Math. Proc. Camb. Phil. Soc. (1995), **117**, 21
Printed in Great Britain

Chaos map for the universal enveloping algebra of $U(N)$*

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(*Received* 14 *July* 1993)

Abstract

It is shown that the family of representations $\{j_t, t \in \mathbb{R}_+\}$ of the universal enveloping algebra $\mathcal{U}$ of the N-dimensional unitary group which is generated by the N-dimensional number process of quantum stochastic calculus can be expressed in the form

$$j_t = I_t \circ \psi,$$

where ψ is a bijective linear map from $\mathcal{U}$ onto the space $\mathcal{S}$ of symmetric tensors over the Lie algebra, and I_t is the iterated (chaotic) integral on $\mathcal{S}$. The chaotic product $*$ is defined by the formula

$$\psi(LM) = \psi(L) * \psi(M)$$

and satisfies

$$I_t(S * T) = I_t(S) I_t(T).$$

This work generalizes and completes earlier results on the centre of $\mathcal{U}$.

1. *Introduction*

In N-dimensional quantum stochastic calculus, the generalized number processes [**Par**] $(\Lambda(H,t), t \geqslant 0)$, labelled by skew-symmetric linear transformations H on the ambient space $\mathbb{C}^N$, consist of operators of differential second quantization. Thus they form the infinitesimal representation j_t of a representation of the group $U(N)$ of $N \times N$ unitary matrices

$$u \mapsto \pi_t(u) := \Gamma(M_{\chi_{[0,t]}} \otimes u + M_{\chi_{(t,\infty)}} \otimes 1). \tag{1·1}$$

Here M_{χ_s} denotes the operator on $L^2(\mathbb{R}_+)$ of multiplication by the indicator function χ_s of $S \subseteq \mathbb{R}_+$, and Γ is the second quantization map from unitary operators on the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}_+; \mathbb{C}^N)$ of vector-valued functions on $\mathbb{R}_+$, canonically identified with the tensor product $L^2(\mathbb{R}_+) \otimes \mathbb{C}^N$, to unitary operators on the Fock space $\mathcal{H} = \text{Fock}\,(L^2(\mathbb{R}_+; \mathbb{C}^N))$ of quantum stochastic calculus [**Par**]. Thus, if V is a unitary operator on $\mathfrak{h}$, $\Gamma(V)$ is the unitary operator on $\mathcal{H}$ whose action on exponential vectors $e(f), f \in \mathfrak{h}$ is

$$\Gamma(V)\, e(f) = e(Vf). \tag{1·2}$$

In this paper we investigate the family $(j_t, t \in \mathbb{R}_+)$, regarded as representations of the complex universal enveloping algebra $\mathcal{U}$ of $U(N)$. The processes $(\pi_t(u), t \in \mathbb{R}_+)$ satisfy the stochastic differential equations

$$d\pi_t(u) = \pi_t(u)\, d\Lambda(u-1, t), \quad \pi_0(u) = 1$$

* Work supported by SERC grant GR1H25317.

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from which we derive the chaotic expansion

$$\pi_t(U) = I + \sum_{r=1}^{\infty} I_t((u-1)^{\otimes^r}), \tag{1·3}$$

where I_t is the iterated integral map from the tensor algebra $\mathscr{T}$ over the space $\mathscr{L}$ of linear transformations on $\mathbb{C}^N$ which extends the map

$$H \mapsto \Lambda(H,t) = I_t(H).$$

Identifying elements of $\mathscr{U}$ with left-invariant differential operators, we may differentiate the relation (1·3) in the Lie group theoretic sense, to obtain a corresponding (finite) chaotic expansion for elements L of $\mathscr{U}$

$$j_t(L) = I_t(\psi(L)), \tag{1·4}$$

where $\psi(L)$ is an element of the symmetric tensor algebra $\mathscr{S}$ over $\mathscr{L}$. The linear map ψ from $\mathscr{U}$ to $\mathscr{S}$ is bijective.

