Indian J. pure appl. Math., 25(3); 345-352, March 1994

EFFECT OF A MAGNETIC FIELD ON BLOOD FLOW THROUGH AN INDENTED TUBE IN THE PRESENCE OF ERYTHROCYTES

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(Received 24 *February* 1993; *after revision* 16 *August* 1993; *accepted* 12 *October* 1993*)*

The aim of the present investigation is to study the effect of an externally applied homogeneous magnetic field on the flow characteristics of blood in a single constricted blood vessel. The necessary theoretical results such as velocity, minute expenditure of the blood, wall shear stress and pressure gradient have been obtained in this analysis. Out of these theoretical results the numerical solutions of wall shear and pressure gradient are shown graphically for better understanding of the problem

INTRODUCTION

Many cardiovascular diseases—particularly artherosclerois (medically called stenosis) which are responsible for the deaths of people, are closely related to the nature of blood movement and the dynamic behaviour of blood vessel. From medical survey it is well known fact that more than eighty percent of the total deaths of people are due to the diseases of blood vessel walls.

The actual reason for formation of stenosis in the lumen of an artery is not known but its effect over the flow characteristics has been studied by many workers. Haldar[1] has given a review of model studies of blood flow through constricted single arteries. Haldar[2] has studied the effects of erythrocytes on the flow characteristics of blood in an indented tube.

In all the above works in the review of Haldar[1] the effect of an externally applied magnetic field is not considered. But it has been reported by Barnothy[3] that the biological systems, in general, are affected by the application of an external magnetic field. In recent years, Rao and Deshikachar[5] have given an excellent reveiw of a good number of works concerning the effect of a magnetic field on the flow characteristics of blood through non-constricted single tubes. Sud and Sekhon[5] have used the finite-element method to analyse the effect of a magnetic field on blood flow through the human arterial system. In the present work, the effect of an eternally applied magnetic filed over the flow characteristics of blood in a single stenosed artery has been considered.

Mathematical Model

Consider steady, laminar and axially symmetric flow of blood through an artery provided with a mild stenosis under the influence of an externally applied hamogeneous magnetic field. The blood flowing in the tube is assumed to be a suspension of red blood cells in plasma. It is also assumed that the density of the fluid is constant but the viscosity varies radially and that the electromagnetic force produced is very small. Under the assumption of small electrical conductivity, the one dimensional equation of motion is

$$\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rx}) + \beta_0^2\,\sigma_e\,u = 0. \qquad \text{... (1)}$$

Here p is the fluid pressure, u the axial velocity component, $\beta_0 = (\mu_e H_0)$ the electro-magnetic induction, μ_e the magnetic permeability, H_0 the intensity of magnetic field and σ_e is the conductivity of fluid.

The shear stress τ_{rx} is given by

$$\tau_{rx} = -\mu(r)\frac{du}{dr} \qquad \text{... (2)}$$

where $\mu(r)$ is the co-efficient of viscosity of blood proposed by Einstein

$$\mu(r) = \mu_0\,[1 + \beta\,h(r)\,]. \qquad \text{... (3)}$$

Here μ_0 is the co-efficient of viscosity of plasma, β is a constant equal to 2.5 and $h(r)$ is the hematocrit described by an empirical formula[4]

$$h(r) = h_m\,[1 - (r/R_0)^n\,] \qquad \text{... (4)}$$

in which R_0 is the radius of normal tube, h_m is the maximum hematocrit at the centre of the tube and $n\,(\geq 2)$ is a parameter determining the exact shape of profile. The relation (4) is valid only for a very dilute suspension of red cells which are supposed to be spherical in shape.

The stenosis develops symmetrically about the tube axis but it is nonsymmetric with respect to radial co-ordinates and its geometry is described by

$$\frac{R(x)}{R_0} = 1 - A\,[\,l_0^{s-1}\,(x-d) - (x-d)^s\,],\ d \leq x \leq d + l_0. \qquad \text{... (5)}$$

where $S(\geq 2)$ is a parameter determining the shape of slenosis. $R(x)$ is the radius of stenosed artery, l_0 is the length of stenosis, d indicates its location and A is given by

$$A = \frac{\varepsilon}{R_0\, l_0^{s}}\ ; \quad \frac{S^{S/(S-1)}}{S-1}. \qquad \text{... (6)}$$

