Sankhyā : The Indian Journal of Statistics
1986, Volume 48, Series B, Pt. 3, pp. 392-400

A NOTE ON ASYMPTOTIC SOLUTION TO BAYESIAN THREE-DECISION PLAN BY ATTRIBUTES

By R. J. PANDEY
Indian Statistical Institute

SUMMARY. A simple approximate formula to obtain three-decision Bayesian plans are derived by using asymptotic results. A numerical comparison of exact and approximate plans, in the sense of Hald (1981), is carried out for a chosen set of cost parameters.

1. Introduction

Asymptotic solutions to Bayesian three-decision plan by attributes, discussed by Pandey (1974), are derived. Approximate formula which may be used in practice are obtained. The numerical solutions obtained by using approximate formula are compared with the exact solutions tabulated in Pandey (1974). Asymptotic result shows that the sample size increases with lot size. The approximate formula is found quite handy in practice.

2. An asymptotic expansion for binomial distribution

Using Hald's (1981, p. 220) technique, $B(c_j\,;\,n,\,p^{(k)})$, $j=1,\,2$ where $c_j/n = p_0+\varepsilon_j$, $\varepsilon_j\to 0$ for $n\to\infty$ and $p^{(k)}>0$ can be expanded. The result without proof is stated in the Theorem 1.

Theorem 1 : *Let* $c_j/n = p_0+\sum_{i=1}^{4} a_{ij}n^{-i/2}+O(n^{-5/2})$. *For* $p^{(k)}>p_0$ *we have*

$$B(c_j\,;\,n,\,p^{(k)}) = q_0p^{(k)}\,exp\,[-n\phi(p_0,\,p^{(k)})+\sum_{i=1}^{4} b_{ijk}\,(p_0,\,p^{(k)})\,n^{-i/2}+O(n^{-5/2})]/[\,|p^{(k)}-p_0|\,\sqrt{2\pi np_0q_0}] \quad \ldots \quad (1)$$

where

$$p_0 = [ln(p^{(1)}/q^{(2)})]/[ln(p^{(2)}\,q^{(1)}/p^{(1)}\,q^{(2)})] \quad \ldots \quad (2)$$

$$\phi(p_0,\,p^{(k)}) = p_0\,ln(p_0/p^{(k)})+q_0\,ln(q_0/q^{(k)})$$

$$b_{1jk} = a_{1j}\,ln(p_0\,q^{(k)}/q_0\,p^{(k)})$$

$$b_{2jk} = a_{2j}\,ln(p_0q^{(k)}/q_0\,p^{(k)})+a_{1j}^2/2p_0\,q_0$$

$$b_{3jk} = a_{3j}\,ln(p_0\,q^{(k)}/q_0\,p^{(k)})-a_{1j}/(p^{(k)}-p_0)+a_{1j}(1+2a_{2j})/(2p_0\,q_0)-a_{1j}^3\,(q_0-p_0)/(6(p_0\,q_0)^2)$$

$$b_{4jk} = a_{4j}\,ln(p_0\,q^{(k)}/q_0p^{(k)})-a_{2j}/(p^{(k)}-p_0)-(a_{1j}^2-2p_0\,q_0)/2(p^{(k)}-p_0)^2+(a_{2j}+a_{2j}^2+2a_{1j}\,a_{3j})/(2p_0\,q_0)+(1-p_0\,q_0)/(12p_0\,q_0)-(q_0-p_0)a_{1j}^2\,(1+a_{2j})/4(p_0\,q_0)^2+(p_0^3+q_0^3)a_{1j}^4/\,12(p_0\,q_0).$$

AMS (1980) *subject classification :* 62N10

Key words and phrases : ASR plan, Asymptotic solution, Bayesian three-decision plan, Efficiency of approximate plans.

The same expression is valid for $1-B(c_j\,;\,n,\,p^{(k)})$ when $p^{(k)} < p_0$. In the subsequent section we shall use this expression in obtaining an asymptotic solution.