A new product formula for iterated integrals

$$I_t(S)\, I_t(T) = I_t(S * T), \quad S, T \in \mathscr{S}$$

enables us to define a 'chaotic product' $*$ in $\mathscr{S}$ such that

$$\psi(LM) = \psi(L) * \psi(M), \quad L, M \in \mathscr{U}.$$

When restricted to $\mathscr{S}$ the quotient map from $\mathscr{T}$ to $\mathscr{U}$ is bijective. Composing ψ with this map, we obtain a bijective map $C : \mathscr{U} \to \mathscr{U}$. The centre $\mathscr{Z}$ of $\mathscr{U}$ is invariant under C, so that C extends the chaos map for the centre found in a preliminary version of this work [**HP2**].

2. *A product formula for iterated stochastic integrals*

We regard the space $\mathscr{L}$ of all linear transformations of $\mathbb{C}^N$ as the complexified Lie algebra of $U(N)$, by equipping it with the commutator bracket

$$[H, K] = HK - KH$$

and the natural involution $\dagger$ of adjunction with respect to the canonical inner product in $\mathbb{C}^N$. The space $\mathscr{T}$ of tensors over $\mathscr{L}$ becomes an associative algebra equipped with the multiplication which extends the formation of tensor products. Its symmetric subspace $\mathscr{S}$ is spanned by tensors of form

$$H^{\otimes^r} = H \otimes \ldots \otimes H^{(r)}, \quad H \in \mathscr{L}, \quad r = 0, 1, 2, \ldots.$$

Here $H^{\otimes^0}$ is the identity I.

For $H \in \mathscr{L}$ the generalized number process $\Lambda(H,t)$ is defined by the action on exponential vectors

$$\Lambda(H,t)\, e(f) = \frac{d}{d\epsilon} e((M_{\chi_{[0,t]}} \otimes \exp(\epsilon H) + M_{\chi_{(t,\infty)}} \otimes 1) f)|_{\epsilon=0}.$$

Then we may define iterated stochastic integrals [**Par**] against the $\Lambda(H,t)$ as integrators. Because the map

$$\mathscr{L}^r \ni (H_1, \ldots, H_r) \mapsto \int_{0 < t_1 < \ldots < t_r < t} d\Lambda(H_1, t_1) \ldots d\Lambda(H_r, t_r)$$

is multilinear it extends uniquely to a linear map $I_t^{(r)}$ on the subspace of $\mathscr{T}$ comprising homogeneous tensors of rank r. Combining the maps $I_t^{(r)}$, $r = 0, 1, \dots$ (where $I_t^{(0)}(z) = z1$, $z \in \mathbb{C}$) we obtain a linear map I_t on $\mathscr{T}$ for which, for $H_1, \dots, H_r \in \mathscr{L}$,

$$I_t(H_1 \otimes \dots \otimes H_r) = \int_{0<t_1<\dots<t_r<t} d\Lambda(H_1, t) \dots d\Lambda(H_r, t_r).$$

Note that, for $H \in \mathscr{L}$,

$$I_t(H) = \Lambda(H, t)$$

and that, for $T \in \mathscr{T}$,

$$I_t(T)^\dagger = I_t(T^\dagger) \tag{2·1}$$

where, on the left, † denotes the restriction to the exponential domain of the Hilbert space adjoint and, on the right, † is the extension to $\mathscr{T}$ of the involution on $\mathscr{L}$, for which, for $H_1, \dots, H_r \in \mathscr{L}$,

$$(H_1 \otimes \dots \otimes H_r)^\dagger = H_1^\dagger \otimes \dots \otimes H_r^\dagger. \tag{2·2}$$

