Effects of ion and electron drifts on large amplitude solitary waves in a relativistic plasma

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(Received 24 June 1997; accepted 12 September 1997)

The effect of ion and electron drifts on the existence of arbitrary amplitude solitary waves is studied using Sagdeev's pseudopotential method. It is found that if the electron drift velocity u_0 is finite, solitary waves may exist for relatively large values of v_0/c, where v_0 is the ion drift velocity and c is the velocity of light.

I. INTRODUCTION

When the speeds of particles are comparable to those of light, relativistic effects play an important part in the formation of solitary waves. For example, ions with very high speed are frequently observed in the solar atmosphere and interplanetary space. High energy ion beams also occur in the plasma sheet boundary layers of the Earth's atmosphere and in the Van Allen radiation belts.[1,2] Although there have been some experimental[3–6] and numerical[7,8] studies in this field, comparatively few theoretical works exist on this subject. During the last two decades or so, some authors have studied ion acoustic solitary waves in relativistic plasmas.[9–13] It was Roychoudhury and Bhattacharyya[14] who first found the exact pseudopotential for a relativistic plasma. They, however, neglected the electron inertia. Later Kuehl and Zhang[15] showed that electron inertia restricts the region for existence of solitary waves, and because of this restriction relativistic effects on solitary waves are negligible. Recently, Chatterjee and Roychoudhury[16] extended their results in the case of warm ions and found that finite ion temperature further restricts the region of existence of solitary waves. They, however, like Kuehl and Zhang,[15] neglected electron drift. Relativistic effects on formation of solitons were studied by several authors using the reduction perturbation technique. Unfortunately, as pointed out by Kuehl and Zhang,[15] most of these authors considered values of v_0/c too large to be permitted by analytical constraints if one neglects electron drift effect. Very recently Kalita *et al.*[17] studied weakly relativistic solitons in a cold plasma with electron inertia. Unfortunately, their work also has some serious shortcomings. They are:

(1) Their analysis is valid for low amplitude solitons only, yet they give results for amplitudes of order unity or more.
(2) They considered only weak relativistic effects, but values taken for v_0/c (0.2–0.5) are rather too large to be weakly relativistic. Also, they have normalized the velocities twice. This means c, the velocity of light, is normalized to c_s, the ion acoustic speed. So any numerical value of c is related to the electron temperature, which is not mentioned in their work. When neglecting terms of order higher than $(v_0/c)^2$ it is not proper to take into account the extremely small difference $u_0/c - v_0/c \sim 10^{-7}$.
(3) In the limit $u_0 \to 0$ and $v_0 \to 0$ their results do not reproduce the well-known nonrelativistic results. In fact, even in the limit $u_0 \to 0$, their results do not reproduce the small amplitude relativistic results of Kuehl and Zhang.[15]

To overcome these drawbacks and to study large amplitude solitary waves, we use here Sagdeev's pseudopotential approach to study the effects of ion and electron drifts on the ion acoustic solitary waves.

We found, somewhat surprisingly, that it was easier to deal with the exact relativistic terms than to use weak relativistic assumptions. Our work can be considered an extension of the work of Kuehl and Zhang[15] and reproduces their result (apart from a misprint in their paper) in the limit $u_0 \to 0$. However, our conclusion is very much different from that of Ref. 15. We found that the presence of the u_0 term allows rather large values of v_0/c and u_0/c for solitary waves. We found an analytic constraint on the soliton velocity in terms of the electron inertia parameter $\mu(=m_e/m_i)$. The plan of the paper is as follows.

In Sec. II we write down the basic equations and derive the pseudopotential ψ. In Sec. III we discuss the conditions for existence of solitons. In Sec. IV a small amplitude expansion of ψ is given. Our results are also compared with those of Ref. 15, in the limit $u_0 \to 0$. Section V contains discussion and conclusion.