3. Unrestricted Bayesian single sampling three-decision plan by attributes

Exact solutions to a wide variety of three-decision plans were discussed by Pandey (1974). We have chosen only ASR (Accept-Screen-Reject) Bayesian three-decision plan with two point prior distribution for obtaining asymptotic solution. The asymptotic results for other three-decision plans have been attempted on the similar lines and are omitted from the present discussions.

In Theorem 2 we give an asymptotic solution to the chosen problem.

Theorem 2 : *For the Bayesian single sampling three-decision plan* $(n,\, c_1,\, c_2)$ *we have*

$$c_j = np_0+a_{2j}+a_{4j}/n+O(n^{-2}), \quad j = 1, 2 \qquad \ldots \quad (3)$$

where

$$a_{2j} = ln\,\{(\lambda_{j1}\,\phi_1')/(\lambda_{j2}(-\phi_2'))\}/\delta' \qquad \ldots \quad (4)$$

$$a_{4j} = -[\{(a_{2j}/(p_0-p^{(1)})\}+\{a_{2j}/(p^{(2)}-p_0)\}$$
$$+\{p_0q^{(1)}/(p_0-p^{(1)})\}-\{p_0q^{(2)}/(p^{(2)}-p_0)^2\}]/\delta' \qquad \ldots \quad (5)$$

$$\delta = \phi(p_0,\, p^{(1)})-\phi(p_0,\, p^{(2)}),\ \delta' = d\delta/dp_0. \qquad \ldots \quad (6)$$

The minimum value of regret function R is given by

$$R_0 = n+[1-1/(2n\alpha_0)]/\alpha_0+O(n^{-3/2}) \qquad \ldots \quad (7)$$

and the lot size N and the sample size n are related as

$$N = n+h(n)/(w\,\alpha_0) \qquad \ldots \quad (8)$$

where $h(n)$ is function of n and, α_0 and w are constants depending on $p^{(k)}$, $k = 1,\ 2$.

Proof : From Theorem 1 and expansion of $b(c_j\,;\,n,\,p^{(k)})$ we get for $p_0 < p^{(k)}$

$$B(c_j\,;\,n,\,p^{(k)}) = q_0p^{(k)}\,exp\,[-n\phi(p_0+\epsilon_j,\,p^{(k)})+O(\epsilon_j)$$
$$+O(n^{-1})]/[\,|\,p^{(k)}-p_0|\,\sqrt{2\pi np_0q_0}] \qquad \ldots \quad (9)$$

and the same expression for $1-B(c_j\,;\,n,\,p^{(k)})$ for $p_0 > p^{(k)}$ as stated in the Theorem 1.

For $p^{(1)} < p_0 < p^{(2)}$ we have the following asymptotic expansion for the regret function considered by Pandey (1974)

$$R = n+(N-n)n^{-1/2} \sum_{j=1}^{2} \sum_{k=1}^{2} \lambda_{jk} \exp[-n\phi(p_0, p^{(k)})-n\varepsilon_j \phi_k'+O(\varepsilon_k)+O(n^{-1})] \quad \ldots \quad (10)$$

where

$$\lambda_{jk} = q_0 \, p^{(k)} \mu_{jk}/[\,|p_0-p^{(k)}|\sqrt{2\pi \, p_0 \, q_0}] \quad \ldots \quad (11)$$

$$\phi_k' = \phi_k'(p_0, q^{(k)}) = \ln (p_0 \, q^{(k)}/q_0 \, p^{(k)})$$

and μ_{jk}, $j, k = 1, 2$ are the constants depending on the cost parameters and the chosen prior distribution and are defined in Pandey (1974).

It can be seen that the optimal value of p_0 should satisfy the equation $\phi(p_0, p^{(1)}) = \phi(p_0, p^{(2)})$ which gives (2), for any other choice of p_0, will make one of the terms in (10) greater than $\exp(-n\phi_0)$.