In the theorem which follows, for $H_1, \dots, H_s \in \mathscr{L}$ and $r_1, \dots, r_s \in \mathbb{N}$,

$$\operatorname{Sym}(H_1 \otimes \dots \otimes H_1^{(r_1)} \otimes H_2 \otimes \dots \otimes H_2^{(r_2)} \otimes \dots \otimes H_s \otimes \dots \otimes H_s^{(r_s)})$$

denotes the unnormalized symmetrization of the tensor

$$H_1 \otimes \dots \otimes H_1^{(r_1)} \otimes H_2 \otimes \dots \otimes H_2^{(r_2)} \otimes \dots \otimes H_s \otimes \dots \otimes H_s^{(r_s)}$$

consisting of the sum of the $(r_1 + \dots + r_s)!/(r_1! \dots r_s!)$ distinct permutations of the original tensor. The product of operators on the exponential domain spanned by the $e(f), f \in \mathfrak{h}$ may be understood in the sense that $R = ST$ means that $\langle e(f), Re(g)\rangle = \langle S^\dagger e(f), Te(g)\rangle$ for arbitrary $f, g \in \mathfrak{h}$; alternatively the exponential domain may be enlarged in such a way that it reduces to ordinary multiplication of operators.

THEOREM 2·1. *Let $H, K \in \mathscr{L}$, $m, n \in \mathbb{N} \cup \{0\}$. Then*

$$I_t(H \otimes \dots \otimes \overset{(m)}{H})\, I_t(K \otimes \dots \otimes \overset{(n)}{K}) = I_t\left(\sum_{r=0}^{\min\{m,n\}} \operatorname{sym}(HK \otimes \dots \otimes \overset{(r)}{HK} \otimes H \otimes \dots \otimes \overset{(m-r)}{H} \otimes K \otimes \dots \otimes \overset{(n-r)}{K})\right). \tag{2·3}$$

Proof. There is nothing to prove when m or $n = 0$. The proof is by induction on $m+n$. By the quantum Itô formula [**HP1**], for $m, n \geqslant 1$,

$$\begin{aligned} d(I_t(H \otimes \dots \otimes \overset{(m)}{H})\, I_t(K \otimes \dots \otimes \overset{(n)}{K})) &= I_t(H \otimes \dots \otimes \overset{(m-1)}{H})\, I_t(K \otimes \dots \otimes \overset{(n)}{K})\, d\Lambda(H, t) \\ &\quad + I_t(H \otimes \dots \otimes \overset{(m)}{H})\, I_t(K \otimes \dots \otimes \overset{(n-1)}{K})\, d\Lambda(K, t) \\ &\quad + I_t(H \otimes \dots \otimes \overset{(m-1)}{H})\, I_t(K \otimes \dots \otimes \overset{(n-1)}{K})\, d\Lambda(HK, t). \end{aligned} \tag{2·4}$$

Making the inductive assumption that the Theorem holds when (m, n) is replaced by any of $(m-1, n)$, $(m, n-1)$ or $(m-1, n-1)$, we obtain the result by integrating, noting that the three terms on the right hand side of (2·4) account for the terms in the right hand side of (2·3) in which the final entry in the product tensor is respectively H, K and HK respectively. ∎

COROLLARY 2·2. *There exists an associative bilinear product* $*$ *in the linear space* $\mathscr{S}$ *such that, for arbitrary* $S,\ T\in\mathscr{S}$,

$$I(S)\,I(T) = I(S*T). \tag{2·5}$$

Proof. We define $*$ by

$$H^{\otimes^m} * K^{\otimes^n} = \sum_{r=0}^{\min\{m,n\}} \mathrm{sym}\,(\overbrace{HK\otimes\ldots\otimes HK}^{(r)}\otimes \overbrace{H\otimes\ldots\otimes H}^{(m-r)}\otimes \overbrace{K\otimes\ldots\otimes K}^{(n-r)}) \tag{2·6}$$

and extends it to all of $\mathscr{S}$ by bilinearity. Since the map I is linear it is evident from the Theorem that (2·5) holds. That $*$ is associative may be verified directly from (2·6); alternatively it follows from (2·5) and the associativity of operator multiplication on the extended exponential domain together with non-degeneracy of the iterated integral map I. ∎

Note. In [**HP2**] we showed that $*$ can be defined on the whole of the tensor algebra $\mathscr{T}$.

