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Kolmogorov's existence theorem for Markov processes in C^* algebras

B V RAJARAMA BHAT and K R PARTHASARATHY

Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi 110016, India

Dedicated to the memory of Professor K G Ramanathan

Abstract. Given a family of transition probability functions between measure spaces and an initial distribution Kolmogorov's existence theorem associates a unique Markov process on the product space. Here a canonical non-commutative analogue of this result is established for families of completely positive maps between C^* algebras satisfying the Chapman-Kolmogorov equations. This could be the starting point for a theory of quantum Markov processes.

Keywords. Completely positive map; Markov process; GNS principle.

1. Introduction

Let $(X_i, \mathscr{F}_i)$, $i = 0, 1, 2, \ldots$ be Polish measurable spaces and let $P_i(x_i, dx_{i+1})$ be a transition probability from $(X_i, \mathscr{F}_i)$ to $(X_{i+1}, \mathscr{F}_{i+1})$ for each i. Given a probability measure μ on $(X_0, \mathscr{F}_0)$ it follows from Kolmogorov's extension theorem that there exists a unique probability measure P_μ on the infinite product space $(\Omega, \mathscr{F}) = \bigotimes_{i=0}^{\infty}(X_i, \mathscr{F}_i)$ such that, for every finite n, its projection or marginal distribution P_μ^n in $\bigotimes_{i=0}^{n}(X_i, \mathscr{F}_i)$ is given by

$$P_\mu^n(E_0 \times E_1 \times \cdots \times E_n) = \int_{E_0 \times E_1 \times \cdots \times E_n} \mu(dx_0) P_0(x_0, dx_1) P_1(x_1, dx_2) \cdots P_n(x_{n-1}, dx_n) \tag{1.1}$$

for all $E_i \in \mathscr{F}_i$, $i = 0, 1, 2, \ldots, n$. The probability space $(\Omega, \mathscr{F}, P_\mu)$ describes the Markov process with initial distribution μ and transition probability $P_i(\cdot,\cdot)$ for transition from a state at time i to a new state at time $i+1$. This can be described in a * algebraic language as follows. Denote by $\mathscr{A}_i$ the commutative * algebra of all complex valued bounded measurable functions on $(X_i, \mathscr{F}_i)$. Introduce the positive unital operator $T(i, i+1)$: $\mathscr{A}_{i+1} \to \mathscr{A}_i$ by

$$(T(i, i+1)g)(x_i) = \int g(x_{i+1}) P_i(x_i, dx_{i+1}).$$

For any $i \leqslant k$ define $T(i,k)$: $\mathscr{A}_k \to \mathscr{A}_i$ by

$$T(i,k) = \begin{cases} \text{identity} & \text{if } i = k, \\ T(i, i+1)\, T(i+1, i+2) \cdots T(k-1, k) & \text{if } i < k. \end{cases}$$

The family $\{T(i,k), i \leqslant k\}$ of transition operators obeys the Chapman-Kolmogorov equations:

$$T(i,k)\,T(k,\ell) = T(i,\ell) \quad \text{for } i \leqslant k \leqslant \ell.$$

Let $\mathscr{H}$ be the Hilbert space $L^2(P_\mu)$ and $F(i)$ denote the Hilbert space projection on the subspace of functions depending only on the first $i+1$ coordinates $(x_0, x_1, \ldots, x_i)$ of $\omega = (x_0, x_1, x_2, \ldots,)$ in Ω. Then $\{F(i)\}$ is an increasing sequence of projections in $\mathscr{H}$. For any $g \in \mathscr{A}_i$ define the operator $j_i(g)$ in $\mathscr{H}$ by

$$(j_i(g)\phi)(\omega) = g(x_i)(F(i)\phi)(\omega), \quad \omega = (x_0, x_1, \ldots,).$$

Then j_i is a * homomorphism from $\mathscr{A}_i$ into the * algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators in $\mathscr{H}$. The Markov property of the stochastic process $(\Omega, \mathscr{F}, P_\mu)$ is encapsulated in the operator relations

$$j_k(1) = F(k), \tag{1.2}$$

$$F(i) j_k(g) F(i) = j_i(T(i,k)g), \quad g \in \mathscr{A}_k, \quad i \leqslant k. \tag{1.3}$$

The relations (1.1) can be expressed as

$$\langle u, j_0(g_0) j_1(g_1) \cdots j_n(g_n) v \rangle = \int (\bar{u} v g_0)(x_0) g_1(x_1) \cdots g_n(x_n) \mathrm{d}P_\mu(\omega) \tag{1.4}$$

for all u, v in the range of $F(0)$ and $g_i \in \mathscr{A}_i$, $i = 0, 1, 2, \ldots, n$. Here ω denotes the sequence $(x_0, x_1, \ldots)$. We may call the triple $(\mathscr{H}, F, j_k, k = 0, 1, 2, \ldots)$ consisting of the Hilbert space $\mathscr{H}$, the filtration of projections $F(k)$ increasing in k and the family $\{j_k, k = 0, 1, 2, \ldots\}$ of * (but nonunital) homomorphisms, a Markov process with transition operators $\{T(i,j), i \leqslant j\}$. A similar description of a Markov process in continuous time is also possible.