Now, $dR/d\varepsilon_j = 0$ gives

$$\sum_{k=1}^{2} \phi_k' \lambda_{jk} \exp[-n\phi(p_0, p^{(k)})-n\varepsilon_j \phi_k'+O(\varepsilon_j)+O(n^{-1})] = 0$$

which gives

$$n \, \varepsilon_j \, \phi_1'-\ln(\phi_1' \, \lambda_{j1}) = n \, \varepsilon_j \, \phi_2'-\ln\{(-\phi_2') \, \lambda_{j2}\}, \quad j = 1, 2. \quad \ldots \quad (12)$$

Solving for $a_{2j} = n \, \varepsilon_j$ we get

$$a_{2j} = \ln\{\lambda_{j1} \, \phi_1'/\lambda_{j2} \, (-\phi_2')\}/\delta', \quad j = 1, 2 \quad \ldots \quad (13)$$

and hence we get $c_j = np_0+a_{2j}+O(1)$.

Write

$$c_j/n = p_0+(a_{2j}+\varepsilon_j)/n, \; \varepsilon_j \to 0.$$

From Theorem 1 write (10) as

$$R = n+(N-n) \, n^{-1/2} \sum_{j=1}^{2} \sum_{k=1}^{2} \lambda_{jk} \exp[-n\phi(p_0, p^{(k)})-a_{2j} \, \phi_k'$$
$$-(n \, \varepsilon_j \, \phi_k'+\beta_{jk})/n+O(\varepsilon_j/n)+O(n^{-2})] \quad \ldots \quad (14)$$

where

$$\beta_{jk} = (a_{2j}^2+a_{2j})/(2p_0 \, q_0)+(1-p_0 \, q_0)/(12p_0 \, q_0)-a_{2j}/(p^{(k)}-p_0)$$
$$+(p_0 q^{(k)})/(p^{(k)}-p_0)^2. \quad \ldots \quad (15)$$

From $dR/d\varepsilon_j = 0$ for (13) we get

$$n \, \varepsilon_j \, \phi_1'+\beta_{j1} = n \, \varepsilon_j \, \phi_2'+\beta_{j2}, \quad j = 1, 2 \quad \ldots \quad (16)$$

and solving $a_{4j} = n \, \varepsilon_j$ gives

$$a_{4j} = -[\{a_{2j}/(p_0-p^{(1)})\}+\{a_{2j}/(p^{(2)}-p_0)\}+\{p_0 q^{(1)}/(p_0-p^{(1)})^2\}$$
$$-\{p_0 q^{(2)}/(p^{(2)}-p_0)^2\}]/\delta', \quad j = 1, 2$$

i.e., $\qquad c_j = np_0+a_{2j}+a_{4j}/n+O(n^{-2}).$

Writing (14) as

$$R = n+(N-n)n^{-1/2} \sum_{j=1}^{2} \sum_{k=1}^{2} \lambda_{jk} \exp [-n\, \phi(p_0, p^{(k)})$$

$$-a_{2j}\, \phi_k' - (a_{4j}\, \phi_k' + \beta_{jk})n^{-1} + O(\epsilon_j/n) + O(n^{-2})]$$

and using (11) and (15) and simplifying we write

$$R = n+(N-n)n^{-1/2}\, w \sum_{j=1}^{2} \exp \left[-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right] \quad \dots \quad (17)$$

where

$$w = \delta'/(-\phi_2') \quad \dots \quad (18)$$

$$\alpha_{0j} = \alpha_0 = \phi(p_0, p^{(k)}),\ k = 1, 2 \quad \dots \quad (19)$$

$$\alpha_{1j} = \alpha_{3j} = 0$$

$$\alpha_{2j} = -a_{2j}\, \phi_1' + \ln \lambda_{j1} = -a_{2j}\, \phi_2' + \ln \lambda_{j2} - \ln \left\{ \frac{\phi_1'}{(-\phi_2')} \right\},\ \ j = 1, 2 \quad \dots \quad (20)$$

$$\alpha_{4j} = a_{4j}\, \phi_1' + \beta_{j1} = a_{4j}\, \phi_2' + \beta_{j2},\ \ j = 1, 2. \quad \dots \quad (21)$$

