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Reflection of water waves by a nearly vertical wall

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The problem of reflection of water waves by a nearly vertical wall is studied. A simplified perturbational analysis followed by Havelock's expansion of water wave potential is employed to tackle the problem. Assuming some particular shapes of the nearly vertical wall, first-order correction to the reflection coefficient is calculated for deep water as well as for uniform finite depth of water.

1. Introduction

Kachoyan and Mckee [1] considered the problem of reflection of a two-dimensional surface wave train by a sloping sea wall. Assuming the slope to be large they set up a perturbational scheme and calculated the velocity potential and the force on the wall approximately.

In the present paper, the problem of reflection of a two-dimensional surface water wave train by a nearly vertical rigid wall is considered, by using a simplified perturbational analysis followed by an appropriate Havelock's expansion of water wave potential. The first-order correction to the reflection coefficient is obtained in terms of an integral involving the shape function for deep water as well as finite depth of water. Finally this is calculated explicitly by assuming some typical shapes of the wall.

2. Formulation of the problem

We assume that the water is bounded on the left side by a nearly vertical wall $x=\varepsilon c(y)$, $(\varepsilon \ll 1)$ where $c(y)$ is a bounded and continuous function and $c(0)=0$, y being taken vertically downwards and $y=0 (x>0)$ is the position of the mean free surface. Assuming linear theory and the motion to be irrotational, the velocity potential φ satisfies

$$\nabla^2\varphi=0 \qquad \text{in the fluid region} \tag{1}$$

with boundary conditions

$$K\varphi+\frac{\partial\varphi}{\partial y}=0 \qquad \text{on } y=0 (K>0) \tag{2}$$

$$\frac{\partial\varphi}{\partial n}=0 \qquad \text{on } x=\varepsilon c(y) \tag{3}$$

(n denotes normal to the wall)

$$\nabla\varphi\to 0 \qquad \text{as } y\to\infty \tag{4}$$

Here the condition (2) is the linearized free surface condition and $K=\sigma^2/g$ where σ is the frequency of the incident wave with time dependence $\exp(-i\sigma t)$ (suppressed throughout), g is the acceleration due to gravity. In addition to the above boundary conditions, φ is also required to satisfy the requirement that

$$\varphi \sim \exp(-Ky-iKx)+R\exp(-Ky+iKx) \qquad \text{as } x\to+\infty \tag{5}$$

where $\exp(-Ky-iKx)$ denotes a normally incident wave train from positive infinity and R is the reflection coefficient.

Assuming the parameter ε to be very small, and neglecting $O(\varepsilon^2)$ terms, the boundary condition (3) can be expressed in approximate form on $x=0$ as Mandal and Chakrabarti [2]

$$\frac{\partial\varphi}{\partial x}(0,y)-\varepsilon\frac{\mathrm{d}}{\mathrm{d}y}\left\{c(y)\frac{\partial\varphi}{\partial y}(0,y)\right\}=0 \qquad \text{for } y>0 \tag{6}$$

3. Method of solution

The form of the approximate boundary condition (6) suggests that we may take the following straightforward perturbational expansion in terms of the small parameter ε for φ and R respectively.

$$\left.\begin{aligned}\varphi(x,y;\varepsilon)&=\varphi_0(x,y)+\varepsilon\varphi_1(x,y)+O(\varepsilon^2)\\ R(\varepsilon)&=R_0+\varepsilon R_1+O(\varepsilon^2)\end{aligned}\right\} \tag{7}$$

We shall content ourselves with the determination of φ_0, R_0 and φ_1, R_1. Substituting the expansion (7) into equations (1), (2), (4), (5) and (6) we find after equating the coefficients of identical powers of ε^0 and ε^1 on both sides, the functions φ_0 and φ_1 must be the solution of the following two independent boundary value problems (BVP).