In the context of quantum or non-commutative probability theory there have been several partial attempts (for example, by Accardi, Frigerio and Lewis [AFL], Emch [E], Sauvageot [S] and Vincent-Smith [Vi–S]) to construct Markov processes when transition probabilities between measurable spaces, or equivalently, the transition operators between the corresponding commutative * algebras of bounded measurable functions are replaced by unital and completely positive linear maps between unital * algebras of operators in Hilbert spaces. In the present paper we shall start with a family of completely positive maps between C^* algebras which obey the Chapman–Kolmogorov equations and build a unique canonical minimal Markov process, using the GNS principle. Rather remarkably, this minimal process, when restricted to the centres of the different C^* algebras that are involved, can be obtained as a conditional expectation of a completely commutative process. The definition of a Markov process that we shall adopt is inspired by the equations (1.2)–(1.4).

2. The basic construction

Let $\mathscr{A}_t$ be a unital C^* algebra of bounded operators in a complex Hilbert space $\mathscr{K}_t$ for every $t \geqslant 0$. The time index t here may be discrete or continuous. It is useful to imagine any hermitian element $x \in \mathscr{A}_t$ as a real valued observable concerning a system at time t. For every $0 \leqslant s \leqslant t < \infty$ let $T(s,t)\colon \mathscr{A}_t \to \mathscr{A}_s$ be a linear, unital and completely positive map (hereafter called simply a c.p. map) satisfying the following: (i) $T(s,s)$ is the identity map on $\mathscr{A}_s$; (ii) $T(r,t) = T(r,s)\,T(s,t)$ for all $0 \leqslant r \leqslant s \leqslant t < \infty$. When (i) and (ii) hold we say that the family $\{T(s,t)\}$ of c.p. maps obeys the Chapman–Kolmogorov equations and call it a family of *transition operators.* Complete positivity is equivalent to the condition

$$\sum_{i,j} X_i^*\{T(s,t)(Y_i^* Y_j)\}X_j \geqslant 0$$

for all bounded operators X_i in $\mathscr{H}_s$ and elements $Y_i \in \mathscr{A}_t$, the summation being over any finite index set. Another equivalent description of complete positivity is that, for every finite n, the matrix $((T(s,t)(Y_{ij})))_{1\leqslant i,j \leqslant n}$, viewed as an operator in the n-fold direct sum $\mathscr{H}_s \oplus \cdots \oplus \mathscr{H}_s$, is positive whenever $((Y_{ij}))_{1\leqslant i,j\leqslant n}$ is positive in the n-fold direct sum $\mathscr{H}_t \oplus \cdots \oplus \mathscr{H}_t$ with $Y_{ij} \in \mathscr{A}_t$ for each i,j.

Denote by $\Gamma_0(\mathbb{R}_+) = \Gamma_0$ the set $\{\sigma \mid \sigma \subset \mathbb{R}_+,\ 0 \in \sigma,\ \#\sigma < \infty\}$, where $\#\sigma$ denotes the cardinality of σ. When $\#\sigma = n$ and $t_i \in \sigma$, $i = 1,2,\ldots,n$ are distinct we always express it as $\sigma = \{t_1, t_2, \ldots, t_n\}$ with $t_1 > t_2 > \cdots > t_n = 0$. When $X_{t_i} \in \mathscr{A}_{t_i}$ for each $i = 1,2,\ldots,n$ we denote the n-length sequence $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ by $X(\sigma)$. Suppose that $\sigma = \{s_1, s_2, \ldots, s_m\}$, $\delta = \{t_1, t_2, \ldots, t_n\}$ and $\sigma \cup \delta = \{r_1, r_2, \ldots, r_k\}$ are in Γ_0. For any $X(\sigma)$ with $X_{s_i} \in \mathscr{A}_{s_i}$ we write $X^\sigma(\sigma \cup \delta)$ for the sequence $Y(\sigma \cup \delta)$ defined by

$$Y_{r_i} = \begin{cases} X_{s_j} & \text{if} \quad r_i = s_j \text{ for some } j = 1,2,\ldots,n, \\ I_{r_i} & \text{otherwise,} \end{cases}$$

where I_r is the identity element in $\mathscr{A}_r$. Denote by $\tilde{A}$ the set of all sequences of the form $X(\sigma)$ with σ varying in Γ_0 and write

$$\mathscr{M} = \tilde{A} \times \mathscr{H}_0, \tag{2.1}$$

$$\mathscr{M}_t = \begin{cases} \{(X(\sigma), u) \in \mathscr{M}, \sigma = (t, t_2, \ldots, t_n), n = 2,3,\ldots,\} & \text{if } t > 0 \\ \mathscr{A}_0 \times \mathscr{H}_0 & \text{if } t = 0 \end{cases} \tag{2.2}$$

To the family $\{T(s,t)\}$ of transition operators we now associate a function L_T on the set $\mathscr{M} \times \mathscr{M}$ as follows:

$$\begin{aligned} L_T((X(\sigma),u),(Y(\sigma),v)) = \langle u, X_0^*\{T(0,t_{n-1})(X_{t_{n-1}}^*\{T(t_{n-1},t_{n-2}) \\ (\cdots X_{t_2}^*\{T(t_2,t_1)(X_{t_1}^* Y_{t_1})\}\, Y_{t_2}\cdots)\}\, Y_{t_{n-1}})\, Y_0)v\rangle \\ \text{if } \sigma = \{t_1, t_2, \ldots, t_n\}, \end{aligned} \tag{2.3}$$

and

$$L_T((X(\sigma),u),(Y(\delta),v)) = L_T((X^\sigma(\sigma\cup\delta),u),(Y^\delta(\sigma\cup\delta),v)). \tag{2.4}$$

PROPOSITION 2.1.