Consider

$$R'(n) = 1 - n^{-1/2}\, w \sum_{j=1}^{2} \exp \left[-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right]$$

$$+(N-n) \left[(-n^{-3/2}/2)\, w \sum_{j=1}^{2} \exp \left(-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right)\right.$$

$$\left. + n^{-1/2}\, w \sum_{j=1}^{2} \exp \left(-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right) \left\{ - \sum_{i=0}^{4} \alpha_{ij}(-i/2+1) n^{-i/2} \right\} \right] = 0$$

$$= 1 - n^{-1/2}\, w \sum_{j=1}^{2} \exp \left[-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right]$$

$$-(N-n)n^{-1/2}\, w \sum_{j=1}^{2} \exp \left(-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right) \left\{ \alpha_0 - \alpha_{4j}\, n^{-2} + \frac{1}{2n} \right\} = 0.$$

Therefore

$$(N-n)n^{-1/2}\, w \sum_{j=1}^{2} \exp \left(-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right)$$

$$= \left[\alpha_0 + \frac{1}{2n}\right]^{-1} \left[1 - n^{-1/2}\, w \sum_{j=1}^{2} \exp \left(-n \sum_{i=0}^{4} \alpha_{ij}\, n^{-i/2}\right)\right].$$

Since $1-n^{-1/2}\, w \sum_{j=1}^{2} \exp\left(-n \sum_{i=0}^{4} \alpha_{ij} n^{-i/2}\right) = 1+o(e^{-n})$ we have

$$(N-n)n^{-1/2}\, w \sum_{j=1}^{2} \exp\left(-n \sum_{i=0}^{4} \alpha_{ij} n^{-i/2}\right)$$

$$=\left[\alpha_0+\frac{1}{2n}+o\,(n^{-3/2})\right]^{-1}$$

$$=\frac{1}{\alpha_0}\left(1-\frac{1}{2n\alpha_0}\right)+o(n^{-3/2}) \qquad \ldots \quad (22)$$

and $\min R = n+\frac{1}{\alpha_0}\left(1-\frac{1}{2n\alpha_0}\right)+o(n^{-3/2})$ ... (23)

Taking logarithm on both the sides of (22) we get

$$\log_e(N-n) = -\log_e\, w\, n^{-1/2}+\log_e \frac{1}{\alpha_0}\left(1-\frac{1}{2n\alpha_0}\right)$$

$$-\log_e\left\{\sum_{j=1}^{2} \exp\left(-n \sum_{i=0}^{4} \alpha_{ij} n^{-1/2}\right)\right\}+O(n^{-2})$$

which gives the following equation to determine n

$$N = n+h(n)/w\alpha_0 \qquad \ldots \quad (24)$$

where

$$h(n) = \sqrt{n}\left(1-\frac{1}{2n\alpha_0}\right)\Big/\left\{\sum_{j=1}^{2} \exp\,(-n\alpha_0-\alpha_{2j}-\alpha_{4j}\, n^{-1})\right\}.$$

Numerical examples show that the values of N obtained from (24) are very much smaller as compared to the corresponding exact values given by $\bar{N}$. Investigations led to a correction factor $K(n,\, p^{(1)},\, p^{(2)})$ given by exp $(2.5281+1.25n\; p^{(1)}\, q^{(2)})$ as a multiplier of $h(n)/w\; \alpha_0$ in (24) to have a corrected value of N closer to $\bar{N}$. Thus the corrected value of N is given by

$$N = n+h(n)/\{w\; \alpha_0\; K(n,\, p^{(1)},\, p^{(2)})\}. \qquad \ldots \quad (25)$$

It is easy to show that $R'(n) = 0$ gives the minimum value of $R(n)$. The expression (23) for minimum R is similar to one by Hald (1967, page 14).

