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A note on derivation of the generating function for the right truncated Rayleigh distribution

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Abstract

An expression is obtained for the probability that a Weibull random variable falls after the truncation and within a finite interval. However small, the truncation in the Weibull distribution (when the value of the shape parameter is two, it is called the Rayleigh distribution) has an impact. An attempt is made to obtain generating functions for two fixed shape parameters.

Keywords: Weibull distribution; Incomplete gamma; Kummer's function

1. Preliminaries

The p.d.f. of a Weibull r.v. Y is given by

$$f(y) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{y}{\alpha}\right)^{\beta}}, \qquad \text{for all } y \geq 0. \tag{1.1}$$

Here α is a scale and β is a shape parameter. There are other forms for the function in (1.1) (see [1]). The characteristics of this function are well known. However, the characteristics of this function are different if some of the values of the r.v. Y are right truncated [2,3]. For better understanding of the progression of a disease, e.g., for HIV, earlier researchers considered right truncated Weibull distributions [3]. Here they assumed that the progression of the disease is not an increasing function, but will stabilize after a certain time point. This time point is called the truncation time δ and the truncated Weibull p.d.f. [3] is given below:

$$f_{\leq\delta}(y) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{y}{\alpha}\right)^{\beta}}; \qquad \text{for all } 0 \leq y < \delta \text{ and } \alpha, \beta > 0 \tag{1.2}$$

$$f_{\geq\delta}(y) = \frac{\beta}{\alpha}\mathrm{e}^{\left\{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\frac{y}{\alpha}\right)^{\beta}\right\}} \left\{\left(\frac{t}{\alpha}\right)^{\beta-1} + \left(\frac{y}{\alpha}\right)^{\beta-1}\right\}, \quad \text{for all } \delta \leq y < \infty \text{ and } \alpha, \beta > 0, \delta \leq t < \infty. \tag{1.3}$$

Define $f_\delta(y)$ as follows:

$$f_\delta(y) = \begin{cases} f_{\leq\delta}(y) & \text{for all } 0 \leq y < \delta \\ f_{\geq\delta}(y) & \text{for all } \delta \leq y < \infty. \end{cases} \tag{1.4}$$

There have been attempts to estimate the parameters of the truncated Weibull distribution, in some cases censoring and also fitting the survival data [4–6]. For $\beta = 2$, the Weibull distribution is called a Rayleigh distribution [7]. In Section 2 technicalities of the effect of truncation on the Weibull distribution are given and a result useful in dealing with such truncated Weibull distributions is also proved. Section 3 deals with an attempt at deriving the generating functions in a couple of cases. Section 4 concludes with brief outline of applications of truncated Weibull distributions and potential applications for the future.

2. Importance of truncation

Theorem 2.1. *For all $\alpha, \beta, \delta > 0$, a definite interval of length σ and for a given $k > \delta$, the probability*

$$P(Y \leq k) = P(Y \leq \delta) + \phi_\delta(k\sigma) \tag{2.1}$$

where

$$\phi_\delta(k\sigma) = \mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \mathrm{e}^{-\frac{k\beta\sigma}{\delta}\left(\frac{\delta}{\alpha}\right)^\beta - \left(\frac{\delta+k\sigma}{\alpha}\right)^\beta}.$$

Proof. Take $P(Y \leq k)$. This can be expressed as

$$= P(Y \leq \delta) + P(\delta \leq Y \leq k) \qquad (\text{since } k > \delta) \tag{2.2}$$

$$= \int_0^\delta f_{\leq\delta}(y)\mathrm{d}y + \int_\delta^k f_{\geq\delta}(y)\mathrm{d}y. \tag{2.3}$$

Distributing the second term in the RHS of (2.3) into two finite integrals of equal length σ and evaluating them, we get