L_T is a positive definite kernel on $\mathscr{M} \times \mathscr{M}$, i.e., for any $n = 1,2,\ldots,$ complex scalars c_i and elements $(X_i(\sigma_i), u_i) \in \mathscr{M}$, $i = 1,2,\ldots,n$ the following inequality holds:

$$\sum_{1\leqslant i,j\leqslant n} \bar{c}_i c_j L_T((X_i(\sigma_i),u_i),(X_j(\sigma_j),u_j)) \geqslant 0 \tag{2.5}$$

Proof. We claim that for a pair of elements of the form $(X(\sigma), u)$, $(Y(\sigma), v)$ in $\mathcal{M}$ and $\delta \in \Gamma_0$

$$L_T((X(\sigma), u), (Y(\sigma), v)) = L_T((X^\sigma(\sigma \cup \delta), u), (Y^\delta(\sigma \cup \delta), v)). \tag{2.6}$$

It suffices to prove this relation when $\delta = \{t, 0\}$, $\sigma = \{t_1, t_2, \ldots, t_{n-1}, 0\}$, $t \neq t_i$ for every i, since the more general case would follow by induction. In this special case (2.6) follows easily from (2.3) with σ replaced by $\sigma \cup \delta$ and the Chapman–Kolmogorov equations. In view of (2.4) it is enough to prove (2.5) when $\sigma_i = \sigma$ for each i, for otherwise, we may replace all the σ_i's by $\sigma = \bigcup_i \sigma_i$. Let $\sigma = \{t_1, t_2, \ldots, t_{m-1}, t_m = 0\}$ and

$$X_i(\sigma) = (X_{it_1}, X_{it_2}, \ldots, X_{it_m}), \quad i = 1, 2, \ldots, n.$$

Define inductively the following operators:

$$Z_{ij}(t_1) = X^*_{it_1} X_{jt_1}$$
$$Z_{ij}(t_r) = X^*_{it_r} T(t_r, t_{r-1})(Z_{ij}(t_{r-1})) X_{jt_r},$$
$$r = 2, 3, \ldots, m.$$

Clearly, the matrix $((Z_{ij}(t_1))$ is a positive operator in the n-fold direct sum $\mathcal{K}_{t_1} \oplus \cdots \oplus \mathcal{K}_{t_1}$. If $((Z_{ij}(t_{r-1})))$ is a positive operator in $\mathcal{K}_{t_{r-1}} \oplus \cdots \oplus \mathcal{K}_{t_{r-1}}$ the complete positivity of $T(t_r, t_{r-1})$ implies that $((Z_{ij}(t_r)))$ is positive in $\mathcal{K}_{t_r} \oplus \cdots \oplus \mathcal{K}_{t_r}$. Thus, by induction, $((Z_{ij}(t_m)))$ is a positive operator in $\mathcal{K}_0 \oplus \cdots \oplus \mathcal{K}_0$. If we write $\xi = \bigoplus_{i=1}^{n} c_i u_i$ in $\mathcal{K}_0 \oplus \cdots \oplus \mathcal{K}_0$ we have

$$\sum_{1 \leqslant i, j \leqslant n} \bar{c}_i c_j L_T((X_i(\sigma), u_i), (X_j(\sigma), u_j)) = \langle \xi, ((Z_{ij}(t_m)))\xi \rangle \geqslant 0. \qquad \blacksquare$$

PROPOSITION 2.2.

There exists a Hilbert space $\mathcal{H}$ and a map $\lambda: \mathcal{M} \to \mathcal{H}$ satisfying the following:

(i) $\langle \lambda(X(\sigma), u), \lambda(Y(\delta), v) \rangle \equiv L_T((X(\sigma), u), (Y(\delta), v))$;
(ii) *The set* $\{\lambda(X(\sigma), u) | (X(\sigma), u) \in \mathcal{M}\}$ *is total in* $\mathcal{H}$;
(iii) *If* $\mathcal{H}'$ *is another Hilbert space and* λ': $\mathcal{M} \to \mathcal{H}'$ *satisfying* (i) *and* (ii) *with* $(\mathcal{H}, \lambda)$ *replaced by* $(\mathcal{H}', \lambda')$ *then there exists a unitary operator* $W: \mathcal{H} \to \mathcal{H}'$ *such that* $W \circ \lambda = \lambda'$;
(iv) $\lambda((X(\sigma), u)) = \lambda(X^\sigma(\sigma \cup \delta), u)$ *for all* $(X(\sigma), u) \in \mathcal{M}$ *and* $\delta \in \Gamma_0$.

Proof. (i), (ii) and (iii) are immediate from Proposition 2.1 and the G.N.S. principle. (See, for example, Proposition 15.4, [P]). By (2.3) and (2.4) we have

$$\begin{aligned} L_T((X(\sigma), u), (X(\sigma), u)) &= L_T((X(\sigma), u), (X^\sigma(\sigma \cup \delta), u) \\ &= L_T((X^\sigma(\sigma \cup \delta), u), (X^\sigma(\sigma \cup \delta), u)) \end{aligned}$$

and hence by (i) in the proposition

$$\begin{aligned} \|\lambda(X(\sigma), u) - \lambda(X^\sigma(\sigma \cup \delta), u)\|^2 &= \|\lambda(X(\sigma), u)\|^2 + \|\lambda(X^\sigma(\sigma \cup \delta), u)\|^2 \\ &\quad - 2\mathrm{Re}\langle \lambda(X(\sigma), u), \lambda(X^\sigma(\sigma \cup \delta), u) \rangle = 0. \qquad \blacksquare \end{aligned}$$

Remark. When $\sigma = \{t_1, t_2, \ldots, t_n\}$ is fixed it is a consequence of (i) in Proposition 2.2 that $\lambda((X_{t_1}, X_{t_2}, \ldots, X_{t_n}), u)$ is multilinear on $\mathscr{A}_{t_1} \times \cdots \times \mathscr{A}_{t_n} \times \mathscr{K}_0$.

