Algebraic Geometry Solves an Old Matrix Problem

R Bhatia

Let A, B be $n \times n$ Hermitian matrices, and let $C = A + B$. Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be the eigenvalues of A, B, and C, respectively. Mathematicians, physicists, and numerical analysts have long been interested in knowing all possible relations between the n-tuples $\{\alpha_j\}, \{\beta_j\}$ and $\{\gamma_j\}$.

Since tr $C =$ tr $A +$ tr B, where tr stands for the trace of a matrix, we have

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} (\alpha_i + \beta_i). \tag{1}$$

H Weyl (1912) was the first to discover several non-trivial relations between these numbers; these are the inequalities

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad \text{for} \quad i + j - 1 \leq n. \tag{2}$$

(See [1, Chapter 3]) for a proof and discussion of this and some of the other results described below.)
When $n = 2$, this yields three inequalities

$$\gamma_1 \leq \alpha_1 + \beta_1, \quad \gamma_2 \leq \alpha_1 + \beta_2, \quad \gamma_2 \leq \alpha_2 + \beta_1. \tag{3}$$

It turns out that, together with the equality (1), these three inequalities are sufficient to characterise the possible eigenvalues of A, B, and C; i.e., if three pairs of real numbers $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$, each ordered decreasingly ($\alpha_1 \geq \alpha_2$, etc.), satisfy these relations, then there exist 2×2 Hermitian matrices A and B such that these pairs are the eigenvalues of A, B and $A + B$.

When $n \geq 3$, more relations exist. The first one due to Ky Fan (1949) says

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \alpha_j + \sum_{j=1}^{k} \beta_j, \quad \text{for} \quad 1 \leq k \leq n. \tag{4}$$

When $k = n$, the two sides of (4) are equal; that is just the equality (1). A substantial generalisation of this was obtained by V B Lidskii (1950). For brevity, let $[1, n]$ denote the set $\{1, 2, \ldots, n\}$. Lidskii's theorem says that for every subset $I \subset [1, n]$ with cardinality $|I| = k$, we have

$$\sum_{i \in I} \gamma_i \leq \sum_{i \in I} \alpha_i + \sum_{j \leq k} \beta_j. \tag{5}$$

Note that these inequalities include (4) as a special case – choose $I = [1, k]$.

Lidskii's theorem has an interesting history. It was first proved by F Berezin and I M Gel'fand in connection with their work on Lie groups. On their suggestion Lidskii provided an elementary proof. Others had difficulty following this proof. It was H W Wielandt (1955) who supplied a proof that was understood by others. Now several proofs of this theorem are known; see [1].

When $n = 3$, we get six relations from Weyl's inequalities :

$$\begin{array}{lll} \gamma_1 \leq \alpha_1 + \beta_1, & \gamma_2 \leq \alpha_1 + \beta_2, & \gamma_2 \leq \alpha_2 + \beta_1, \\ \gamma_3 \leq \alpha_1 + \beta_3, & \gamma_3 \leq \alpha_3 + \beta_1, & \gamma_3 \leq \alpha_2 + \beta_2. \end{array} \tag{6}$$

Five more follow from the inequalities (5) :

$$\begin{aligned} \gamma_1 + \gamma_2 &\leq \alpha_1 + \alpha_2 + \beta_1 + \beta_2, \\ \gamma_1 + \gamma_3 &\leq \alpha_1 + \alpha_3 + \beta_1 + \beta_2, \\ \gamma_2 + \gamma_3 &\leq \alpha_2 + \alpha_3 + \beta_1 + \beta_2, \\ \gamma_1 + \gamma_3 &\leq \alpha_1 + \alpha_2 + \beta_1 + \beta_3, \\ \gamma_2 + \gamma_3 &\leq \alpha_1 + \alpha_2 + \beta_2 + \beta_3. \end{aligned} \tag{7}$$

(use the symmetry in A, B). It turns out that one more relation

$$\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3, \tag{8}$$

is valid. Further, the relations (1), (6), (7) and (8) are sufficient to characterise the possible eignevalues of A, B and C.