$$\begin{aligned} f_\delta(1\sigma) &= \int_\delta^{\delta+\sigma} f_{\geq\delta}(y)\mathrm{d}y + \int_{\delta+\sigma}^{\delta+2\sigma} f_{\geq\delta}(y)\mathrm{d}y \\ &= \frac{\beta}{\alpha}\left[\int_\delta^{\delta+\sigma} \mathrm{e}^{\left\{\frac{\beta}{\alpha}\left(\frac{\delta}{\alpha}\right)^{\beta-1}\{\delta-y\}-\{\frac{y}{\alpha}\}^\beta\right\}}\mathrm{d}y\right] \\ &\quad + \frac{\beta}{\alpha}\left[\int_{\delta+\sigma}^{\delta+2\sigma} \mathrm{e}^{\left\{\frac{\beta}{\alpha}\left(\frac{\delta}{\alpha}\right)^{\beta-1}\{\delta-y\}-\{\frac{y}{\alpha}\}^\beta\right\}}\left\{\left(\frac{\delta}{\alpha}\right)^{\beta-1} + \left(\frac{y}{\alpha}\right)^{\beta-1}\right\}\mathrm{d}y\right] \\ &= \frac{\beta}{\alpha}\left[\frac{\alpha}{\beta}\left\{\mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \mathrm{e}^{\beta\left(\frac{\delta}{\alpha}\right)^\beta - \beta\left(\frac{\delta}{\alpha}\right)^\beta(1+\frac{\sigma}{\delta}) - \left(\frac{\delta+\sigma}{\alpha}\right)^\beta}\right\}\right] \\ &\quad + \frac{\beta}{\alpha}\left[\frac{\alpha}{\beta}\left\{\mathrm{e}^{\beta\left(\frac{\delta}{\alpha}\right)^\beta - \beta\left(\frac{\delta}{\alpha}\right)^\beta(1+\frac{\sigma}{\delta}) - \left(\frac{\delta+\sigma}{\alpha}\right)^\beta}\right\}\right] \\ &\quad - \frac{\beta}{\alpha}\left[\frac{\alpha}{\beta}\left\{\mathrm{e}^{\beta\left(\frac{\delta}{\alpha}\right)^\beta - \beta\left(\frac{\delta}{\alpha}\right)^\beta(1+\frac{\sigma}{\delta}) - \left(\frac{\delta+2\sigma}{\alpha}\right)^\beta}\right\}\right]. \end{aligned}$$

After canceling and simplifying these terms, we get (2.4)

$$f_\delta(1\sigma) = \mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \mathrm{e}^{-\frac{2\beta\sigma}{\delta}\left(\frac{\delta}{\alpha}\right)^\beta - \left(\frac{\delta+2\sigma}{\alpha}\right)^\beta}. \tag{2.4}$$

Again, distributing the second term in the RHS of (2.3) into three finite integrals of equal length σ and evaluating that term as above, we get

$$f_\delta(2\sigma) = \int_\delta^{\delta+\sigma} f_{\geq\delta}(y)\mathrm{d}y + \int_{\delta+\sigma}^{\delta+2\sigma} f_{\geq\delta}(y)\mathrm{d}y + \int_{\delta+2\sigma}^{\delta+3\sigma} f_{\geq\delta}(y)\mathrm{d}y$$
$$= \mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \mathrm{e}^{-\frac{3\beta\sigma}{\delta}\left(\frac{\delta}{\alpha}\right)^\beta - \left(\frac{\delta+3\sigma}{\alpha}\right)^\beta}. \tag{2.5}$$

There is a symmetry in the expression and terms cancel in the order shown for deriving (2.4). By this rule, if we divide the same integral, from (2.3), into k finite integrals of equal length σ, then the integral evaluated will be of the form given in (2.6), i.e.

$$f_\delta(k\sigma) = \int_\delta^{\delta+\sigma} f_\delta(y)\mathrm{d}y + \int_{\delta+\sigma}^{\delta+2\sigma} f_\delta(y)\mathrm{d}y \cdots + \int_{\delta+(k-1)\sigma}^{\delta+k\sigma} f_\delta(y)\mathrm{d}y$$
$$= \mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \mathrm{e}^{-\frac{k\beta\sigma}{\delta}\left(\frac{\delta}{\alpha}\right)^\beta - \left(\frac{\delta+k\sigma}{\alpha}\right)^\beta}. \tag{2.6}$$

Hence the proof. □

Remark 2.2. As the value of the truncation point increases, (2.6) is a decreasing function except where $\beta \to 0$. When $\beta \to 0$ and δ is increasing the values become oscillatory, and then it looks like a step function. When $\beta > 1$ and δ is increasing the function looks like a quadratic exponential with a uni-peak.