PROPOSITION 2.3.

In Proposition 2.2 let $\mathscr{H}_t$ be the closed linear span of the set $\{\lambda(X(\sigma), u) | (X(\sigma), u) \in \mathscr{M}_t\}$ where $\mathscr{M}_t$ is defined by (2.1) and (2.2). Then $\{\mathscr{H}_t, t \geqslant 0\}$ is an increasing family of subspaces of $\mathscr{H}$ and the map $V: u \to \lambda(I_0, u)$ is a unitary operator from $\mathscr{K}_0$ to $\mathscr{H}_0$.

Proof. Let $0 \leqslant s < t < \infty$. Suppose $\sigma = \{s, s_2, \ldots, s_m\}$. Then by property (iv) in Proposition 2.2 we have

$$\lambda((X_s, X_{s_2}, \ldots, X_{s_m}), u) = \lambda((I_t, X_s, X_{s_2}, \ldots, X_{s_m}), u)$$

and the right hand side belongs to $\mathscr{H}_t$ by definition. This proves the first part. To prove the second part we first observe that

$$\langle \lambda(I_0, u), \lambda(I_0, v) \rangle_{\mathscr{H}} = \langle u, v \rangle_{\mathscr{K}_0}.$$

Thus V is an isometry from $\mathscr{K}_0$ into $\mathscr{H}_0$. Furthermore (2.3) implies

$$\begin{aligned} &\| \lambda(X_0, u) - \lambda(I_0, X_0 u) \|^2 \\ &\quad = L_T((X_0, u), (X_0, u)) + L_T((I_0, X_0 u), (I_0, X_0 u)) \\ &\qquad - 2\operatorname{Re} L_T((X_0, u), (I_0, X_0 u)) \\ &\quad = \langle u, X_0^* X_0 u \rangle + \langle X_0 u, X_0 u \rangle \\ &\qquad - 2\operatorname{Re} \langle u, X_0^*(X_0 u) \rangle = 0. \end{aligned}$$

■

For any Hilbert space $\mathscr{K}$ we denote by $\mathscr{B}(\mathscr{K})$ the C^* algebra of all bounded operators on $\mathscr{K}$.

PROPOSITION 2.4.

*Let $\mathscr{H}$, $\mathscr{H}_t$, λ, V be as in Proposition 2.3. Then there exists a unique * unital homomorphism $j_t^0 : \mathscr{A}_t \to \mathscr{B}(\mathscr{H}_t)$ for every $t \geqslant 0$ satisfying the relations:*

$$j_t^0(Y) \lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u) = \lambda((YX_t, X_{t_2}, \ldots, X_{t_n}), u) \tag{2.7}$$

for all $Y \in \mathscr{A}_t$, $t > t_2 > \cdots > t_n = 0$, $u \in \mathscr{K}_0$. Furthermore

$$V^* j_0^0(X) V = X \quad \textit{for all } X \in \mathscr{A}_0.$$

Proof. Let $Y \in \mathscr{A}_t$ be unitary. By (2.3) and the fact that $\{T(s, t)\}$ is a family of transition operators it follows immediately that

$$\begin{aligned} &\langle \lambda((YX_t, X_{t_2}, \ldots, X_{t_n}), u), \lambda((YZ_t, Z_{t_2}, \ldots, Z_{t_n}), v) \rangle \\ &\quad = L_T(((YX_t, X_{t_2}, \ldots, X_{t_n}), u), ((YZ_t, Z_{t_2}, \ldots, Z_{t_n}), v)) \\ &\quad = L_T(((X_t, X_{t_2}, \ldots, X_{t_n}), u), ((Z_t, Z_{t_2}, \ldots, Z_{t_n}), v)) \\ &\quad = \langle \lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u), \lambda((Z_t, Z_{t_2}, \ldots, Z_{t_n}), v) \rangle \end{aligned}$$

for all $X_t, Y_t \in \mathscr{A}_t, X_{t_i}, Y_{t_i} \in \mathscr{A}_{t_i}, u, v \in \mathscr{H}_0$. This together with property (iv) of Proposition 2.2 implies that

$$\begin{aligned}&\langle \lambda(YX_t, X_{t_2}, \ldots, X_{t_n}), u), \lambda(YZ_t, Z_{t_1'}, Z_{t_2'}, \ldots, Z_{t_{n'}'}), v)\rangle\\ &= \langle \lambda(X_t, X_{t_2}, \ldots, X_{t_n}), u), \lambda(Z_t, Z_{t_1'}, Z_{t_2'}, \ldots, Z_{t_{n'}'}), v)\rangle\end{aligned}$$