The Lidskii-Wielandt theorem aroused much interest, and several more inequalities were discovered. They all have the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j, \tag{9}$$

where I, J, K are certain subsets of $[1, n]$ all having the same cardinality. Note that the inequalities (2), (4) and (5) all have this form.

This leads to the following questions. What are all the triples (I, J, K) of subsets of $[1, n]$ for which the inequalities (9) are true? Are these inequalities, together with (1), sufficient to characterise the α, β, and γ that can be eigenvalues of Hermitian matrices A, B and $A + B$?

In a fundamental paper in 1962, Alfred Horn made a conjecture that asserted that these inequalities, together with (1), are sufficient and that the set T_r^n of triples (I, J, K) of cardinality r in $[1, n]$ can be described by induction on r as follows. Let us write $I = \{i_1 < i_2 < \cdots < i_r\}$ and likewise for J and K. Then, for $r = 1$, (I, J, K) is in T_1^n if $k_1 = i_1 + j_1 - 1$. For $r > 1$, $(I, J, K) \in T_r^n$ if

$$\sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \binom{r+1}{2} \tag{10}$$

and, for all $1 \leq p \leq r - 1$ and all $(U, V, W) \in T_p^r$

$$\sum_{u \in U} i_u + \sum_{v \in V} j_v \leq \sum_{w \in W} k_w + \binom{p+1}{2} \tag{11}$$

Horn proved his conjecture for $n = 3$ and 4. Note that when $n = 2$, these conditions just reduce to the three inequalities given by (3). When $n = 3$, they reduce to the twelve inequalities (6-8). When $n = 7$, there are 2062 inequalities given by these conditions.

Horn's conjecture has finally been proved by A Klyachko (1998) and A Knutson and T Tao (1999) (see [2], [3]).

It turns out that this problem has some remarkable connections with problems in algebraic geometry and the representation theory of Lie groups. Let us indicate briefly the connection with algebraic geometry.

The classical minimax principle of Courant, Fischer, and Weyl says that the eigenvalues α_j of the Hermitian matrix A are characterised by extremal relations

$$\alpha_j = \max_{\dim V = j} \; \min_{x \in V, \; \|x\| = 1} \operatorname{tr}(A x x^*) \tag{12}$$

Here, dim V stands for the dimension of a subspace V of $\mathbb{C}^n$. Note that xx^* is just the orthogonal projection operator on the 1-dimensional subspace spanned by x. Note also that tr Axx^* is just the number $x^*Ax = \langle x, Ax\rangle$.

The complex Grassmann manifold $G_k(\mathbb{C}^n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$. For $k = 1$, this is just the complex projective space $\mathbb{C}I\!P^{n-1}$, the set of all complex lines through the origin in the space $\mathbb{C}^n$. Each k-dimensional subspace L of $\mathbb{C}^n$ is completely characterised by the orthogonal projection P_L with range L.

Given any Hermitian operator A on $\mathbb{C}^n$, let $A_L = P_L A P_L$. Note that tr $A_L =$ tr $P_L A P_L =$ tr AP_L. To prove the inequality (5), Wielandt invented a remarkable minimax principle. This says that for any $1 \leq i_1 < \cdots < i_k \leq n$

$$\sum_{j=1}^{k} \alpha_{i_j} = \max_{\substack{V_1 \subset \cdots \subset V_k \\ \dim V_j = i_j}} \; \min_{\substack{L \in G_k(\mathbb{C}^n) \\ \dim (L \cap V_j) \geq j}} \operatorname{tr} A_L. \tag{13}$$

Note for $k = 1$, this reduces to (12).