3. *Definition of the chaos map* ψ

Matrix elements between exponential vectors $e(f), e(g), f, g\in h$ of the representation elements (1·1) are found using (1·2) to be

$$\langle e(f), \pi_t(u)\,e(g)\rangle = \exp\left\{\int_0^t \langle f(s), ug(s)\rangle ds + \int_t^\infty \langle f(s), g(s)\rangle\, ds\right\}. \tag{3·1}$$

Differentiating with respect to t gives

$$\frac{d}{dt}\langle e(f), \pi_t(u)\,e(g)\rangle = \langle f(t), (u-1)\,g(t)\rangle\,\langle e(f), \pi_t(u)\,e(g)\rangle, \tag{3·2}$$

where 1 is the identity element in $U(N)$. It follows that the process $\pi_t(u),\ t\in\mathbb{R}_+$, satisfies the stochastic differential equation

$$d\pi_t(u) = \pi_t(u)\,d\Lambda(u-1, t).$$

Since $\pi_0(u)$ is the identity operator I, this may be solved formally as

$$\pi_t(u) = \sum_{r=0}^{\infty} I_t((u-1)^{\otimes^r}), \tag{3·3}$$

where $(u-1)^{\otimes^r}$ denotes the tensor product of r copies of $(u-1)$, which is 1 when $r=0$.

The universal enveloping algebra $\mathscr{U}$ is conveniently defined as the quotient of the tensor algebra $\mathscr{T}$ by the two sided ideal $\mathscr{J}$ generated by all elements of the form $H\otimes K - K\otimes H - [H,K]$ where $H, K\in\mathscr{L}$. Identifying $\mathscr{L}$ with the Lie algebra of all left-invariant vector fields on the Lie group $U(N)$, $\mathscr{U}$ becomes correspondingly the algebra of left invariant differential operators on $C^\infty(U(N))$. We may pass from the representation π_t of $U(N)$ to a representation j_t of $\mathscr{U}$ by differentiating at the element $1\in U(N)$; thus

$$j_t(L) = (L\pi_t)\,(1) \tag{3·4}$$

in the sense that, for arbitrary ϕ_1, ϕ_2 in the exponential domain

$$\langle \phi_1, j_t(L)\,\phi_2\rangle = (L\langle \phi_1, \pi_t(\,.\,)\,\phi_2\rangle)\,(1).$$

We wish to apply (3·4) to (3·3) to obtain

$$j_t(L) = \sum_{r=0}^{\infty} L(I_t((.-1)^{\otimes^r})(1). \tag{3·5}$$

Although the sum (3·5) is finite, (3·3) is only a formal expression. To derive (3·5) rigorously we use (3·1) in the form

$$\langle e(f), \pi_t(u)\, e(g)\rangle = \langle e(f), e(g)\rangle \sum_{n=0}^{\infty} (n!)^{-1} \left(\int_0^t \langle f(s), (u-1)\, g(s)\rangle\, ds\right)^n.$$

It is evident that, on application of the differential operator L and evaluating at $u = 1$, only terms for which n does not exceed the degree of L survive on the right hand side, and we recover (3·5) using the identity

$$\left(\int_0^t \phi(s)\, ds\right)^n = n! \int_{0<s_1<\ldots<s_n<t} \phi(s_1)\,\phi(s_2)\ldots\phi(s_n)\, ds_1\, ds_2 \ldots ds_n.$$

Let ϵ_i, $i = 1, \ldots, N$ be the canonical basis in $\mathbb{C}^N$ and let ϵ^i_j be the dyad $\epsilon^*_i\, \epsilon_j$ so that, for each $u = ((u^i_j)) \in U(N)$,

$$u = \sum_{i,j} u^i_j\, \epsilon^j_i. \tag{3·6}$$

If R is an $N \times N$ matrix of non-negative integers, we denote by f_R the product of coordinate functions

$$f_R(u) = \sum_{i,j=1}^{N} (u^i_j)^{R^i_j} \tag{3·7}$$

and by $\tilde{f}_R$ the function, got by formal translation by 1,

$$\tilde{f}_R(u) = \text{“}f_R(u-1)\text{”} = \prod_{i,j=1}^{N} (u^i_j - \delta^i_j)^{R^i_j}. \tag{3·8}$$

We put $|R| = \sum_{i,j} r^i_j$.