II. BASIC EQUATIONS AND PSEUDOPOTENTIAL APPROACH

The system of equations governing the plasma in unidirectional motion is given by

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v_\alpha = -\frac{\partial \phi}{\partial x}, \tag{2}$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u) = 0, \tag{3}$$

$$\mu\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u_\alpha=\left(\frac{\partial\phi}{\partial n}-\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right), \tag{4}$$

$$\frac{\partial^2\phi}{\partial x^2}=n_e-n, \tag{5}$$

where $v_\alpha=\gamma v$, $u_\alpha=\gamma_e u$, and

$$\gamma=(1-v^2/c^2)^{-1/2},\quad \gamma_e=(1-u^2/c^2)^{-1/2}.$$

Here, n, n_e are the ion and electron densities, respectively, normalized to n_{e0}, the unperturbed electron density.

Also, $\mu=m_e/m_i$, m_e,m_i being the electron and ion masses, respectively, v, u, and c are the ion electron fluid velocities and the velocity of light, respectively.

The velocities are normalized to the ion acoustic speed $c_s=(kT_e/m_i)^{1/2}$, where k is the Boltzmann's constant and T_e is the electron temperature. The potential ϕ is normalized to kT_e, e being the electron charge. The spatial coordinate x is normalized to the Debye length, while time t is normalized to the ion plasma period.

To obtain a solitary wave solution, one makes the dependent variables depend on a single independent variable, say $\xi=x-Vt$, where V is the velocity of the solitary waves.

After writing Eqs. (1)–(5) in terms of ξ, one gets ordinary differential equations, which can be solved easily. The integration of the above equations yield the following:

$$n=(V-v_0)/(V-v), \tag{6}$$

$$\phi=(Vu-c^2)\gamma-(Vu_0-c^2)\gamma^0, \tag{7}$$

$$n_e=(V-u_0)/(V-u), \tag{8}$$

$$\phi=\ln\left(\frac{V-u_0}{V-u}\right)+\mu\gamma_e(c^2-Vu)-\mu\gamma_e^0(c^2-Vu_0), \tag{9}$$

where $\gamma^0=(1-v_0/c^2)^{-1/2}$ and $\gamma_e^0=(1-u_0/c^2)^{-1/2}$.

In deriving Eqs. (6)–(9), we have used the following boundary conditions:

$$v\to v_0,\quad u\to u_0,\quad \phi\to 0,\ n\to 1,\ n_e\to 1 \text{ as } \xi\to\infty.$$

To obtain the pseudopotential ψ, we notice that Eq. (5) can be written as

$$\frac{d^2\phi}{d\xi^2}=-\frac{\partial\psi}{\partial\phi}, \tag{10}$$

where $\psi(\phi)$, the pseudopotential, can be written as

$$\psi(\phi)=\psi_i(\phi)+\psi_e(\phi). \tag{11}$$

From Eq. (5) it follows that

$$\psi_i(\phi)=\int n\,d\phi, \tag{12}$$

$$\psi_e(\phi)=-\int n_e\,d\phi. \tag{13}$$

Since n and n_e are explicit functions of v and u, respectively, integrations (12) and (13) are evaluated with respect to u and v, using the functional form of ϕ as given in Eq. (7) and (9). After a few algebraic steps we obtain:

$$\psi_i=(V-v_0)(v\gamma-v_0\gamma^0), \tag{14}$$

$$\psi_e=(V-u_0)\mu u\gamma_e-\frac{(V-u_0)}{V-u}+1-(V-u_0)\mu u_0\gamma_e^0. \tag{15}$$

In the limit $u_0\to 0$, our results completely agree with those of Ref. 7.

III. SOLITARY WAVE SOLUTION

Existence of soliton-like solutions of Eq. (10) will be determined by the form of the pseudopotential ψ. The conditions for the existence of soliton are

$$\left.\frac{\partial^2\psi}{\partial\phi^2}\right|_{\phi=0}<0 \tag{16}$$

and

$$\psi(\phi_m)>0, \tag{16b}$$

where ϕ_m is the value of ϕ, beyond which $\psi(\phi)$ becomes complex. Condition (16) is the condition for existence of a potential well. Condition (16b) ensures that a particle moving in a pseudopotential well will be reflected back at $\phi=\phi_c$, where ϕ_c is the value at which $\psi(\phi)$ cuts the ϕ axis from below.