Here ε denotes the maximum height of stenosis at

$$x = d + \frac{l_0}{S^{1/(S-1)}} \qquad \text{... (7)}$$

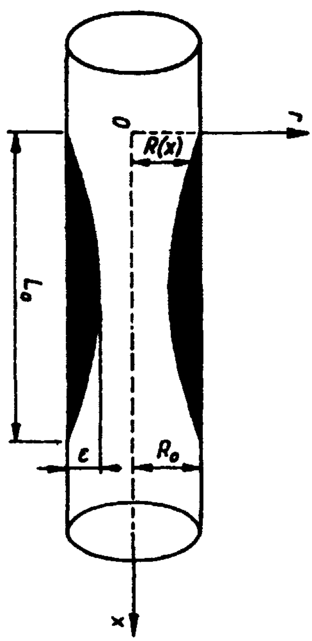

FIG. 1. Geometry of Construction

such that $\varepsilon/R_0 \ll 1$ (Fig.1).

The boundary conditions are

$$u = 0 \quad \text{at} \quad r = R(x) \qquad \text{... (8)}$$

$$\frac{du}{dr} = 0 \quad \text{at } r = 0. \qquad \text{... (9)}$$

Let us next introduce the following transformation.

$$y = r/R_0. \qquad \text{... (10)}$$

Then the governing equation (1) becomes

$$\frac{1}{y}\frac{d}{dy}\left[y\,(a-ky^{n})\frac{du}{dy}\right]-M^{2}u=\frac{R_0^2}{\mu_0}\cdot\frac{d_p}{dx} \qquad \text{... (11)}$$

where

$$k = \beta\, h_m \qquad \text{... (12)}$$

$$a = 1 + k$$

and M is the Hartmann number defined by

$$M = B_0 R_0 (\sigma_e/\mu_0)^{1/2}. \qquad \text{... (13)}$$

The corresponding boundary conditions (8) and (9) take the forms

$$u = 0 \quad \text{at } y = R(x)/R_0 \qquad \text{... (14)}$$

$$\frac{du}{dy} = 0 \quad \text{at } y = 0. \qquad \text{... (15)}$$

Solution of the Problem

Equation (11) can be solved with the help of boundary conditions (14) and (15). In order to solve this equation Frobenius method is applied here. It is required that u is bounded at $y = 0$ and the only admissible series solution of equation (11) is

$$u=D\sum_{m=0}^{\infty} C_m\, y^m+\frac{R_0^2}{4a\,\mu_0}\cdot\frac{dp}{dx}\cdot\sum_{m=0}^{\infty}\overline{C}_m\, y^{m+2} \qquad \text{... (16)}$$

where D is an arbitrary constant to be determined by the boundary condition of the problem. Here C_m and $\overline{C}_m$ involved in this solution are given by

$$C_{m+1}=\frac{k(m+1)\,(m-n+1)\,C_{m-n+1}+M^2 C_{m-1}}{a(m+1)^2} \qquad \text{... (17)}$$

and

$$\overline{C}_{m+1}=\frac{k(m+3)\,(m-n+3)\,\overline{C}_{m-n+1}+M^2\overline{C}_{m-1}}{a(m+3)^2} \qquad \text{... (18)}$$

remembering that C_0 and $\overline{C}_0$ are to be taken as unity and

$$C_{-m}=\overline{C}_{-m} = 0. \qquad \text{... (19)}$$

Applying the boundary condition (14) we have

$$D=-\frac{R_0^2}{4\,\mu_0\,a}\left[\sum_{m=0}^{\infty}\overline{C}_m\;(R/R_0)^{m+2}\right]\Big/\sum_{m=0}^{\infty}C_m\,(R/R_0)^m \qquad \text{... (20)}$$

and the resulting expression for u is

$$u = -\frac{R_0^2}{4\,\mu_0 a}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty} \overline{C}_m (R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty} \overline{C}_m y^m - \sum_{m=0}^{\infty} C_m (R/R_0)^m\cdot\sum_{m=0}^{\infty} \overline{C}_m y^{m+2}\right]\Big/\sum_{m=0}^{\infty} C_m (R/R_0)^m. \quad \text{... (21)}$$