BVP I. The function φ_0 satisfies

$$\nabla^2\varphi_0=0 \qquad y\geqslant 0, x>0 \tag{8a}$$

$$K\varphi_0+\varphi_{0y}=0 \qquad \text{on } y=0 \tag{8b}$$

$$\frac{\partial\varphi_0}{\partial y}=0 \qquad \text{on } x=0 \tag{8c}$$

$$\varphi_0\sim\exp(-Ky-iKx)+R_0\exp(-Ky+iKx) \qquad \text{as } x\to\infty \tag{8d}$$

$$\nabla\varphi_0\to 0 \qquad \text{as } y\to\infty \tag{8e}$$

Obviously

$$\varphi_0=\exp(-Ky-iKx)+\exp(-Ky+iKx)$$

so that $R_0=1$.

BVP II. The function φ_1 satisfies

$$\nabla^2\varphi_1=0 \qquad y\geqslant 0, x>0 \tag{9a}$$

$$K\varphi_1+\varphi_{1y}=0 \qquad \text{on } y=0 \tag{9b}$$

$$\frac{\partial \varphi_1}{\partial x}=\frac{\mathrm{d}}{\mathrm{d}y}\left(c(y)\frac{\partial \varphi_0}{\partial y}\right)\equiv f(y) \qquad \text{on } x=0, y>0 \tag{9c}$$

$$\varphi_1 \sim R_1 \exp(-Ky+\mathrm{i}Kx) \qquad \text{as } x\to\infty \tag{9d}$$

$$\nabla\varphi_1 \to 0 \qquad \text{as} \qquad y\to\infty \tag{9e}$$

We note that the right side $f(y)$ of (9 c) is now known. We employ the Havelock's expansion (Ursell [3]) of water wave potential to solve for $\varphi_1(x, y)$. Thus φ_1 has the expansion given by

$$\varphi_1(x,y)=R_1\exp(-Ky+\mathrm{i}Kx) + \int_0^\infty A(k)\exp(-kx)(k\cos ky-K\sin ky)\,\mathrm{d}k \qquad x>0 \tag{10}$$

Using the condition (9 c) we find

$$f(y)=\mathrm{i}KR_1\exp(-Ky)+\int_0^\infty(-k)A(k)(k\cos ky-K\sin ky)\,\mathrm{d}k \qquad y>0 \tag{11}$$

so that by Havelock's inversion theorem [3]

$$\frac{\mathrm{i}}{2}R_1=\int_0^\infty f(y)\exp(-Ky)\,\mathrm{d}y \tag{12}$$

and

$$A(k)=\frac{-2}{\pi k(K^2+k^2)}\int_0^\infty f(y)(k\cos ky-K\sin ky)\,\mathrm{d}y \tag{13}$$

Thus R_1 and $A(k)$ are obtained when $c(y)$ is given so that φ is obtained up to ε.

As an illustration we consider the following two particular shapes of the nearly vertical cliff:

(i) $c(y)=y\exp(-\lambda y)(y>0)$ (λ is a constant)

then

$$f(y)=2K\exp(-(\lambda+K)y)((\lambda+K)y-1)$$

so that

$$R_1=\mathrm{i}\frac{4K^2}{(\lambda+2K)^2}$$

and

$$A(k)=\frac{4K[K\{(\lambda+K)^2+k^2\}+2\lambda k^2]}{\pi(k^2+K^2)(k^2+(K+\lambda)^2)^2}$$

(ii) $c(y)=a\sin\lambda y$

then

$$f(y)=2aK\exp(-Ky)(K\sin\lambda y-\lambda\cos\lambda y)$$

so that

$$R_1=\mathrm{i}\frac{4a\lambda K^2}{(\lambda^2+4K^2)}$$

and

$$A(k)=\frac{4a\lambda K^2(K^2+k^2+\lambda^2)}{\pi(K^2+k^2)(K^2+(k+\lambda)^2)(K^2+(k-\lambda)^2)}$$