Now Eq. (7) can be inverted and v, u can be written in terms of ϕ. The expression for v is

$$v=\frac{V-(\phi'-A_0)\sqrt{V^2-c^2+(\phi'-A_0)^2c^2}}{V^2/c^2+(\phi'-A_0)^2}, \tag{17}$$

where

$$A_0=\gamma^0(1-Vu_0/c^2) \tag{18}$$

and

$$\phi'=\phi/c. \tag{19}$$

It is seen from Eq. (17) that

$$\phi_m\equiv(c^2-Vu_0)\gamma^0-c^2(1-V^2/c^2)^{1/2}. \tag{20}$$

Again, using the results

$$\frac{\partial\psi}{\partial\phi}=n-n_e \tag{21}$$

and

$$\frac{\partial u}{\partial\phi}=\frac{V-u}{1-\mu\gamma_e^3(V-u)^2}, \tag{22}$$

one finds that

$$\left.\frac{\partial^2\psi}{\partial\phi^2}\right|_{\phi=0}=\frac{(\gamma^0)^{-3}}{(V-v_0)^2}-\frac{1}{1-\mu(\gamma_e^0)^3(V-u_0)^2}. \tag{23}$$

Hence, condition (16) reduces to

$$(V-v_0)^2>(\gamma^0)^{-3}(1-\mu(\gamma_e^0)^3(V-u_0)^2). \tag{24}$$

If one takes $v_0\sim u_0$ then Eq. (24) can be simplified to give

$$(V-v_0)>\frac{(\gamma^0)^{-3/2}}{\{(1+\mu(\gamma_e^0)^3(\gamma^0)^{-3}\}^{1/2}}. \tag{25}$$

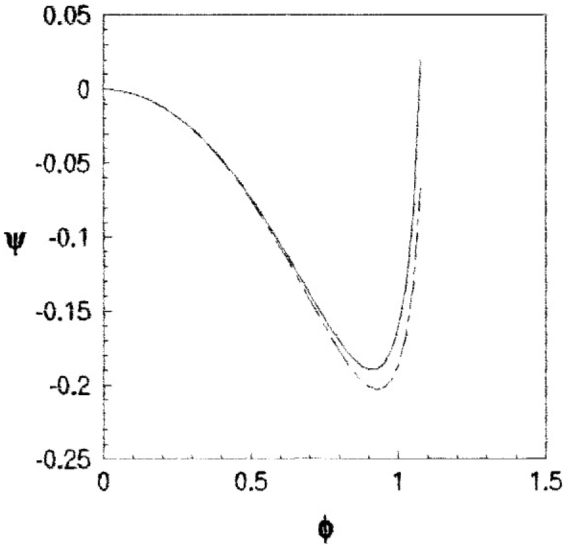

FIG. 1. Plot of ψ against ϕ. The solid line is for $V=51.47$ and the broken line is for $V=51.48$. In both cases $v_0=u_0=50$.

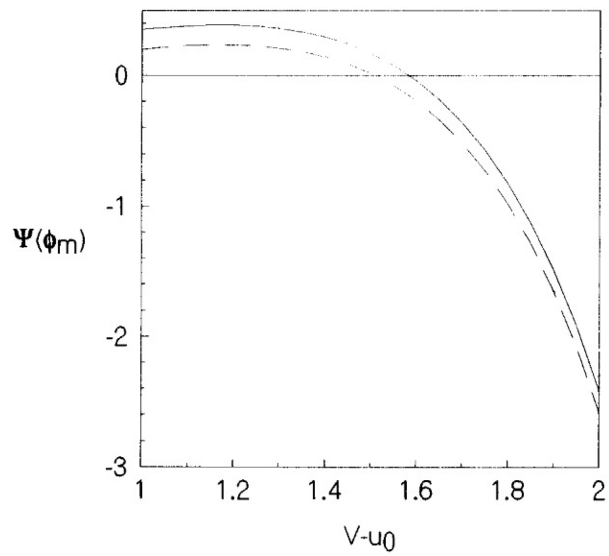

FIG. 2. Plot of $\psi(\phi_m)$ against $V-u_0$. Here ϕ_m is the value of ϕ beyond which $\psi(\phi)$ becomes complex. The solid line is for $u_0=40$, the broken line is for $u_0=50$.