Now consider (3·5). For $u \in U(N)$, using (3·6),

$$\begin{aligned}(u-1)^{\otimes^r} &= \left(\sum_{i,j=1}^{N} (u^i_j - \delta^i_j)\, \epsilon^j_i\right)^{\otimes^r} \\ &= \sum_{|R|=r} \prod_{i,j=1}^{N} (u^i_j - \delta^i_j)^{R^i_j}\, \mathrm{Sym} \bigotimes_{i,j=1}^{N} (\epsilon^{j\otimes^{R^i_j}}_i) \\ &= \sum_{|R|=r} \tilde{f}_R(u)\, E(R),\end{aligned}$$

where we introduce the notation

$$E(R) = \mathrm{Sym} \bigotimes_{i,j=1}^{N} (\epsilon^{j\otimes^{R^i_j}}_i).$$

It follows that

$$\pi_t(L) = \sum_{|R|=0}^{\deg l} L\tilde{f}_R(1)\, I_t(E(R)). \tag{3·9}$$

We define $\psi: \mathscr{U} \to \mathscr{T}$ by

$$\psi(L) = \sum_{|R|=0}^{\deg L} L\tilde{f}_R(1)\,E(R) \tag{3·10}$$

and embody our result in:

THEOREM 3·1. *There exists a linear map* $\psi: \mathscr{U} \to \mathscr{T}$ such that, for arbitrary $L \in \mathscr{U}$ and $t \in \mathbb{R}_+$,

$$j_t(L) = I_t(\psi(L)). \tag{3·11}$$

COROLLARY 3·2. ψ satisfies

$$\psi(LM) = \psi(L) * \psi(M) \tag{3·12}$$

for arbitrary $L, M \in \mathscr{U}$.

Proof. This is evident from (3·11) and (2·6).

4. *Bijectivity of* ψ

We denote by X^i_j the (complex) left-invariant vector field on $U(N)$ determined by the dyad $\epsilon^i_j = \epsilon^*_i \epsilon_j$. The action of X^i_j on $f \in C^\infty(U(N))$ is

$$(X^i_j f)(u) = \frac{1}{2}\frac{d}{dt}\{f(u\exp(t(\epsilon^i_j - \epsilon^j_i)) - if(u\exp(it(\epsilon^i_j + \epsilon^j_i))\}|_{t=0}.$$

For the coordinate functions

$$f^i_j(u) = f_{\epsilon^i_j}(u) = u^i_j$$

we find that

$$X^i_j f^k_l = \delta^i_l f^k_j.$$

It follows that

$$X^i_j f_R = \sum_{k=1}^{N} R^k_i f_{R-\epsilon^k_i+\epsilon^k_j} \tag{4·1}$$

and hence that

$$X^i_j \tilde{f}_R = \sum_{k=1}^{N} R^k_i \tilde{f}_{R-\epsilon^k_i+\epsilon^k_j} + R^j_i \tilde{f}_{R-\epsilon^j_i}. \tag{4·2}$$

Note that $f_{R-\epsilon^k_i+\epsilon^k_j}$ and $\tilde{f}_{R-\epsilon^k_i+\epsilon^k_j}$ are undefined if $R^k_i = 0$, but if this is so the coefficients in the sums (4·1) and (3·2) vanish so that the sums are well defined. The same applies to the last term in (4·2). We set $R! = \prod_{i,j}(R^i_j!)$.