4. Numerical example

Consider the determination of three-decision ASR plan with two point prior distribution as discussed in Pandey (1974, 371-377) with the following values

$$p' = p^{(1)} = 0.01\;;\; p'' = p^{(2)} = 0.15\;;\; w_1 = 0.93,\; w_2 = 0.07.$$

The minimum unavoidable cost of decision (k_m) = 8.096 and average cost of inspection per unit (k_s) = 23.693 and the constant μ_{ij}, i, $j = 1, 2$ as defined in the above paper are

$$\mu_{11} = 0.217638 \; ; \qquad \mu_{12} = 0.382606$$

$$\mu_{21} = 0.545586 \; ; \qquad \mu_{22} = 0.012342.$$

Using the above values and formula in Section 3 we obtain the values of $p_0, \phi_1', \phi_2', \delta', \lambda_{ij}, a_{21}, a_{22}, a_{41}, a_{42}, \beta_{ij}, \alpha_0, \alpha_{21}, \alpha_{22}, \alpha_{41}, \alpha_{42}$ etc. as follows :

$p_0 = 0.053301$; $\phi_1' = 1.718093$; $\phi_2' = -1.142445$;
$\delta' = 2.860538$; $\lambda_{11} = 0.084475$; $\lambda_{12} = 0.997668$;
$\lambda_{21} = 0.211800$; $\lambda_{22} = 0.032171$;
$a_{21} = -0.720468$; $a_{22} = 0.801465$; $a_{41} = 0.276638$;
$a_{42} = -17.512360$; $\beta_{11} = 11.076893$; $\beta_{12} = 11.868745$;
$\beta_{21} = 62.526719$; $\beta_{22} = 12.431946$; $\alpha_0 = \phi(p_0, p^{(k)}) = 0.046852$;
$\alpha_{21} = -1.233468$; $\alpha_{22} = -2.929104$; $\alpha_{41} = 11.552183$;
$\alpha_{42} = 32.438856$; $w = 2.503874.$

Choosing the values of $c_1=0, 1, 2, \ldots 10$ systematically in the expression (3) for $j = 1$ we compute the corresponding values of sample size n. Substituting these values of n in (3) for $j = 2$ we obtain the corresponding values of c_2 systematically. The values of correction factor $K(n, .01, .15)$ are computed for values of n obtained above. The corrected values of N are obtained by (25).

Table 1 provides the asymptotic values of n, c_2 and N. The values of $K(n, p^{(1)}, p^{(2)})$ are given in Table 2.

TABLE 1. ASYMPTOTIC VALUES OF n, c_2 AND N

c_1	n	c_2	$h(n)/w\alpha_0$	N
0	13.1214	0.1662	3.4251	59.9
1	32.1167	1.9680	15.6749	276.1
2	50.9378	3.1707	41.5336	787.1
3	69.7266	4.2668	109.3707	2275.8
4	88.5038	5.3209	271.9654	6287.2
5	107.2764	6.3561	706.3721	18428.7
6	126.0440	7.3808	1823.2031	53937.8
7	144.8107	8.3991	4678.8065	157284.5
8	163.5762	9.4132	11985.4462	432339.7
9	182.3408	10.4244	28961.0861	1251124.0
10	201.1049	11.4335	73735.1256	8624363.5

TABLE 2. THE VALUES OF $K(n, p^{(1)}, p^{(2)})$ FOR $p^{(1)} = 0.01$ AND $p^{(2)} = 0.15$

n	$K(n, 0.01, 0.15)$
13	15.6877
32	15.5755
51	17.7236
70	20.1680
88	22.7940
107	25.9377
126	29.5150
145	33.5856
163	37.9587
182	43.1939
201	49.1511

The exact ASR Bayesian plan for the same set of cost parameters and prior distribution as in the present example are given in Table 3 in Pandey (1974).

The value of regret function R^* for exact ASR plans is given by

$$R^* = n^* + (\bar{N} - n^*)\, D(n^*, c_1^*, c_2^*) \qquad \ldots \quad (26)$$

where

$$D(n^*, c_1^*, c_2^*) = \mu_{11} + \mu_{21} - \delta_1(n^*, c_1^*) - \delta_2(n^*, c_2^*)$$

$$\delta_i(n^*, c_j^*) = \mu_{i1}\, B(c_j^*, n^*, p^{(1)}) - \mu_{i2} B(c_j^*, n^*, p^{(2)}),\ i = 1, 2$$

and $\bar{N}$ denotes the geometric mean of the lower and the upper limit of lot range.