If one neglects relativistic effects, then one gets from Eq. (25)

$$(V-v_0)>\frac{1}{\sqrt{1+\mu}}. \tag{26}$$

Hence, for finite μ one gets solitary waves even for $V-u_0<1$ but

$$>\frac{1}{\sqrt{1+\mu}}.$$

Also the poles of Eq. (23) occur at

$$V-u_0=\pm\frac{1}{\sqrt{\mu}}. \tag{27}$$

Hence

$$V-v_0=\pm\frac{1}{\sqrt{\mu}}+(u_0-v_0). \tag{28}$$

This shows that if $u_0\simeq v_0$, there is no restriction on v_0/c and u_0/c for occurrence of solitary waves (apart from the obvious restrictions that $v_0/c<1, u_0/c<$). This result differs from that of Ref. 7 and is due to the presence of electron drift. To show the limiting value of V for relatively large values of v_0 and u_0, we have plotted in Fig. 1 $\psi(\phi)$ against ϕ for $v_0=u_0=50$ and $V=51.47$ and 51.48, taking $c/c_s=1000$, which corresponds to $v_0/c=0.05$, $v_{Te}/c(=(kT_e/m_e)^{1/2})=0.043$.

It is seen that for $v<51.48$, a compressive soliton solution exists even for $v_0/c>v_{Te}/c$. In fact a soliton solution exists for higher values of v_0/c also. But we emphasize here that for such large values of v_0/c the weak relativistic effect assumption breaks down and one must find the exact pseudopotential, without the weak relativistic assumption, to study the effects of large ion and electron drifts.

To test the inequality $\psi(\phi_m)>0$ numerically, we have plotted ϕ_m against $V-u_0$ in Fig. 2. The solid line is for line $v_0=u_0=40$ and the dotted line is for $v_0=u_0=50$.

This shows that there exist critical values of $V-u_0$ beyond which $\psi(\phi_m)$ becomes negative. The solitary wave solution would not exist if $V-u_0$ exceeds these values.

IV. SMALL AMPLITUDE SOLITARY WAVES

To obtain small amplitude solitary waves that can be compared to the solution of the KdV equation, we expand $\psi(\phi)$ in power series. Neglecting terms of order $0(\phi^3)$, and noting that

$$\psi(\phi)=\frac{\partial\psi}{\partial\phi}=0 \quad \text{at } \phi=0$$

we can write

$$\frac{d^2\phi}{d\xi^2}=-\frac{\partial\psi}{\partial\phi}=a\phi-b\phi^2, \tag{29}$$

where

$$a=-\left.\frac{\partial^2\psi}{\partial\phi^2}\right|_{\phi=0} \quad \text{and } b=\left.\frac{1}{2}\frac{\partial^3\psi}{\partial\phi^3}\right|_{\phi=0}. \tag{30}$$

From Eq.(23) we have

$$a=-\frac{(\gamma^0)^3}{(V-v_0)^2}+\frac{1}{1-\mu(\gamma_e^0)^3(V-u_0)^2}. \tag{31}$$

Also

$$b=\left.\frac{1}{2}\frac{\partial^3\psi}{\partial\phi^3}\right|_{\phi=0}=\frac{3}{2}\frac{(1-Vv_0/c^2)(\gamma^0)^{-4}}{(V-v_0)^4}$$
$$-\frac{1}{2}\frac{[1-3\mu(\gamma_e^0)^5(V-u_0)^2(1-Vu_0/c^2)]}{(1-\mu(V-u_0)^2(\gamma_e^0)^3)^3}. \tag{32}$$

To compare with the result of the reduction perturbation technique, we put $V=\lambda+dV$, where dV is small compared to λ.