LEMMA 4·1. *Suppose that* $|R| \geqslant n$. *Then*

$$X^{i_n}_{j_n} \dots X^{i_1}_{j_1} \tilde{f}_R(1) = \begin{cases} R! & \text{if } R = \epsilon^{j_1}_{i_1} + \epsilon^{j_2}_{i_2} + \dots + \epsilon^{j_n}_{i_n}, \\ 0 & \text{otherwise.} \end{cases} \tag{4·3}$$

Proof. It is evident from the definition that

$$\tilde{f}_R(1) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4·4}$$

Making the inductive assumption that, whenever $k < n$, and $|R| \geqslant k$,

$$(X^{i_k}_{j_k} \dots X^{i_1}_{j_1})\tilde{f}_R(1) = \begin{cases} R! & \text{if } R = \epsilon^{i_1}_{i_1} + \dots + \epsilon^{i_k}_{i_k}, \\ 0 & \text{otherwise,} \end{cases}$$

we use (4·2) to write

$$X^{i_n}_{j_n}\dots X^{i_1}_{j_1}\tilde{f}_R = \sum_S C_S \prod_{(i,j)\in S} X^i_j \tilde{f}_{R(S)} + R^{j_1}_{i_1}(R-\epsilon^{j_1}_{i_1})^{j_2}_{i_2}\dots(R-\epsilon^{j_1}_{i_1}-\epsilon^{j_2}_{i_2}-\dots-\epsilon^{j_{n-1}}_{i_{n-1}})^{j_n}_{i_n}\tilde{f}_{R-\epsilon^{j_1}_{i_1}-\dots-\epsilon^{j_n}_{i_n}},$$

where the sum is over ordered r-tuples S of elements of $\{1,\dots,N\}\times\{1,\dots,N\}$ with $r=|S|<n$, $|R(S)|>|S|$ and the C_S are non-negative integers. Evaluation at $1\in U(\mathbb{N})$ annihilates the sum in view of the inductive hypothesis and we obtain

$$\begin{aligned} X^{i_n}_{j_n}\dots X^{i_1}_{j_1}\tilde{f}_R(1) &= R^{j_1}_{i_1}(R-\epsilon^{j_1}_{i_1})^{j_2}_{i_2}\dots(R-\epsilon^{j_1}_{i_1}-\epsilon^{j_2}_{i_2}-\dots-\epsilon^{j_{n-1}}_{i_{n-1}})^{j_n}_{i_n}\tilde{f}_{R-\epsilon^{j_1}_{i_1}-\dots-\epsilon^{j_n}_{i_n}}(1) \\ &= \begin{cases} R! & \text{if } R=\epsilon^{j_1}_{i_1}+\dots+\epsilon^{j_n}_{i_n}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

in view of (4·4).

THEOREM 4·2. *ψ is bijective.*

Proof. From (3·10) and (4·3) we have

$$\psi(X^{i_n}_{j_n}\dots X^{i_1}_{j_1}) = R!\,E(R)+F, \tag{4·5}$$

where $R=\epsilon^{j_1}_{i_1}+\dots+\epsilon^{j_n}_{i_n}$ and F is a symmetric tensor of order less than n. Since the tensors $E(R)$ of this form generate $\mathscr{S}$ linearly, it follows that ψ is surjective. On the other hand suppose $\psi(L)=0$ for $L\in\mathscr{U}$, $L\neq 0$. Write $L=L_n+K$ where K consists of terms of order $<n$ in the Poincaré–Birkhoff–Witt expansion of L and

$$L_n = \sum_{|R|=n} a_n(R)\,(X^N_N)^{R^N_N}(X^N_{N-1})^{R^N_{N-1}}\dots(X^1_1)^{R^1_1} \neq 0. \tag{4·6}$$

From (4·5) we have

$$\psi(L) = \sum_{|R|=n} a_n(R)\,R!\,E_n(R)+F$$

where F is a symmetric tensor of order $<n$, hence we have

$$\sum_{|R|=n} a_n(R)\,R!\,E_n(R)=0.$$

But the tensors $E_n(R)$ are linearly independent. Hence each $a_n(R)=0$ which contradicts (4·6).

5. *Casimir chaos map*

Recall that $\mathscr{J}$ is the two-sided ideal in $\mathscr{T}$ generated by all elements of form $H\otimes K-K\otimes H-[H,K]$, H, $K\in\mathscr{L}$, so that the universal enveloping algebra $\mathscr{U}$ is canonically identified with the quotient algebra $\mathscr{T}/\mathscr{J}$.

PROPOSITION 5·1. *Every element $L\in\mathscr{U}$ can be represented uniquely in the form*

$$L = T+\mathscr{J}, \tag{5·1}$$

where $T\in\mathscr{S}$.