Thus for any unitary Y in A_t there exists a unitary operator $j_t^0(Y)$ in $\mathscr{H}_t$ satisfying (2.7). If Y_1, Y_2 are unitary elements in A_t it follows from the definitions that $j_t^0(Y_1) j_t^0(Y_2) = j_t^0(Y_1 Y_2)$. Since $\lambda((X_t, X_{t_1}, \ldots, X_{t_n}), u)$ is linear in the variable X_t and any element in $\mathscr{A}_t$ is a linear combination of at most four unitary elements in $\mathscr{A}_t$ it follows that $j_t^0(\cdot)$ defined for unitary elements extends linearly to $\mathscr{A}_t$ as a * unital homomorphism from $\mathscr{A}_t$ into $\mathscr{B}(\mathscr{H}_t)$. The uniqueness part is obvious. To prove the last part we have to only note that by the definition of V in Proposition 2.3 and the last part of its proof

$$\begin{aligned} j_0^0(X)\, Vu &= j_0^0(X)\lambda(I_0, u) = \lambda(X, u)\\ &= \lambda(I_0, Xu) = VXu\end{aligned}$$

for all $u \in \mathscr{H}_0$. ■

Theorem 2.5. *Let $\mathscr{A}_t$ be a unital C^* algebra of operators in a Hilbert space $\mathscr{H}_t$ for every $t \geqslant 0$ and let $T(s,t)$: $\mathscr{A}_t \to \mathscr{A}_s$, $s \leqslant t$ be a family of transition operators. Then there exists a Hilbert space $\mathscr{H}$, an increasing family $\{F(t), t \geqslant 0\}$ of projection operators on $\mathscr{H}$, a family of contractive * homomorphisms $j_t: \mathscr{A}_t \to \mathscr{B}(\mathscr{H})$, $t \geqslant 0$ and a unitary isomorphism V from $\mathscr{H}_0$ onto the range of $F(0)$ satisfying the following:*

(i) $j_t(I_t) = F(t)$, *I_t being the identity operator in $\mathscr{H}_t$;*
(ii) *for any* $0 \leqslant s \leqslant t < \infty$, $X \in \mathscr{A}_t$

$$F(s) j_t(X) F(s) = j_s(T(s,t)(X));$$

(iii) *the set* $\{j_{t_1}(X_1) \cdots j_{t_n}(X_n)\, Vu, t_1 > t_2 > \cdots > t_n = 0, X_i \in \mathscr{A}_{t_i}$ *for each* i, $n = 1, 2, \ldots, u \in \mathscr{H}_0\}$ *is total in* $\mathscr{H}$;
(iv) $j_0(X)\, V = VX$ *for all* $X \in \mathscr{A}_0$ *and for any* u, $v \in \mathscr{H}_0$, $\sigma = \{s_1 > s_2 > \cdots > s_m = 0\}$, $\delta = \{t_1 > t_2 > \cdots > t_n = 0\}$,

$$\begin{aligned}&X_i \in \mathscr{A}_{s_i}, Y_j \in \mathscr{A}_{t_j}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\\ &\langle j_{s_1}(X_1) j_{s_2}(X_2) \cdots j_{s_m}(X_m)\, Vu, j_{t_1}(Y_1) j_{t_2}(Y_2) \cdots j_{t_n}(Y_n)\, Vv\rangle\\ &= L_T((X(\sigma), u), (Y(\delta), v)),\end{aligned}$$

where L_T is given by (2.3) *and* (2.4).

Proof. Let $\mathscr{H}, \mathscr{H}_t, \lambda, V$ and j_t^0 be as in Proposition 2.4. Define $F(t)$ to be the projection on the subspace $\mathscr{H}_t$. By Proposition 2.3, $F(t)$ is increasing in t. Define, for any $X \in \mathscr{A}_t$, the operator $j_t(X)$ in $\mathscr{H}$ by

$$j_t(X) = j_t^0(X) F(t) \quad \text{for any } t \geqslant 0.$$

Since j_t^0 is a * unital homomorphism from $\mathscr{A}_t$ into $\mathscr{B}(\mathscr{H}_t)$ and $F(t)$ is a projection it follows that $\|j_t(X)\| \leqslant \|X\|$ and $j_t(I_t) = F(t)$. To check that $j_t(X) j_t(Y) = j_t(XY)$ it is enough to verify this on vectors of the form $\lambda((X_t, X_{t_1}, \ldots, X_{t_n}), u)$. This is immediate from (2.7). Since $j_t^0(X)F(t) = F(t)j_t^0(X)F(t)$ it follows that $j_t(X)^* = j_t(X^*)$.

To prove (ii) it is enough to check that, for $s < t$,

$$\langle \lambda((X_s, X_{s_2}, \ldots, X_{s_m}), u), \quad j_t^0(X)\lambda((Y_s, Y_{s_2}, \ldots, Y_{s_m}), v)\rangle =$$
$$\langle \lambda((X_s, X_{s_2}, \ldots, X_{s_m}), u), \quad \lambda((T(s,t)(X)\, Y_s, Y_{s_2}, \ldots, Y_{s_m}), v)\rangle$$

for all $X \in \mathscr{A}_t$. By definitions the left hand side is equal to

$$\langle \lambda((I_t, X_s, X_{s_2}, \ldots, X_{s_m}), u), \lambda((X, Y_s, Y_{s_2}, \ldots, Y_{s_m}), v)\rangle$$

which, by property (i) in Proposition 2.2 and 2.3, is equal to the right hand side.

(iii) is just a restatement of property (ii) in Proposition 2.2 because

$$j_{t_1}(X_1) \cdots j_{t_n}(X_n)\, Vu = \lambda(X(\sigma), u)$$

with $\sigma = \{t_1, t_2, \ldots, t_n\}$.