4. Finite depth of water

For finite depth of water we can extend the problem easily. In this case water is bounded below by a flat bottom $y=h$ and on the left by a nearly vertical wall $x=\varepsilon c(y)$, $\varepsilon \ll 1$, $0<y<h$ ($c(0)=0$). Then the velocity potential φ satisfies the following BVP:

$$\nabla^2\varphi=0 \qquad x>0, 0<y<h \tag{14a}$$

$$K\varphi+\frac{\partial\varphi}{\partial y}=0 \qquad \text{on } y=0 \tag{14b}$$

$$\frac{\partial\varphi}{\partial n}=0 \qquad \text{on } x=\varepsilon c(y) \qquad 0<y<h \tag{14c}$$

$$\frac{\partial\varphi}{\partial y}=0 \qquad \text{on } y=h \tag{14d}$$

$$\varphi\sim\varphi_{\text{inc}}(x,y)+R\varphi_{\text{inc}}(-x,y) \qquad \text{as } x\to\infty \tag{14e}$$

where

$$\varphi_{\text{inc}}(x,y)=\frac{\cosh k_0(h-y)}{\cosh k_0 h}\exp(-ik_0x)$$

and k_0 is the positive root of $K=k\tanh(kh)$.

Following a similar technique we find that φ_0 and φ_1 are the solutions of the following BVPs.

BVP I. The function φ_0 satisfies

$$\nabla^2\varphi_0=0 \qquad y\geqslant 0, x>0 \tag{15a}$$

$$K\varphi_0+\varphi_{0y}=0 \qquad \text{on } y=0 \tag{15b}$$

$$\varphi_0\sim\varphi_{\text{inc}}(x,y)+R_0\varphi_{\text{inc}}(-x,y) \qquad \text{as } x\to\infty \tag{15c}$$

$$\frac{\partial\varphi_0}{\partial x}=0 \qquad \text{on } x=0 \tag{15d}$$

$$\frac{\partial\varphi_0}{\partial y}=0 \qquad \text{on } y=h \tag{15e}$$

Obviously

$$\varphi_0=\frac{2\cosh k_0(h-y)}{\cosh k_0 h}\cos k_0x$$

so that $R_0=1$.

BVP II. The function φ_1 satisfies

$$\nabla^2\varphi_1=0 \qquad x>0, 0<y<h \tag{16a}$$

$$K\varphi_1+\varphi_{1y}=0 \qquad \text{on } y=0 \tag{16b}$$

$$\varphi_1\sim R_1\varphi_{\text{inc}}(-x,y) \qquad \text{as } x\to\infty \tag{16c}$$

$$\frac{\partial\varphi_1}{\partial x}=\frac{\mathrm{d}}{\mathrm{d}y}\left\{c(y)\frac{\partial\varphi_0}{\partial y}\right\} \qquad \text{on } x=0 \qquad 0<y<h \tag{16d}$$

$$\frac{\partial\varphi_1}{\partial y}=0 \qquad \text{on } y=h \tag{16e}$$

From (16 *d*) we obtain

$$\frac{\partial\varphi_1}{\partial x}=\frac{\mathrm{d}}{\mathrm{d}y}\left\{c(y)\frac{\partial\varphi_0}{\partial y}\right\}=F(y)\qquad 0<y<h\qquad \text{on } x=0 \tag{17}$$

By Havelock's expansion φ_1 has the representation [3]

$$\varphi_1(x,y)=R_1\frac{\cosh k_0(h-y)}{\cosh k_0h}\exp(\mathrm{i}k_0x)+\sum_n B_n\cos k_n(h-y)\exp(-k_nx) \tag{18}$$

where the summation extends over the real positive roots of $K+k\tan(kh)=0$.