Another such principle was discovered by Hersch and Zwahlen. Let v_j be the eigenvectors of the Hermitian matrix A corresponding to its eigenvalues α_j. For $m = 1, \ldots, n$, let V_m be the linear span of $v_1, \ldots, v_m$. Then, for any $1 \leq i_1 < \cdots < i_k \leq n$,

$$\sum_{j=1}^{k} \alpha_{i_j} = \min_{L \in G_k(\mathbb{C}^n)} \left\{ \operatorname{tr} A_L : \ \dim (L \cap V_{i_j}) \geq j, j = 1, \ldots, k \right\}. \tag{14}$$

The Grassmannian $G_k(\mathbb{C}^n)$ is a smooth compact manifold of real dimension $2k(n-k)$. There is a famous embedding called Plücker embedding via which $G_k(\mathbb{C}^n)$ is realised as a projective variety in the space $\mathbb{C}I\!P^N$, where $N = \binom{n}{k} - 1$.

A sequence of nested subspaces $\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n$, where dim $V_j = j$, is called a *flag* . Given a flag $\mathcal{F}$ and a set of indices $1 \leq i_1 < \cdots < i_k \leq n$ the subset

$$\{W \in G_k(\mathbb{C}^n) : \ \dim (W \cap V_{i_j}) \geq j, \quad j = 1, \ldots, k\}$$

of the Grassmanian is called a *Schubert variety* .

The principle (14) thus says that the sum $\sum \alpha_{i_j}$ is characterised as the minimal value of tr A_L evaluated on the Schubert variety corresponding to the flag constructed from the eigenvectors of A.

This suggests that inequalities like the ones conjectured by Horn could be related to Schubert calculus, a component of algebraic geometry dealing with intersection properties of flags. This line was pursued vigorously by R. C. Thompson beginning in the early seventies. Finally, the problem has now been solved by the efforts of several others using Schubert calculus.

There are other ways to look at Horn's inequalities. The matrices X and Y are said to be *unitarily equivalent* if there exists a unitary matrix U such that $X = UYU^*$. Two Hermitian matrices are unitarily equivalent if and only if they have the same eigenvalues. It is easy to see that Horn's conjecture (now proved) amounts to the following. Given Hermitian matrices A, B, consider the collection of all n-tuples that arise as eigenvalues of $A + UBU^*$ as U varies over all unitary matrices (with the convention that the eigenvalues of a Hermitian matrix are counted in decreasing order). Horn's inequalities assert that this is a convex polytope in $\mathbb{R}^n$ whose faces are characterised by the conditions (1), (10) and (11).

Suggested Reading

[1] R Bhatia, Matrix Analysis, Springer-Verlag, 1997

[2] A A Klyachko, Stable bundles, representation theory and Hermitian operators, *Selecta Mathematica*, New Series, 4 (1998) 419-445.

[3] A Knutson and T Tao, The Honeycomb model of the Berenstein-Zelevinsky cone I : Proof of the saturation conjecture, preprint dated January 3, 1999.

[4] W Fulton, Eigenvalues of sums of Hermitian matrices (after A.Klyachko), *Séminaire Bourbaki*, 1998.

R Bhatia, Statistics and Mathematics Unit,Indian Statistical Institute, 7, SJS Sansanwal Marg, New Delhi 110 016, India.

Addendum

In the general article on Galois theory (*Resonance*,Vol. 4, No. 10, pp.47-60, 1999), one of the statements occurring at the end was stated incorrectly due to an oversight and should be qualified to read as follows: *The solvability of a separable polynomial f by radicals amounts to getting a tower of fields $K = K_0 \subseteq \cdots \subseteq L$ where L contains the splitting field of f and each $K_{i+1} = K_i(t_i)$ with t_i satisfying either (a) $t_i^{m_i} \in K_i$ for some m_i not divisible by the characteristic of K or (b) $t_i^p - t_i \in K_i$ when $0 < p = \mathrm{char}(K) \leq \deg(f)$.*The last condition was missed out in the article and the corresponding extensions are known as Artin–Schreier extensions. It could perhaps be pointed out that mathematicians like Dedekind, Artin and Schreier were responsible for the modern formulation of Galois theory which was discussed in the article.

B Sury