The first part of (iv) is contained in the last part of Proposition 2.4. The remaining part of (iv) follows from property (i) in Proposition 2.2. ■

Remark. It is interesting to compare the properties of $\{F(t)\}$ and $(j_t\}$ in Theorem 2.5 with (1.2)–(1.4) in the case of classical Markov processes. This motivates the following definition: suppose $\mathscr{A}_t, \mathscr{K}_t$ and $T(s,t)$, $s \leqslant t$ are as in Theorem 2.5. Then any quadruple $(\mathscr{H}, F, \{j_t\}, V)$ consisting of a Hilbert space $\mathscr{H}$, an increasing family $\{F(t)\}$ of projections in $\mathscr{H}$, contractive * homomorphisms j_t from $\mathscr{A}_t$ into $\mathscr{B}(\mathscr{H})$ and a unitary isomorphism V from $\mathscr{K}_0$ onto the range of $F(0)$ is called a *conservative Markov flow* with transition operators $T(\cdot,\cdot)$ if

$$j_t(I_t) = F(t), \quad F(s)j_t(X)F(s) = j_s(T(s,t)(X)) \text{ for } 0 \leqslant s \leqslant t < \infty$$

and $j_0(X)\, V = VX$ for all $X \in \mathscr{A}_0$, the flow is said to be *minimal* if, in addition, property (iii) of Theorem 2.5 holds. Two such minimal conservative Markov flows $(\mathscr{H}, F, \{j_t\}, V)$ and $(\mathscr{H}', F', \{j_t'\}, V')$ with the same transition operators $T(\cdot,\cdot)$ are called *equivalent* if there exists a unitary isomorphism $W: \mathscr{H} \to \mathscr{H}'$ such that

$$WF(t)\, W^{-1} = F'(t), \quad Wj_t(X)\, W^{-1} = j_t'(X), \quad WV = V'$$

for all $t \geqslant 0$, $X \in \mathscr{A}_t$ [BP], [M].

We shall establish soon that upto equivalence the minimal Markov flow constructed in Theorem 2.5 is unique.

PROPOSITION 2.6.

Let $(\mathscr{H}, F, \{j_t\}, V)$ be a minimal conservative Markov flow with transition operators $T(\cdot,\cdot)$ then the following hold:

(i) *Let $0 \leqslant t_1 < t_2 > t_3 < \infty$. Then for any $X_i \in \mathscr{A}_{t_i}$, $i = 1, 2, 3$*

$$j_{t_1}(X_1)j_{t_2}(X_2)j_{t_3}(X_3) = \begin{cases} j_{t_1}(X_1\, T(t_1,t_2)(X_2))j_{t_3}(X_3) & \text{if} \quad t_1 \geqslant t_3 \\ j_{t_1}(X_1)j_{t_3}(T(t_3,t_2)(X_2)X_3) & \text{if} \quad t_1 < t_3 \end{cases}$$

(ii) *Let $\mathcal{N}$ be the set of all pairs of sequences of the form* $(t_1, t_2, \ldots, t_n; X_1, X_2, \ldots, X_n)$ *where* $0 \leqslant t_1, t_2, \ldots, t_n < \infty$, $X_i \in \mathcal{A}_{t_i}$, $i = 1, 2, \ldots, n, n = 1, 2, \ldots$ *Then there exists a map* $\alpha: \mathcal{N} \to \mathcal{A}_0$ *independent of the Markov flow such that*

$$F(0) j_{t_1}(X_1) j_{t_2}(X_2) \cdots j_{t_n}(X_n) F(0) = j_0(\alpha(\mathbf{t}, \mathbf{X})) \tag{2.8}$$

for all $(\mathbf{t}, \mathbf{X}) = (t_1, t_2, \ldots, t_n; X_1, X_2, \ldots, X_n) \in \mathcal{N}$.

Proof. Let t_1, t_2, t_3 be as in (i) and $t_1 \geqslant t_3$. Then

$$\begin{aligned} & j_{t_1}(X_1) j_{t_2}(X_2) j_{t_3}(X_3) \\ & = j_{t_1}(X_1) F(t_1) j_{t_2}(X_2) F(t_1) j_{t_3}(X_3) \\ & = j_{t_1}(X_1) j_{t_1}(T(t_1, t_2)(X_2)) j_{t_3}(X_3) \\ & = j_{t_1}(X_1\, T(t_1, t_2)(X_2)) j_{t_3}(X_3), \end{aligned}$$

which proves the first part of (i). Its second part is proved in the same manner.

To prove (ii) observe that

$$\begin{aligned} & F(0) j_{t_1}(X_1) j_{t_2}(X_2) \cdots j_{t_n}(X_n) F(0) \\ & = j_0(I_0) j_{t_1}(X_1) j_{t_2}(X_2) \cdots j_{t_n}(X_n) j_0(I_0). \end{aligned} \tag{2.9}$$

Without loss of generality assume that $0 < t_1 < t_2 < \cdots < t_{k-1} > t_k$. Then by (i) the product $j_{t_{k-2}}(X_{k-2}) j_{t_{k-1}}(X_{k-1}) j_{t_k}(X_k)$ can be reduced to a product of size 2 of the form $j_{t_{k-2}}(X'_{k-2}) j_{t_k}(X_k)$ or $j_{t_{k-2}}(X_{k-2}) j_{t_k}(X'_k)$ where the primed operators depend only on $(\mathbf{t}, \mathbf{X})$ and $T(\cdot,\cdot)$ and not on the particular flow under consideration. Thus the n-fold product between the two $j_0(I_0)$'s on the right hand side of (2.9) can be reduced to an $(n-1)$-fold product. A successive reduction of the sequence $(0, t_1, t_2, \ldots, t_n, 0; I_0, X_1, X_2, \ldots, X_n, I_0)$ applying (i) yields in the end an element $\alpha(\mathbf{t}, \mathbf{X})$ satisfying (2.8). ∎

Theorem 2.7. *Let* $\mathcal{A}_t, \mathcal{H}_t, T(s, t)$, $0 \leqslant s \leqslant t < \infty$ *be as in Theorem 2.5. Then any two minimal conservative Markov flows with transition operators* $T(\cdot,\cdot)$ *are equivalent.*