Note 2.3. Suppose $Y_1, Y_2, Y_3, \ldots, Y_n$ are identically independently distributed random variables with Y_i following a truncated Weibull distribution (1.4). Let

$$Z = \mathrm{Min}(Y_1, Y_2, Y_3, \ldots, Y_n)$$

and suppose $\delta \leq z \in Z < \infty$. Then

$$\begin{aligned} P(Z > z) &= P\{\mathrm{Min}(Y_1, Y_2, Y_3, \ldots, Y_n) > z\} \\ &= P(\cap_{i=1}^n Y_i > z) \\ &= \prod P(Y_i > z) \\ &= P(Y_i > z)^n \end{aligned}$$

since $\delta \leq z \in Z < \infty$; we use (1.3) and, for $\beta = 2$, we have

$$\begin{aligned} P(Y_i > z) &= \int_z^\infty f_{\geq\delta}(y)\mathrm{d}y \\ &= \int_z^\infty \frac{2}{\alpha}\mathrm{e}^{\left\{\frac{2}{\alpha}\left(\frac{t}{\alpha}\right)(t-y) - \left(\frac{y}{\alpha}\right)^2\right\}} \left\{\left(\frac{t}{\alpha}\right) + \left(\frac{y}{\alpha}\right)\right\} \mathrm{d}y. \end{aligned}$$

Now substitute $w = \left(\frac{y}{\alpha}\right)^2 - \frac{2t}{\alpha^2}(t-y)$ and change the limits accordingly: $y \to \infty$ *then* $w \to \infty$ and *as* $y \to z$ *then* $w \to \left(\frac{z}{\alpha}\right)^2 - \frac{2t}{\alpha^2}(t-z)$ (say, ϕ). Also, verify that $\mathrm{d}y = \frac{\alpha^2}{2\sqrt{3t^2+\alpha^2 w}}$ and $y = -t + \sqrt{3t^2 + \alpha^2 w}$. Then

$$P(Y_i > z) = \int_\phi^\infty t\mathrm{e}^{-w} \frac{\mathrm{d}w}{\sqrt{3t^2+\alpha^2 w}} + \int_\phi^\infty \left(-t + \sqrt{3t^2+\alpha^2 w}\right)\mathrm{e}^{-w} \frac{\mathrm{d}w}{\sqrt{3t^2+\alpha^2 w}}.$$

Therefore,

$$P(Z > z) = (\Gamma(1, \phi))^n = [\Gamma(1) - \phi\mathrm{e}^{-\phi} M(1, 2, \phi)]^n.$$

Here **M** is called Kummer's function. See Appendix for the relation between lower, upper gamma functions and Kummer's function.

Note 2.4. If

$$\chi = \left\{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y) - \left(\frac{y}{\alpha}\right)^\beta\right\}$$

then (1.3) can be written as

$$
\begin{aligned}
&= \frac{\beta}{\alpha}\left\{\left(\frac{t}{\alpha}\right)^{\beta-1} + \left(\frac{y}{\alpha}\right)^{\beta-1}\right\} \mathrm{e}^{\chi} \\
&= \frac{\beta}{\alpha}\left\{\left(\frac{t}{\alpha}\right)^{\beta-1} + \left(\frac{y}{\alpha}\right)^{\beta-1}\right\}\left(1+\chi+\frac{\chi^2}{2!}+\cdots\right) \\
&= \frac{\beta}{\alpha}\left\{\left(\frac{t}{\alpha}\right)^{\beta-1} + \left(\frac{y}{\alpha}\right)^{\beta-1}\right\}\lim_{n\to\infty}\left(1+\frac{\chi}{n}\right)^n \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{\alpha}\right)^{\beta}\lim_{n\to\infty}\left(1+\frac{\chi}{n}\right)^n + \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{\alpha}\right)^{\beta}\lim_{n\to\infty}\left(1+\frac{\chi}{n}\right)^n. \qquad (2.7)
\end{aligned}
$$

3. Truncated generating functions

Let G_r denote the moment generating function for (1.4); then it is expressed using (1.2)–(1.4) as follows:

$$
\begin{aligned}
G_r &= \int_0^{\infty} \mathrm{e}^{my} f_{\delta}(y)\mathrm{d}y \\
&= \int_0^{\delta} \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} \mathrm{e}^{my-\left(\frac{y}{\alpha}\right)^{\beta}}\mathrm{d}y + \int_{\delta}^{\infty}\frac{\beta}{\alpha}\mathrm{e}^{\left\{my+\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\frac{y}{\alpha}\right)^{\beta}\right\}}\left(\frac{t}{\alpha}\right)^{\beta-1}\mathrm{d}y \\
&\quad + \int_{\delta}^{\infty}\frac{\beta}{\alpha}\mathrm{e}^{\left\{my+\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\frac{y}{\alpha}\right)^{\beta}\right\}}\left(\frac{y}{\alpha}\right)^{\beta-1}\mathrm{d}y. \qquad (3.1)
\end{aligned}
$$

There is a polynomial of degree β in the exponent; hence it will be very difficult to obtain the generating function for this integral. Even for the second degree the integral will be a rational exponential. In this section, with suitable substitutions and using special functions, a moment generating function formula up to second degree, i.e. for M_{r1} and M_{r2}, is evaluated. The formulas could be of potential use in biological and engineering sciences when we are dealing with (1.4): highly flexible in survival and reliability analysis. The value of t indicates the time of truncation. It was shown that when t reaches a certain value, any increase in it after that value will not change the shape of the distribution [3].

