The Electronic Journal of Linear Algebra.

Volume 7, pp. 174-177, September 2000.

AN ELEMENTARY PROOF OF THE HILBERT–MUMFORD CRITERION*

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Abstract. An elementary proof of the Hilbert–Mumford semistability criterion is given that is valid over $\mathbb{C}$. The proof of the criterion is deduced from an elementary lemma in linear algebra that may be of independent interest.

Key words. Semistability criterion, Algebraic one-parameter groups.

AMS subject classifications. 20G20, 14A25

1. Introduction and lemma. A classical result of geometric invariant theory is the Hilbert–Mumford semistability criterion. In one form, it deals with a linear action of a reductive algebraic group G on a vector space over any field k. The references [1], [2], [3], [4] contain proofs over algebraically closed fields, and [5] contains a proof that works over algebraic number fields as well. Here, a transparent elementary proof is given that is valid over $\mathbb{C}$. An elementary positivity lemma in linear algebra is proved and used to deduce the proof of the criterion over $\mathbb{C}$. The lemma may be of independent interest.

We start with the following positivity lemma.

LEMMA 1.1. *Let $m_{ij}, 1 \leq i \leq r, 1 \leq j \leq n$, be integers satisfying the following property:*

If $b_1, \ldots, b_r$ are real numbers (not all zero) such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \ \forall \ j = 1, \ldots, n,$$

then at least two of the b_i must have opposite signs.

Then there are real numbers (and, therefore, also integers) c_i such that

$$m_{i1} c_1 + \cdots + m_{in} c_n > 0 \ \forall \ i \leq r.$$

Proof. The property above means that the kernel of the linear map M from $\mathbb{R}^r$ to $\mathbb{R}^n$ given by

$$(b_1, \ldots, b_r) \mapsto \left(\sum_{i=1}^{r} b_i m_{i1}, \ldots, \sum_{i=1}^{r} b_i m_{in} \right)$$

intersects the "positive orthant" $\mathcal{O}$ in $\mathbb{R}^r$ only in zero. The assertion of the lemma amounts to the statement that the image of the transpose ${}^tM$ of M intersects the

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interior of $\mathcal{O}$. Since $Ker(M)$ and the image of ${}^tM$ are orthogonal complements of each other, it suffices to show that $Ker(M)^\perp$ intersects the interior of $\mathcal{O}$. We show, more generally, that if K is a subspace of $\mathbb{R}^r$ intersecting $\mathcal{O}$ only in zero, then $K^\perp$ intersects the interior of $\mathcal{O}$. We will first show that K can be assumed to be of codimension 1. Suppose that K has codimension $k \geq 2$. Now, D denotes the image of $\mathcal{O}$ in $\mathbb{R}^r/K \equiv \mathbb{R}^k$. Since $\mathbb{R}^k \setminus \{0\}$ is connected, there is a vector $v \neq 0$ in $\mathbb{R}^k \setminus (D \cup -D)$. Hence the line $\mathbb{R}.v$ intersects D only in 0. Pulling back to $\mathbb{R}^r$, we get a subspace L containing K in $\mathbb{R}^r$ of one more dimension such that $L \cap \mathcal{O} = \{0\}$. In the above argument, one could replace $\mathcal{O}$ more generally by a closed cone C in $\mathbb{R}^r$ such that $C \cap -C = \{0\}$. We have used the fact that D is again closed. Proceeding in this way, we can assume that K has codimension 1. Now, let the equation of K be $\sum_{i=1}^r \lambda_i X_i = 0$. Then $K \cap \mathcal{O} = \{0\}$, which evidently forces either all of the $\lambda_i > 0$ or all of the $\lambda_i < 0$. Suppose $\lambda_i > 0 \; \forall \, i$. Then $K^\perp$ is generated by the vector $(\lambda_1, \ldots, \lambda_r)$ and, obviously, $(\lambda_1, \ldots, \lambda_r)$ is in the interior of $\mathcal{O}$. This completes the proof. □

REMARK 1.2. Note that in the above, $\mathcal{O}$ can be replaced by any closed cone subtending an angle $\geq 90°$. The statement is false for cones of smaller angle.

2. The proof of the semistability criterion. Let us see how the lemma applies to the following statement, known as the semistability criterion.