Proof. Let $(\mathcal{H}, F, \{j_t\}, V)$ and $(\mathcal{H}', F', \{j'_t\}, V')$ be two Markov flows satisfying the conditions of the theorem. Suppose that $s_1 > s_2 > \cdots > s_m = 0$, $t_1 > t_2 > \cdots > t_n = 0$, $X_i \in \mathcal{A}_{s_i}$, $Y_j \in \mathcal{A}_{t_j}$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Consider $(\mathbf{r}, \mathbf{Z}) \in \mathcal{N}$ (where $\mathcal{N}$ is as in Proposition 2.6) defined by

$$\begin{aligned} \mathbf{r} &= (s_m, s_{m-1}, \ldots, s_1, t_1, t_2, \ldots, t_n), \\ \mathbf{Z} &= (X^*_m, X^*_{m-1}, \ldots, X^*_1, Y_1, Y_2, \ldots, Y_n). \end{aligned}$$

Since $s_m = t_n = 0$ it follows from Proposition 2.6 that there exists $\alpha(\mathbf{r}, \mathbf{Z}) \in \mathcal{A}_0$ such that

$$\begin{aligned} & j_{s_m}(X^*_m) j_{s_{m-1}}(X^*_{m-1}) \cdots j_{s_1}(X^*_1) j_{t_1}(Y_1) \cdots j_{t_n}(Y_n) = j_0(\alpha(\mathbf{r}, \mathbf{Z})), \\ & j'_{s_m}(X^*_m) j'_{s_{m-1}}(X^*_{m-1}) \cdots j'_{s_1}(X^*_1) j'_{t_1}(Y_1) \cdots j'_{t_n}(Y_n) = j'_0(\alpha(\mathbf{r}, \mathbf{Z})). \end{aligned}$$

Thus for any $u, v \in \mathscr{K}_0$ we have

$$\begin{aligned}&\langle j_{s_1}(X_1)\cdots j_{s_m}(X_m)\,Vu, j_{t_1}(Y_1)\cdots j_{t_n}(Y_n)\,Vv\rangle\\ &\langle j'_{s_1}(X_1)\cdots j'_{s_m}(X'_m)\,V'u, j'_{t_1}(Y_1)\cdots j'_{t_n}(Y_n)\,V'v\rangle\\ &=\langle u, \alpha(\mathbf{r},\mathbf{Z})v\rangle.\end{aligned}$$

From the minimality of the two flows it follows that $\mathscr{H}$ and $\mathscr{H}'$ are spanned by vectors of the form $j_{t_1}(Y_1)\cdots j_{t_n}(Y_n)\,Vu$ and $j'_{t_1}(Y_1)\cdots j'_{t_n}(Y_n)\,V'u$ respectively. Hence there exists a unitary isomorphism $W:\mathscr{H}\to\mathscr{H}'$ such that

$$Wj_{t_1}(Y_1)\cdots j_{t_n}(Y_n)\,Vu = j'_{t_1}(Y_1)\cdots j'_{t_n}(Y_n)\,V'u$$

for all $u\in\mathscr{K}_0$, $t_1>t_2>\cdots>t_n=0$, $Y_i\in\mathscr{A}_{t_i}$, $i=1,2,\ldots,n$. That W is the required isomorphism implementing the equivalence of the two flows is immediate. ■

Remark. Let $(\mathscr{H}, F, \{j_t\}, V)$ be a minimal conservative Markov flow with transition operators $T(\cdot,\cdot)$. Denote by $\mathscr{B}$ and $\mathscr{B}_t$ respectively the C^* algebras generated by $\{j_s(X), X\in\mathscr{A}_s, 0\leqslant s<\infty\}$ and $\{j_s(X), X\in\mathscr{A}_s, 0\leqslant s\leqslant t\}$. By the same arguments as in the proof of Proposition 2.6 it is easy to see that for $t_i\geqslant s$, $i=1,2,\ldots,n$ an expression of the form $F(s)j_{t_1}(X_1)\cdots j_{t_n}(X_n)F(s)$ can be expressed as $j_s(\alpha_s(\mathbf{t},\mathbf{X}))$ where $\alpha_s(\mathbf{t},\mathbf{X})\in\mathscr{A}_s$. In particular the map $\mathbb{E}_{s]}$ defined by

$$\mathbb{E}_{s]}(Z)=F(s)ZF(s),\quad Z\in\mathscr{B}$$

maps $\mathscr{B}$ onto $\mathscr{B}_s$. We may call $\mathbb{E}_{s]}$ the *conditional expectation map* from $\mathscr{B}$ onto $\mathscr{B}_s$. If ρ_0 is a state on $\mathscr{A}_0$ then a state ρ on $\mathscr{B}$ is uniquely determined by

$$\rho(Z)=\rho_0(V^*F(0)ZF(0)\,V),\quad Z\in\mathscr{B}.$$

It is legitimate to call the filtered quantum probability space $(\mathscr{B},\mathscr{B}_t,\rho)$ the Markov process with initial state ρ_0 and transition operators $T(\cdot,\cdot)$.

Let $\mathscr{Z}_t$ denote the centre of $\mathscr{A}_t$ for each t. It is possible that $T(s,t)$ may not map $\mathscr{Z}_t$ into $\mathscr{Z}_s$. In the minimal flow with transition operators $T(\cdot,\cdot)$, the operators $\{j_t(Z), Z\in\mathscr{Z}_t, t\geqslant 0\}$ need not be a commutative family. However, by following an idea in Bhat [B], we shall modify the construction in Proposition 2.4 in order to arrive at a family of * unital homomorphisms $k_t:Z_t\to\mathscr{B}(\mathscr{H})$ so that $\{k_t(Z), Z\in\mathscr{Z}_t, t\geqslant 0\}$ is a commutative family and $j_t(Z)$ is obtained from $k_t(Z)$ by a conditional expectation.