If u_0 be the average velocity given by

$$u_0 = -\frac{R_0^2}{8\,\mu_0}\cdot\left(\frac{dp}{dx}\right)_0 \quad \text{... (22)}$$

where $(dp/dx)_0$ is the pressure gradient of flow in the normal tube in the absense of magnetic field and hematocrit, then the non-dimensional form of u with respect to u_0 is

$$\frac{u}{u_0} = \frac{2}{a}\cdot\frac{(dp/dx)}{(dp/dx)_0}\left[\sum_{m=0}^{\infty} \overline{C}_m (R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty} C_m y^m - \sum_{m=0}^{\infty} C_m (R/R_0)^m\cdot\sum_{m=0}^{\infty} \overline{C}_m y^{m+2}\right]\Big/\sum_{m=0}^{\infty} C_m (R/R_0)^m. \quad \text{... (23)}$$

The volumetric flow rate Q of fluid in the stenotic region is given by

$$Q = 2\pi R_0 \int_0^{R(x)/R_0} u(y)\cdot y\,dy. \quad \text{... (24)}$$

Substituting u from (21) into (24) and then integrating with respect to Y we obtain

$$Q = -\frac{\pi R_0^4}{2\,\mu_0 a}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty} \overline{C}_m (R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty} \frac{C_m}{m+2}(R/R_0)^{m+2} - \sum_{m=0}^{\infty} C_m (R/R_0)^m\cdot\sum_{m=0}^{\infty} \frac{\overline{C}_m}{m+4}(R/R_0)^{m+4}\right]\Big/\sum_{m=0}^{\infty} C_m (R/R_0)^m. \quad \text{... (25)}$$

If Q_0 is the flow rate of plasma fluid in the unconstricted tube in the absence of hematocrit and magnetic field, then

$$Q_0 = \frac{\pi R_0^4}{8\,\mu_0}\cdot\left(\frac{dp}{dx}\right)_0 \quad \text{... (26)}$$

where $(dp/dx)_0$ is the pressure gradient of luid. If the flow is steady and the system is closed, then $(Q/Q_0) = 1$ and we have the relative pressure gradient from (25) and (26) as

$$\frac{(dp/dx)}{(dp/dx)_0}=\frac{a}{4}\,\frac{\sum_{m=0}^{\infty} C_m (R/R_0)^m}{\sum_{m=0}^{\infty} \overline{C}_m (R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty}\frac{C_m}{m+2}\cdot(R/R_0)^{m+2}-\sum_{m=0}^{\infty} C_m (R/R_0)^m\cdot\sum_{m=0}^{\infty}\frac{\overline{C}_m}{m+4}(R/R_0)^{m+4}}. \quad \ldots (27)$$

The wall shear stress is defined by

$$\tau_R=-\left[-\mu(r)\frac{du}{dr}\right]_{r=R(x)} \quad \ldots (28)$$

which, on using (21) gives

$$\tau_R=\frac{R_0}{4^a}\cdot\frac{dp}{dx}\cdot\frac{\sum_{m=0}^{\infty}\overline{C}_m(R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty} m\,C_m(R/R_0)^{m-1}-\sum_{m=0}^{\infty} C_m(R/R_0)^m\cdot\sum_{m=0}^{\infty}(m+2)\overline{C}_m(R/R_0)^{m+1}}{\sum_{m=0}^{\infty} C_m(R/R_0)^m}. \quad \ldots (29)$$

If $\tau_N=-\frac{R_0}{2}(dp/dx)_0$ is the shear stress of plasma fluid at the normal tube wall in the absence of magnetic field, then the non-dimensional form of (29) is

$$\tau=\frac{1}{2a}\cdot\frac{(dp/dx)}{(dp/dx)_0}\cdot\frac{\sum_{m=0}^{\infty} C_m(R/R_0)^m\cdot\sum_{m=0}^{\infty}(m+2)\overline{C}_m(R/R_0)^{m+1}-\sum_{m=0}^{\infty}\overline{C}_m(R/R_0)^{m+2}\cdot\sum_{m=0}^{\infty} m\,\overline{C}_m(R/R_0)^{m-1}}{\sum_{m=0}^{\infty} C_m(R/R_0)^m} \quad \ldots (30)$$

where $(dp/dx)/(dp/dx)_0$ is given by (27).