Proof. We prove that the map $c:\mathscr{S}\mapsto\mathscr{U}$, $T\mapsto T+\mathscr{J}$ is bijective. We compare the basis $E(R)$, $|R|=0,1,2,\dots$ of $\mathscr{S}$ with the basis $L(R)$, $|R|=0,1,2,\dots$ of $\mathscr{U}$ formed from the basis X^i_j, $i,j=1,\dots,N$ of $\mathscr{L}$ by the Poincaré–Birkhoff–Witt theorem; thus

$$L(R) = (X^1_1)^{R^1_1}(X^1_2)^{R^1_2}\dots(X^n_n)^{R^n_n}.$$

Evidently if $|R| \leqslant 1$, $c(E(R)) = L(R)$. More generally, using the commutation relations

$$X_j^i X_l^k - X_l^k X_j^i = \delta_j^i X_j^k - \delta_j^k X_j^i$$

it is evident that $L(R)$ can be symmetrized modulo terms of lower degree, that is, that

$$c(E(R)) = L(R) + \sum_{|S|<|R|} \alpha_{RS} L(S).$$

The triangularity of this relationship shows that the linear map c maps the basis $E(R)$, $|R| = 0, 1, \ldots$ of $\mathscr{S}$ to a basis of $\mathscr{U}$ and is therefore bijective. ∎

Note. Gross has shown [**Gro**] that, under a technical assumption concerning completion of $\mathscr{T}$ in an inner product derived from one on $\mathscr{L}$, the universal enveloping algebra of an arbitrary real Lie algebra $\mathscr{L}$ can be 'parametrized' algebraically by the symmetric subspace of the tensor algebra $\mathscr{T}$ over $\mathscr{L}$. We are grateful to him for communicating his preprint [**Gro**].

We now consider the centre $\mathscr{Z}$ of $\mathscr{U}$, whose elements we call *Casimir elements*. The adjoint action Ad, given by

$$Ad_u(H) = uHu^{-1}, \quad u \in U(N), \quad H \in \mathscr{L}$$

of $U(N)$ on $\mathscr{L}$ extends to the tensor algebra $\mathscr{T}$ and leaves invariant the symmetric subspace $\mathscr{S}$. Since it also leaves $\mathscr{J}$ invariant it lifts to $\mathscr{U}$, and the Casimir elements are characterized as fixed points of the adjoint action in $\mathscr{U}$. For each Casimir element Z, writing

$$Z = D + \mathscr{J}, \quad D \in \mathscr{S}$$

in accordance with Proposition 5·1, we have, for $u \in U(N)$

$$Z = Ad_u Z = Ad_u D + Ad_u \mathscr{J} = Ad_u D + \mathscr{J}.$$

By the uniqueness of the representation 5·1, we have that

$$Ad_u D = D. \tag{5·2}$$

THEOREM 5·2. *There exists a bijective linear map $C: \mathscr{Z} \to \mathscr{Z}$ such that, for arbitrary $Z \in \mathscr{Z}$,*

$$\psi(Z) + \mathscr{J} = CZ.$$

Proof. We have to prove that $\psi(Z) + \mathscr{J} \in \mathscr{Z}$ for arbitrary $Z \in \mathscr{Z}$. In [**HP3**] we showed that, for the family of Casimir elements $Z_0 = I$, $Z_r = D_r + \mathscr{J}$, $r = 1, \ldots, N$ where

$$D_r = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant N} \sum_{\sigma, \tau \in S_r} \operatorname{sign} \sigma\tau \, \epsilon_{i_{\tau(1)}}^{i_{\sigma(1)}} \otimes \ldots \otimes \epsilon_{i_{\tau(r)}}^{i_{\sigma(r)}}, \tag{5·3}$$