Theorem 2.8. *Let $(\mathscr{H}, F, \{j_t\}, V)$ be as in Theorem 2.5. Then there exists a unique *unital homomorphism $k_t:\mathscr{Z}_t\to\mathscr{B}(\mathscr{H})$ satisfying the following:*

(i) *for any $t_1>t_2>\cdots>t_n=0$, $X_{t_i}\in\mathscr{A}_{t_i}$, $i=1,2,\ldots,n$, $Z\in\mathscr{Z}_t$ and $u\in\mathscr{K}_0$*

$$k_t(Z)\lambda((X_{t_1},X_{t_2},\ldots,X_{t_n}),u)=\begin{cases}\lambda((X_{t_1},X_{t_2},\ldots,X_{t_{i-1}},ZX_{t_i},X_{t_{i+1}},\ldots,X_{t_n}),u)\\ \quad \text{if } t=t_i \text{ for some } i\\ \lambda((Z,X_{t_1},\ldots,X_{t_n}),u)\ \text{ if }\ t>t_1,\\ \lambda((X_{t_1},X_{t_2},\ldots,X_{t_{i-1}},Z,X_{t_i},\ldots,X_{t_n}),u)\\ \quad \text{if } t_{i-1}>t>t_i \text{ for some } i;\end{cases}\tag{2.10}$$

(ii) *the family* $\{k_t(Z), Z \in \mathscr{Z}_t, t \geqslant 0\}$ *is commutative;*
(iii) $j_t(Z) = F(t) k_t(Z) F(t)$ *for all* $t \geqslant 0$, $Z \in \mathscr{Z}_t$.

Proof. As in the proof of Proposition 2.4 consider a unitary element $Z \in \mathscr{Z}_t$. Suppose $t = t_i$ for some $i = 1, 2, \ldots, n$. For any $X_{t_i}, Y_{t_i} \in \mathscr{Z}_{t_i}, i = 1, 2, \ldots, n$ we have

$$\begin{aligned} &\langle \lambda((X_{t_1}, X_{t_2}, \ldots, X_{t_{i-1}}, ZX_{t_i}, X_{t_{i+1}}, \ldots, X_{t_n}), u), \\ &\quad \lambda((Y_{t_1}, Y_{t_2}, \ldots, Y_{t_{i-1}}, ZY_{t_i}, Y_{t_{i+1}}, \ldots, Y_{t_n}), v)\rangle = \\ &\quad \langle u, X_{t_n}^*(\cdots X_{t_i}^* Z^* T(t_i, t_{i-1})(\cdots (X_{t_2}^* T(t_2, t_1)(X_{t_1}^* Y_{t_1}) Y_{t_2}) \cdots) Z Y_{t_i} \cdots) Y_{t_n} v\rangle. \end{aligned}$$

Since Z and $Z^* \in \mathscr{Z}_{t_i}$ and $Z^*Z = 1$ it follows that the right hand side is independent of Z. The same argument in the remaining cases together with the Chapman–Kolmogorov equations for $T(\cdot,\cdot)$ and (iv) in Proposition 2.2 imply that $k_t(Z)$ defined by (2.10) on elements of the form $\lambda(X(\sigma), u)$ is scalar product preserving. Hence $k_t(Z)$ extends to a unitary operator on $\mathscr{H}$. Furthermore for any two unitary elements Z, $Z' \in \mathscr{Z}_t$, we have $k_t(Z) k_t(Z') = k_t(ZZ')$. Once again by (iv) in Proposition 2.2, $k_t(I_t)$ is the identity operator in $\mathscr{H}$. Exactly as in the proof of Proposition 2.4 we extend $k_t(\cdot)$ to a * unital homomorphism from $\mathscr{Z}_t$ into $\mathscr{B}(\mathscr{H})$. This proves (i).

If $t \neq t'$, $Z \in \mathscr{Z}_t$, $Z' \in \mathscr{Z}_{t'}$, it follows from (2.10) by straightforward verification that

$$k_t(Z) k_{t'}(Z') \lambda(X(\sigma), u) = k_{t'}(Z') k_t(Z) \lambda(X(\sigma), u)$$

where $\sigma = \{t_1 > t_2 > \cdots > t_n = 0\}$. This proves (ii).

When $t = t_1 > t_2 > \cdots > t_n$, $X_{t_i}, Y_{t_i} \in \mathscr{A}_{t_i}$, $u, v \in \mathscr{K}_0$ we have

$$\begin{aligned} &\langle \lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u), \quad k_t(Z) \lambda((Y_t, Y_{t_2}, \ldots, Y_{t_n}), v)\rangle \\ &= \langle \lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u), \quad \lambda((ZY_t, Y_{t_2}, \ldots, Y_{t_n}), v)\rangle \\ &= \langle \lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u), \quad j_t(Z) \lambda((Y_t, Y_{t_2}, \ldots, Y_{t_n}), v)\rangle. \end{aligned}$$

Since vectors of the form $\lambda((X_t, X_{t_2}, \ldots, X_{t_n}), u)$ span the range $\mathscr{H}_t$ of $F(t)$, property (iii) is immediate. Uniqueness of $\{k_t\}$ follows from the minimality of $\{j_t\}$ and property (i). ■

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