat{b} = \hat{\eta}\hat{\bar{y}}_a$$

and

$$\hat{a} = \hat{\bar{y}}_a(1-\hat{\eta}_g \log \bar{x}_a+\hat{\eta}_g\lambda^2/2),$$

where

$$\hat{\eta}_g = \sqrt{2\pi}[\hat{Q}(\hat{\bar{x}}_g)-0.5]/t_{\hat{q}(\hat{\bar{x}}_g)}$$

and

$$\hat{\lambda} = -t_{\hat{q}(\bar{x}_g)}$$

when Method I is used, and

$$\hat{\eta}_g = \sqrt{\pi/2}\, \hat{L}_s/t_{(1+\hat{L}_0)/2}$$

and

$$\hat{\lambda} = \sqrt{2}\, t_{(1+\hat{L}_0)/2}$$

when Method II is used.

4.2. *Hyperbolic Engel functions* $E(y|x) = a+b/x$ : Based on this function, elasticity at the harmonic mean level $\bar{x}_h$ is given by $\eta_h = -b/(\bar{x}_h\, \bar{y}_a)$ where $\bar{y}_a = a+b/\bar{x}_h$. Therefore, almost consistent estimates of the parameters $a$ and $b$ will be $\hat{b} = -\hat{\eta}_h\, \hat{\bar{y}}_a\, \hat{\bar{x}}_h$ and $\hat{a} = \hat{\bar{y}}_a(1+\hat{\eta}_h)$, where a.c. estimate of $\bar{x}_h$ is $\hat{\bar{x}}_h = \hat{\bar{x}}_a \exp[-t^2_{q(p=0\cdot5)}]$ and that of $\eta_h$ is

$$\hat{\eta}_h = [\hat{Q}(\hat{\bar{x}}_h)-0.5]/\left[2\Phi\left\{\frac{1}{3}\, t_{\hat{q}(\hat{\bar{x}}_h)}\right\}-1\right]-0.5 \text{ by Method I,}$$

and

$$= \hat{L}_s/\hat{L}_0 \text{ by Method II.}$$

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## Appendix

TABLE A.1. FORMULAE FOR ESTIMATING ENGEL ELASTICITIES AT VARIOUS INCOME LEVELS, OBTAINED BY METHODS I AND II AND BASED ON VARIOUS FORMS OF THE ENGEL FUNCTION

| | elasticities | forms of the Engel function | |
|---|---|---|---|
| | | semi-log : $E(y\|x) = a+b\log x$ | hyperbolic : $E(y\|x) = a+b/x$ |
| Method I | $\eta(x)$ | $[Q(x)-p(x)]\div[t_{q(x)}-t_{p(x)}][\phi(t_{p(x)})+\{Q(x)-p(x)\}t_{p(x)}]$ | $[Q(x)-p(x)]\div[\{Q(x)-\Phi(t_{p(x)}+\lambda)\}e^{\lambda t_{p(x)}+\lambda^2/2}-Q(x)+p(x)]$, where $\lambda = t_{p(x)}-t_{p(x)}$ |
| | $\eta_a$ | $[Q(x_a)+q(x_a)-1]\div 2t_{q(\bar{x}_a)}[\phi(t_{q(\bar{x}_a)})+\{Q(x_a)+q(x_a)-1\}t_{q(\bar{x}_a)}]$ | $[Q(x_a)+q(x_a)-1]\div[e^{\lambda^2}\{Q(x_a)-\Phi(3\lambda/2)\}-\{Q(x_a)+q(x_a)-1\}]$, where $\lambda = -2t_{q(\bar{x}_g)}$ |
| | $\eta_g$ | $\sqrt{2\pi}[Q(x_g)-0.5]\div t_{q(\bar{x}_g)}$ | $[Q(x_g)-0.5]\div[\{Q(x_g)+q(x_g)-1\}e^{\lambda^2/2}-Q(x_g)+0.5]$, where $\lambda = t_{q(\bar{x}_g)}$ |
| | $\eta_h$ | $9\left[Q(x_h)-\phi\left\{\frac{1}{3}t_{q(\bar{x}_h)}\right\}\right]\div 2t_{q(\bar{x}_h)}\left[3\Phi\left\{\frac{1}{3}t_{q(\bar{x}_h)}\right\}+\left\{Q(x_h)-\phi\left(\frac{1}{3}t_{q(\bar{x}_h)}\right)\right\}t_{q(\bar{x}_h)}\right]$ | $\left[Q(x_h)-\Phi\left\{\frac{1}{3}t_{q(\bar{x}_h)}\right\}\right]\div\left[2\Phi\left\{\frac{1}{3}t_{q(\bar{x}_h)}\right\}-1\right]$ |
| Method II | $\eta(x)$ | $\sqrt{\pi/2}\,L_s\div[1+\sqrt{\pi}L_s t_{p(x)}]t_{(1+L_0)/2}$ | $L_s\div[(L_o+L_s)e^{\lambda^2/2+\lambda t_{p(x)}}-L_s]$, where $\lambda = \sqrt{2}t_{(1+L_0)/2}$ |
| | $\eta_a$ | $\sqrt{\pi/2}\,L_s\div[1+\sqrt{\pi/2}\,L_s t_{(1+L_0)/2}]t_{(1+L_0)/2}$ | $L_s\div[(L_o+L_s)e^{\lambda^2}-L_s]$, where $\lambda = \sqrt{2}t_{(1+L_0)/2}$ |
| | $\eta_g$ | $\sqrt{\pi/2}\,L_s\div t_{(1+L_0)/2}$ | $L_s\div[L_o+L_s)e^{\lambda^2/2}-L_s]$, where $\lambda = \sqrt{2}t_{(1+L_0)/2}$ |
| | $\eta_h$ | $\sqrt{\pi/2}\,L_s\div[1-\sqrt{\pi/2}\,L_s t_{(1+L_0)/2}]t_{(1+L_0)/2}$ | $L_s\div L_o$ |