NUMERICAL DISCUSSIONS

Most of theoretical results such as velocity, volumetric flow rate, wall shear stress and pressure gradient are obtained in this analysis. Out of these results only the numerical solutions of wall shear stress and relative local pressure gradient are shown graphically for different values of hemotacrit and Hartmann number for better understanding of the problem.

Figure 2 shows the variations of relative local pressure gradient along the length of the tube for different h_m and M considering $n = 2$, $(\varepsilon/R_0) = 0.1$, $l_0 = 1$ and $d = 0$. At the begining the pressure gradient increases rapidly and attains its maximum value. After attaining the maximum the rapid downfall of pressure gradient is observed.

It is also observed that for a particular value of x the pressure gradient increases for both h_m and M. Such increase in the relative pressure gradient indicates the rise

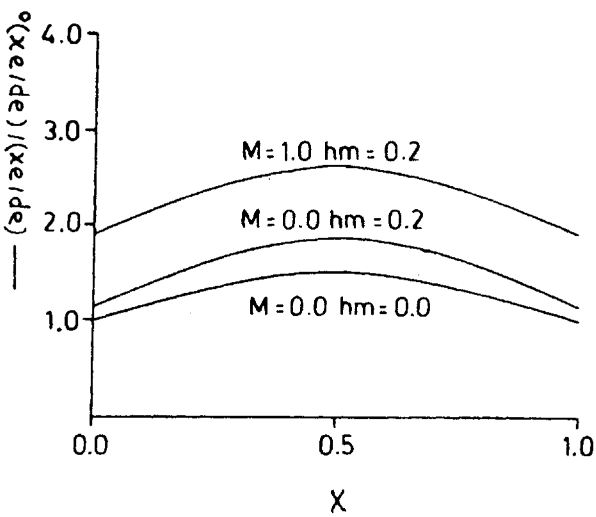

FIG. 2. Variations of local pressure gradient along the length of the tube for different values of b_m and M

in systolic pressure and fall in diastolic pressure which are unfavourable for the diseased and weak heart. But it is interesting to note that the rise for M is much more significant than that for h_m. So it can be concluded that the effect of magnetic field is more destructive for diseased cardiovascular system.

The variations of wall shear stress along the tube length are shown for different numerical values of h_m and M (Fig. 3). From the figure it is seen that for a fixed value of x the rise of wall shear stress is significant for h_m and M. The increase in shear stress indicates the flattening of velocity profile at the centre of the tube and

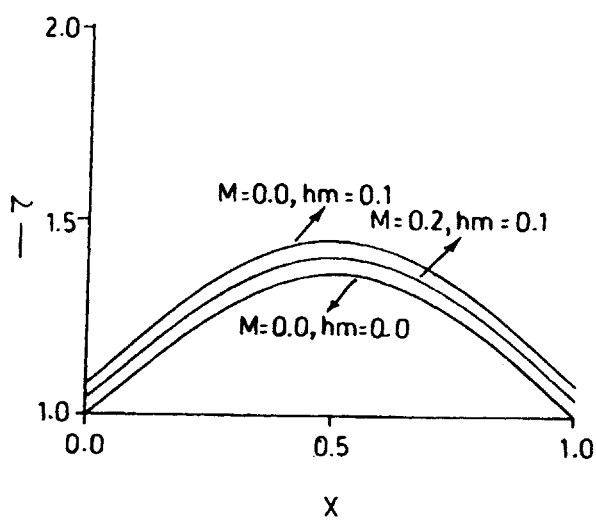

FIG. 3. Variations of wall shear stress along the length of the tube for different values of b_m and M

this flattering is much more pronounced at the centre of the higher values of h_m and M. Again, it is also observed that the stress for M increases more rapidly than that for h_m indicating the possibility of breaking off stenosis and the ultimate result can be paralysis or sudden death.

From the above numerical discussions the following conclusions can be drawn :

(i) The pressure gradient increases for both hematocrit and magnetic field but the increase for magnetic field is more significant than that for hematocrit.

(ii) The increase in pressure gradient indicates the fall in diastolic pressure and rise in systolic pressure. This rise in systolic pressure is destructive for the diseased and weak heart.

(iii) The effects of magnetic field and hematocrit control the velocity and point out the flattening of velocity profile at the central region of the tube.

(iv) The wall shear stress increases with the increase of the strength of magnetic field. At higher magnetic field it increases so significantly that it may cause the stenosis to break off and the ultimate result is paralysis or sudden death.

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