THEOREM 2.1. *Let $G = GL(n, \mathbb{C})$ act linearly on a vector space V. Let $v \in V$ be a point that is not semistable, i.e., the closure $\overline{G.v}$ of the orbit $G.v$ (in the classical topology) contains* 0. *Then there exists an algebraic one-parameter subgroup $A \cong GL_1$ of G such that $0 \in \overline{A.v}$.*

Proof. We have the (Cartan) decomposition $G = KTK$, where K is $U(n)$ and T is the maximal diagonal torus—this can be easily deduced from the spectral theorem for Hermitian operators. From this decomposition, it immediately follows that $0 \in \overline{T.kv}$ for some $k \in K$. It is enough to get a multiplicative one-parameter subgroup A, as in the theorem, for the vector kv, since the group $k^{-1}Ak$ works for v then. So, we rename kv as v and work with it without any loss of generality. Write $v = \sum_{i=1}^r v_{\chi_i}$, where

$$v_{\chi_i} \in V_{\chi_i} := \{w \in V : t.w = \chi_i(t)w \; \forall \; t \in T\}$$

for some algebraic characters $\chi_i : T \to \mathbb{C}^*$. Let $\chi_i = \sum_{j=1}^n m_{ij}\lambda_j$, where $\lambda_j : T \to \mathbb{C}^*$ is the character $\operatorname{diag}(t_1, \ldots, t_n) \mapsto t_j$; here m_{ij} are integers. So, we have

$$t.v = \sum_{i=1}^r t_1^{m_{i1}} \cdots t_n^{m_{in}} v_{\chi_i}$$

for any $t = \operatorname{diag}(t_1, \ldots, t_n) \in T$.

Claim. If $b_1, \ldots, b_n$ are real numbers (not all zero) such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \; \forall \; j = 1, \ldots, n,$$

then at least two of the b_i's are of opposite signs.

To prove the claim, we suppose, on the contrary, that there are b_i (not all zero) all of the same sign such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \ \forall\ j = 1, \ldots, n.$$

Let $t^{(k)} = \mathrm{diag}(t_1^{(k)}, \ldots, t_n^{(k)}) \in T$ be a sequence such that $t^{(k)}.v \to 0$ as $k \to \infty$. Therefore, $\forall\ i \leq r$,

$$(t_1^{(k)})^{m_{i1}} \cdots (t_n^{(k)})^{m_{in}} \to 0 \quad \text{as} \quad k \to \infty\ . \tag{1}$$

Suppose, now, that $b_1 \neq 0$. Then

$$-m_{1j} = \frac{b_2}{b_1} m_{2j} + \cdots + \frac{b_r}{b_1} m_{rj} \ \forall\ j \leq n$$

so that

$$t_1^{-m_{11}} \cdots t_n^{-m_{1n}} = (t_1^{m_{21}} \cdots t_n^{m_{2n}})^{\frac{b_2}{b_1}} \cdots (t_1^{m_{r1}} \cdots t_n^{m_{rn}})^{\frac{b_r}{b_1}}\ . \tag{2}$$

Since $\frac{b_i}{b_1} \geq 0 \ \forall\ i \geq 2$, and not all of them are zero, the right-hand side of (2) tends to 0 as $(t_1, \ldots, t_n)$ runs over the sequence $(t_1^{(k)}, \ldots, t_n^{(k)})$. Looking at the left-hand side of (2), we have a contradiction of (1). This proves the claim.

Let us continue with the proof of the theorem. First, an application of the lemma ensures the existence of integers c_i such that

$$m_{i1} c_1 + \cdots + m_{in} c_n > 0 \ \forall\ i \leq r. \tag{3}$$

Consider the algebraic one-parameter subgroup GL_1 in T given by the homomorphism

$$\theta : GL_1 \to T,\ t \mapsto diag(t^{c_1}, \ldots, t^{c_n}).$$

Note that $\theta(t).v = \sum_{i=1}^{r} t^{m_{i1}c_1 + \cdots + m_{in}c_n} v_{\chi_i}$. By (3), it is clear that $0 \in \overline{\theta(GL_1).v}$. □

The following is a corollary of the proof.

COROLLARY 2.2. *Let* $(,)$ *denote the nondegenerate pairing*

$$X_*(T) \times X^*(T) \to \mathbb{Z},$$

where $X_*(T) = Hom(GL_1, T)$ *is the group of multiplicative one-parameter subgroups and* $X^*(T) = Hom(T, GL_1)$ *is the character group of* T. *Then* $\theta \in X_*(T)$ *satisfies* $0 \in \overline{\theta(GL_1).v}$ *if and only if* $(\theta, \chi_i) > 0 \ \forall\ i \leq r$.

REMARK 2.3. For the other classical groups over $\mathbb{C}$, the proof is completely similar.

Acknowledgments. The author is thankful to the referee for his comments, which were both illuminating and interesting. In particular, the referee points out that the positivity lemma proved in this paper is known to the linear programming cognoscenti as Gordan's theorem; see [6].

REFERENCES

[1] D. Birkes, Orbits of linear algebraic groups, *Ann. of Math.*, 93:459–475, 1971.
[2] D. Hilbert, Uber die vollen Invariantensysteme, *Math. Ann.*, 42:313–373, 1893.
[3] D. Mumford, *Geometric Invariant Theory*, Springer, Berlin, 1965.
[4] P.E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute of Fundamental Research Lecture Notes, Springer, Berlin, 1978.
[5] M.S. Raghunathan, A note on orbits of reductive groups, *J. Indian Math. Soc.*, 38:65–70, 1974.
[6] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, New York, 1986.