Then a can be written as

$$a=a_0+a_1,$$

where

$$a_0=-\frac{(\gamma^0)^3}{(\lambda-v_0)^2}+\frac{1}{1-\mu(\gamma_e^0)^3(\lambda-v_0)^2} \tag{33}$$

and

$$a_1=\left[\frac{2(\gamma^0)^3}{(1-v_0)^3}+\frac{2\mu(\gamma_c^0)^3(\lambda-u_0)}{(1-\mu(\gamma_e^0)^3(\lambda-u_0)^2)^3}\right]dV. \tag{34}$$

If λ is the linear ion acoustic velocity,[15,17] then a_0 is identically zero. This value of a agrees with that of Ref. 15. Finally, the soliton solution for small amplitude soliton is given by

$$\phi=\frac{3a}{2b}\,\mathrm{sech}^2(3/\delta),$$

where $\delta=2/\sqrt{a}$ is the width and $3a/2b$ is the amplitude of the soliton.

V. DISCUSSION AND CONCLUSION

In this paper we have studied the effect of ion and electron drifts on the existence of solitary waves using Sagdeev's pseudopotential approach. It is found that, with finite electron drift, the restriction on the value of v_0/c for the existence of solitary wave solutions, as found in Ref. 15, goes away. However, for $u_0\rightarrow 0$, our results reduce to the results of Ref. 15. But, our results for small ϕ disagree with the small amplitude results of Kalita *et al.*[17] Our approach is valid for arbitrary amplitude solitary waves and for relatively large values of v_0/c.

ACKNOWLEDGMENTS

R. R. and S. K. V. are indebted to Department of Science and Technology, (DST), Government of India for financial support. C. D. is grateful to Council of Scientific and Industrial Research (CSIR), India, for financial help.

[1] C. Grable, J. Geophys. Res. **94**, 299 (1989).
[2] J. I. Vette, *Summary of Particles in the Magnetosphere* (Reidel, Dordrecht, 1970), p. 305.
[3] M. Ashour-Abdulla, J. N. Leboeuf, T. Tajima, J. M. Dawson, and C. F. Kennel, Phys. Rev. **23**, 1906 (1981).
[4] E. Graybill and J. K. Uglam, J. Appl. Phys. **41**, 236 (1981).
[5] J. Rander, B. Ecker, G. Younas, and D. J. Drickes, Phys. Rev. Lett. **24**, 283 (1970).
[6] S. J. Stephanakis, D. Mosher, G. Cooperatein, J. R. Boller, J. Golden, and A. S. Goldstein, Phys. Rev. Lett. **37**, 1543 (1976).
[7] R. C. Davidson, N. A. Krall, K. Papadopoulas, and R. Shanny, Phys. Rev. Lett. **24**, 579 (1970).
[8] B. Lembege and J. M. Dawson, Phys. Fluids B **15**, 1001 (1989).
[9] G. C. Das and S. N. Paul, Phys. Fluids **28**, 823 (1985).
[10] Y. Nejoh, J. Plasma Phys. **37**, 487 (1987).
[11] S. Sing and R. P. Dahiya, Phys. Fluids B **2**, 901 (1990).
[12] G. Pakira, A. Roychoudhury, and S. N. Paul, J. Plasma Phys. **40**, 359 (1988).
[13] G. C. Das, B. Karmakar, and S. N. Paul, IEEE Trans. Plasma Sci. **PS-16**, 22 (1988).
[14] R. K. Roychoudhury and S. Bhattacharyya, Phys. Fluids **30**, 2582 (1987).
[15] H. H. Kuehl and C. Y. Zhang, Phys. Fluids B **3**, 555 (1991).
[16] P. Chatterjee and R. Roychoudhury, Phys. Plasmas **1**, 2148 (1994).
[17] B. C. Kalita, S. N. Barman, and G. Goswami, Phys. Plasmas **3**, 45 (1996).