the chaotic expansion of $j_t(Z_r)$ is of form

$$j_t(Z_r) = I_t\left(\sum_{s=0}^{r} \alpha_{rs} D_s\right)$$

where the α_{rs} are non-zero rational numbers. Thus

$$\psi(Z_r) + \mathscr{G} = \sum_{s=0}^{r} \alpha_{rs} Z_s \in \mathscr{Z}.$$

Now the Casimir elements $Z_0, \ldots, Z_N$ generate the algebra $\mathscr{Z}$. The theorem will therefore be proved if, using the characterization (5·2) of symmetric tensors corresponding to Casimir elements, we can show that, if D and $\tilde{D}$ are elements of $\mathscr{T}$ point-wise invariant under the adjoint action, then so too is $D * \tilde{D}$. To do this, we note first that, for arbitrary $T \in \mathscr{T}$,

$$I_t(Ad_u T) = \pi_t(u) I_t(T) \pi_t(u)^{-1}$$

as is evident from (1·1) and (1·2). It follows that

$$\begin{aligned} I_t(Ad_u(D * \tilde{D})) &= \pi_t(u) I_t(D * \tilde{D}) \pi_t(u)^{-1} \\ &= \pi_t(u) I_t(D) I_t(\tilde{D}) \pi_t(u)^{-1} \\ &= \pi_t(u) I_t(D) \pi_t(u)^{-1} \pi_t(u) I_t(\tilde{D}) \pi_t(u)^{-1} \\ &= I_t(ad_u D) I_t(ad_u \tilde{D}) \\ &= I_t(D) I_t(\tilde{D}) \\ &= I_t(D * \tilde{D}). \end{aligned}$$

Hence, by the independence of stochastic integrators,

$$Ad_u(D * \tilde{D}) = D * \tilde{D}$$

as required. (Note that in general, even if D and $\tilde{D}$ are symmetric, $D * \tilde{D}$ is not. But $D * \tilde{D} + \mathscr{J}$ is a Casimir element.) ∎

We call the map C of Theorem 5·2 the *Casimir chaos map.*

The centre $\mathscr{Z}$ of $\mathscr{U}$ is isomorphic to the algebra $\mathscr{Z}_N$ of symmetric polynomials in N commuting indeterminates, under an isomorphism $\eta: Z \mapsto z$, where, for integers $l_1, \ldots, l_N$ with $l_1 > l_2 > \ldots > l_N$, in the irreducible representation of $U(N)$ of highest weight $(l_1 - N + 1, l_2 - N + 2, \ldots, l_N)$, Z corresponds to the multiple $z(l_1, \ldots, l_N) I$ of the identity operator. We may use this isomorphism to construct a chaos map $C_N = \eta C \eta^{-1}$ on $\mathscr{Z}_N$. Its action on a family of generators of the algebra $\mathscr{Z}_N$ is given by

Proposition 5·3. Denote by σ_r, $r = 0, 1, \ldots, N$ the elementary symmetric polynomials $\sigma_0 = 1$,

$$\sigma_r(l_1, \ldots, l_N) = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant N} l_{i_1} \ldots l_{i_r}, \quad r = 1, \ldots, N.$$

Then there is a triangular array of rational numbers

$$\beta_{rs}, \quad r = 0, 1, \ldots, N, \quad s = 0, 1, \ldots, r$$

such that each $\beta_{rr} \neq 0$ and

$$C_N \sigma_r = \sum_{s=0}^{r} \beta_{rs} \sigma_s, \quad r = 0, 1, \ldots, n.$$

Proof. For the Casimir elements $Z_r = D_r + \mathscr{J}$ defined by (5·3) it is known [**Hud**] that $z_r = \eta(Z_r)$ is given by

$$z_r = \sum_{s=0}^{r} \gamma_{rs} \sigma_s$$

where the triangular array γ_{rs}, $r = 0, \ldots, N$, $s = 0, \ldots, r$ of rational numbers is such that each $\gamma_{rr} \neq 0$. On the other hand, as we noted in the proof of Theorem 5·2,

$$C(Z_r) = \sum_{s=0}^{r} \alpha_{rs} Z_s,$$

where (α_{rs}) is another such triangular array. The proposition follows now from the fact that products and inverses of non-singular triangular matrices are matrices of the same type.

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