When $\beta = 1$ then (3.1) will become

$$
\int_0^{\delta}\frac{1}{\alpha}\mathrm{e}^{my-\left(\frac{y}{\alpha}\right)}\mathrm{d}y + 2\int_{\delta}^{\infty}\frac{1}{\alpha}\mathrm{e}^{\left\{my+\frac{1}{\alpha}(t-y)-\left(\frac{y}{\alpha}\right)\right\}}\mathrm{d}y. \qquad (3.2)
$$

Substitute $u = y/\alpha - 1/\alpha(t-y) - my$ in (3.2), and change the limits accordingly; then we get the following equations:

$$
= \frac{1}{\alpha(2/\alpha - m)}\left\{\int_{-t/\alpha}^{g}\mathrm{e}^{-u-\frac{1}{\alpha}(t-y)}\mathrm{d}u + 2\int_{g}^{\infty}\mathrm{e}^{-u}\mathrm{d}u\right\}.
$$

Since *as* $y \to 0$ *then* $u \to -t/\alpha$ and *as* $y \to \delta$ *then* $u \to \delta/\alpha - 1/\alpha(t-\delta) - m\delta$ *(say,* g*) and* $\mathrm{d}y = \mathrm{d}u(2/\alpha - m)^{-1}$ where,

$$
\begin{aligned}
y &= \frac{u+\frac{t}{\alpha}}{\frac{2}{\alpha}-m} \\
&= \frac{1}{\alpha(2/\alpha - m)}\left[\mathrm{e}^{-\frac{t}{\alpha}\left\{\frac{t(1-m\alpha)-u\alpha}{2-m\alpha}\right\}}\left(\mathrm{e}^{-\frac{t}{\alpha}} - \mathrm{e}^{-g}\right) + 2\Gamma(1,g)\right]
\end{aligned}
$$

then,

$$
M_{r1} = \frac{1}{\alpha(2/\alpha - m)}\left[\mathrm{e}^{-\frac{t}{\alpha}\left\{\frac{t(1-m\alpha)-u\alpha}{2-m\alpha}\right\}}\left(\mathrm{e}^{-\frac{t}{\alpha}} - \mathrm{e}^{-g}\right) + 2\{1 - g\mathrm{e}^{-g}\mathbf{M}(1,2,g)\}\right]. \qquad (3.3)
$$

When $\beta = 2$, (3.1) will become

$$\int_0^{\delta} \frac{2y}{\alpha^2} e^{my-\left(\frac{y}{\alpha}\right)^2} dy + \int_{\delta}^{\infty} \frac{2t}{\alpha^2} e^{\left\{my+\frac{2t}{\alpha^2}(t-y)-\left(\frac{y}{\alpha}\right)^2\right\}} dy + \int_{\delta}^{\infty} \frac{2y}{\alpha^2} e^{\left\{my+\frac{2t}{\alpha^2}(t-y)-\left(\frac{y}{\alpha}\right)^2\right\}} dy. \tag{3.4}$$

Take

$$v = \left(\frac{y}{\alpha}\right)^2 - my - \frac{2t}{\alpha^2}(t-y); \tag{3.5}$$

then

$$dy = \frac{\alpha^2}{\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}} dv. \tag{3.6}$$

Also,

$$y \to 0 \Rightarrow v \to -\frac{2t^2}{\alpha^2}$$
$$y \to \delta \Rightarrow v \to \left(\frac{\delta}{\alpha}\right)^2 - \frac{2t}{\alpha^2}(t-\delta) - my \ (say, h)$$
$$y \to \infty \Rightarrow t \to \infty.$$

Consider only the first term of (3.4) and make the above changes; then this term with new limits can be written as

$$\frac{2}{\alpha^2}\left[\int_0^h e^{\frac{(m\alpha)^2}{2} - tm + \frac{m}{2}\left(\sqrt{4\alpha^2 v+\alpha^4 m^2-4\alpha^2 mt+12t^2}\right) - \left\{\frac{m\alpha}{2} - \frac{t}{\alpha} + \frac{1}{2\alpha}\left(\sqrt{4\alpha^2 v+\alpha^4 m^2-4\alpha^2 mt+12t^2}\right)\right\}^2}\right.$$
$$\left. \times \left\{\frac{m\alpha^2}{2} - t + \frac{t}{2}\left(\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}\right)\right\}\left(\frac{\alpha^2}{\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}}\right) dv\right]. \tag{3.7}$$