Now using the condition (16 *d*) we find

$$F(y)=\mathrm{i}k_0R_1\frac{\cosh k_0(h-y)}{\cosh k_0h}-\sum_n k_nB_n\cos k_n(h-y) \tag{19}$$

so that by Havelock's inversion theorem [3] we find

$$R_1=\frac{-4\mathrm{i}\cosh k_0h\int_0^h F(\alpha)\cosh k_0(\alpha-h)\,\mathrm{d}\alpha}{2k_0h+\sinh 2k_0h}$$

and

$$B_n=\frac{-4\int_0^h F(\alpha)\cos k_n(\alpha-h)\,\mathrm{d}\alpha}{2k_nh+\sin 2k_nh}$$

Thus R_1 and B_n are found when $c(y)$ is given. Hence we find φ up to the first order.

As an illustration we consider the following two particular shapes of the nearly vertical cliff:

(i) $c(y)=y\exp(-\lambda y),\ 0\leqslant y\leqslant h$

then

$$F(y)=-\frac{2k_0\exp(-\lambda y)}{\cosh k_0h}[\sinh k_0(h-y)-k_0y\cosh k_0(h-y)-\lambda y\sinh k_0(h-y)]$$

so that

$$R_1=\frac{-4\mathrm{i}k_0^2}{(2k_0h+\sinh 2k_0h)(\lambda^2-4k_0^2)^2}\left[\frac{1}{\lambda^2}(\lambda^2-4k_0^2)^2+8k_0^2\right.$$
$$\left.\times\exp(-\lambda h)-(\lambda^2+4k_0^2)\cosh 2k_0h+4\lambda k_0\sinh 2k_0h\right]$$

and

$$B_n=\frac{4k_0\exp(-\lambda h)}{\cosh k_0h(2k_nh+\sin 2k_nh)}\left[\frac{\lambda+k_0}{\{(\lambda+k_0)^2+k_n^2\}^2}\right.$$

$$\times\{(\lambda+k_0)h((\lambda+k_0)^2+k_n^2)-2k_n^2\}$$

$$-\frac{\lambda-k_0}{\{(\lambda-k_0)^2+k_n^2\}^2}\{(\lambda-k_0)h((\lambda-k_0)^2+k_n^2)-2k_n^2\}$$

$$+\frac{\exp\{(\lambda+k_0)h\}k_n}{\{(\lambda+k_0)^2+k_n^2\}^2}\{2(\lambda+k_0)k_n\cos k_nh-((\lambda+k_0)^2-k_n^2)\sin k_nh\}$$

$$\left.-\frac{\exp\{(\lambda-k_0)h\}k_n}{\{(\lambda-k_0)^2+k_n^2\}^2}\{2(\lambda-k_0)k_n\cos k_nh-((\lambda-k_0)^2-k_n^2)\sin k_nh\}\right]$$

(ii) $c(y)=a\sin\lambda y, 0\leqslant y\leqslant h$

then

$$F(y)=\frac{-ak_0}{\cosh k_0h}[(\lambda-k_0)\sinh((\lambda-k_0)y+k_0h)-(\lambda+k_0)\sinh((\lambda+k_0)y-k_0h))]$$

so that

$$R_1=\frac{-4iak_0}{(2k_0h+\sinh 2k_0h)}\left[1+\frac{\lambda k_0\cosh 2k_0h}{\lambda^2-4k_0^2}-\frac{\lambda^2+\lambda k_0-4k_0^2}{\lambda^2-4k_0^2}\cosh\lambda h\right]$$

and

$$B_n=\frac{8\lambda ak_0k_n}{\cosh k_0h}(2k_nh+\sin 2k_nh)^{-1}((\lambda-k_0)^2+k_n^2)^{-1}((\lambda+k_0)^2+k_n^2)^{-1}$$

$$\times[2k_0(\cosh\lambda h-\cosh k_0h\cos k_nh)+(\lambda^2-4k_0^2+k_n^2)\sinh k_0h\sin k_nh]$$

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References

[1] Kachoyan, P. J., and McKee, W. D., 1985, *Engg. Math.*, **19**, 351.
[2] Mandal, B. N., and Chakrabarti, A., *IMA. J. Appl. Math.*, **43**, 157.
[3] Ursell, F., 1947, *Proc. Camb. Phil. Soc.*, **43**, 374.