The values of δ_i's are provided in Table 10 in Pandey (1984).

The exact and approximate ASR plans are given in Table 3. The approximate plan (n, c_1, c_2, N) and exact plan $(n^*, c_1^*, c_2^*, \bar{N})$ both are obtained and listed according to the same values of c_1. The values of $\bar{N}$ in Table 3 are the geometric mean of the lower and upper lmits of the corresponding lot range. For example, the value of $\bar{N}$ for the exact plan (8, 0, 1) is equal to 34 which is the geometric mean of 8 and 146. Following Hald (1981, page 362) efficiency $e(n, c_1, c_2)$ of the approximate plan (n, c_1, c_2, N) in Table 3 is defined by R^*/R where R is computed from asymptotic formula (23).

TABLE 3. EXACT AND APPROXIMATE BAYESIAN THREE-DECISION ASR PLANS

exact						approximate					$e(n, c_1, c_2)$
N	n^*	c_1^*	c_2^*	$\bar{N}$	R^*	n	c_1	c_2	N	R	(%)
8-146	8	0	1	34	11.39	13	0	1	60	16.82	68
147-609	27	1	2	299	37.27	32	1	2	276	46.22	81
610-1782	46	2	3	1042	58.82	51	2	3	787	67.88	87
1783-4721	64	3	4	2901	78.17	70	3	4	2276	88.09	89
4722-13140	83	4	5	7877	97.34	88	4	5	6287	106.75	91
13141-35801	102	5	6	21690	116.93	107	5	6	18429	126.21	93
35802-97192	121	6	7	58988	136.48	126	6	7	53938	145.53	94
97193-409222	141	7	8	199433	159.93	145	7	8	157284	164.77	97
409223-555699	158	8	9	476870	178.02	163	8	9	432340	182.94	97
555700-1428735	177	9	10	891038	191.25	182	9	10	1251124	202.09	95
1428736-5000175	196	10	11	2672813	212.03	201	10	11	3624363	221.21	96

It can be seen from Table 3 that the relation between the sample size n and the decision numbers c_1 and c_2 for the approximate plans and the corresponding values for the exact plans is quite closely correct. The relation between the corrected lot size N for the approximate plan and the corresponding value of lot size $\bar{N}$ for the exact plan is also fairly correct. The efficiency of the approximate plans measured as ratio R^*/R expressed in percentage is found to be generally increasing and is well above 80 for lot size above 300.

Approximate ASR three-decision Bayesian plans for two-point prior distribution can be tabulated exhaustively in the following four steps :

Step 1. Using the formula (2), (4)-(6), (11), (15), (18)-(21) compute the value of p_0, δ', a_{2j}, a_{ij} β_{jk}, w, α_0, α_{2j}, α_{ij}, $j = 1, 2$.

Step 2. Choosing the values of $c_1 = 0, 1, 2, \ldots$systematically compute the values of sample size n from (3) for $j = 1$.

Step 3. Substituting, the value of n obtained above compute the value of c_2 from (3) for $j = 2$.

Step 4. Using the values obtained in the step 1 obtain the value of corrected lot size from (25).

5. Concluding remarks

Approximate formula obtained here is useful to obtain fairly accurate ASR plan in practice. Such formula developed for other type of three-decision, restricted and unrestricted cases also provide fairly accurate approximate plans.

Acknowledgement. The author is grateful to the referee for his valuable suggestions.

References

Hald, A. (1967) : On the theory of single sampling inspection by attributes based on two quality levels. *Rev. Intern. Statist. Inst.*, **35**, 1-29.

——— (1981) : *Statistical Theory of Sampling Inspection by Attributes*, Academic Press, Inc. London, New York.

Pandey, R. J. (1974) : Bayesian single sampling plan by attributes with three-decision criteria for discrete prior distribution. *Sankhyā*, B, **36**, 348-390.

——— (1984) : Certain generalisations of acceptance sampling plans by attributes *Unpublished Doctoral Thesis*, Indian Statistical Institute.

Paper received : December, 1984.

Revised : July, 1986.