Consider the second and third terms of (3.4) and following the steps as above; then these terms with new limits can be written as

$$\int_h^{\infty} \frac{2t}{\alpha^2} e^{-v} \frac{\alpha^2}{\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}} dv$$
$$+ \int_h^{\infty} \frac{2}{\alpha^2} e^{-v} \frac{\alpha^2\left\{\frac{m\alpha^2}{2} - t + \frac{t}{2}\left(\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}\right)\right\}}{\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}} dv. \tag{3.8}$$

Now as $v \to -\frac{2t^2}{\alpha^2}$, we have

$$\left\{\frac{m\alpha^2}{2} - t + \frac{t}{2}\left(\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}\right)\right\} \to 0$$

and

$$\sqrt{4\alpha^2 v + \alpha^4 m^2 - 4\alpha^2 mt + 12t^2}\Big/\alpha^2 \to \frac{2t}{\alpha^2} - m.$$

Therefore the sum of (3.7) and (3.8) will lead to the generating function in the special conditions discussed in this section. This can be written as follows:

$$M_{r2} = \frac{2te^{2t^2/\alpha^2}}{2t - m\alpha^2}\{\Gamma(1) - he^{-h} M(1, 2, h)\}.$$

4. Conclusions

It is already established that when $\beta = 1$ the ordinary Weibull distribution becomes an exponential distribution. It is also known that when the shape parameter is two, it will be a Rayleigh distribution [7]. Hence the generating function obtained in this work can be called a generating function for the truncated Rayleigh distribution, which is not available in the literature. The discrete form derived here for the general right truncated Weibull distribution for the probability that a random variable falls in the finite interval after the truncation time can be practically explored.

Theorem 2.1 can be applied to find the probability that a random variable falls after the truncation and within a specified interval. If Y_i are independent and identically distributed Weibull random variables, then the probability that a minimum value greater than z (say), where z falls after the truncation time point, is obtained in this work is an incomplete gamma function.

One biological application of such a truncated distribution arises in studies where the variable of interest is the duration between infection with a virus and development of symptoms of a disease. Given an infection time for an individual, it is not always possible to assess the time of onset of symptoms or development of the disease. It is proved here that the truncation of this kind of distribution has an impact [3]. The risk of development of the disease will increase as $t \to \infty$ and the chance of survival will decline after the optimal truncation point.

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Appendix

The lower and upper gamma functions utilized in this work are from the following Pearson forms of incomplete gamma functions (4.1) and (4.2); Kummer gave a formula (4.3) for the incomplete gamma function as a confluent hypergeometric function [8]:

$$1^{-r-1}\Gamma(r+1, 2\delta) = \int_{2\delta}^{\infty} \mathrm{e}^{-0.1.t} t^{r+1-1} \mathrm{d}t \tag{4.1}$$

$$\gamma\left(r+1, \frac{t}{\alpha}\right) = \Gamma(r+1) - \Gamma\left(r+1, \frac{t}{\alpha}\right) \tag{4.2}$$

$$\gamma(r, h) = r^{-1} h^r \mathrm{e}^{-h} \mathbf{M}(1, 1+r, h). \tag{4.3}$$

In (4.3), $\mathbf{M}$ is called Kummer's function.

References

[1] D.R. Cox, Survival Analysis, Chapman & Hall, 1972.
[2] R.W. Dallas, Methods for fitting the right-truncated Weibull distribution to life-test and survival data, Biom. J. 30 (5) (1988) 545–551.
[3] A.S.R.S. Rao, M Kakehashi, Incubation time distribution in back-calculation applied to HIV/AIDS in India, Math. Biosci. Eng. 2 (2) (2005) 263–277.
[4] R.U. Khan, H. Athar, Ratio and inverse moments of order statistics from doubly truncated Weibull distribution, J. Statist. Res. 34 (1) (2000) 33–38.
[5] B.J. Lee, E. Max, Inferences on the parameters and current system reliability for a time truncated Weibull process, Technometrics 22 (3) (1980) 421–426.
[6] M. Mittal, R.C. Dahiya, Estimating the parameters of a truncated Weibull distribution, Comm. Statist. Theory Methods 18 (6) (1989) 2027–2042.
[7] N.L. Johnson, S. Kotz, Distributions in Statistics, in: Continuous Univariate Distributions, vol. 2, John Wiley and Sons, New York, 1970.
[8] M. Abramowitch, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1972.