Fermion doubling on a lattice and topological aspects of chiral anomaly

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The problem of fermion doubling on a lattice has been discussed here from the specific geometrical properties of a lattice structure and topological aspects of chiral anomaly. It is argued that there cannot be chiral anomaly on a lattice and as such there cannot be any conserved charge. This unveils the root cause of fermion doubling, and the unwanted fermions just reflect the geometrical properties of a lattice and may be viewed as to represent the ''fictitious'' chiral spinors associated with the lattice structure which make chiral fermions anomaly free.

I. INTRODUCTION

As is well known, there appears an equal number of species of left- and right-handed Weyl fermions for a general class of lattice fermion theories for each combination of quantum numbers. Indeed, a theorem by Nielsen and Ninomiya[1] states that a space cubic lattice, with a bilinear Hamiltonian which is local, Hermitian, translation invariant and with bilinear locally defined conserved charges, has fermions appearing in pairs with opposite chirality and the same internal quantum numbers. In view of this, a lattice fermion formulation without species doubling and with explicit chiral symmetry appears to be impossible.

Several attempts have been made to eliminate the unwanted fermions. Wilson[2] proposed a way to remove these extra particles by giving them a mass of the order of the cutoff so that we have just one fermion with a relativistic spectrum in the continuum limit. The disadvantage of using Wilson fermion is that chiral symmetry is not an exact symmetry. Even for vanishing mass, the fermion lattice action with the Wilson term is not chiral invariant. Thus the use of Wilson fermions in a lattice formulation of field theories where chiral invariance is supposed to play an important role is not possible. Susskind[3] has proposed a lattice fermion formulation where the fermion field function has just only one component instead of all four components defined on a site of the lattice. This solves part of the naive degeneracy and one has a discrete γ_5 invariance. However, continuous chiral transformation cannot be defined. Moreover, because of the fact that there is likely no Goldstone theorem for discrete symmetries the pion mass comes out too high in strong coupling calculations.[4] Thus for the Susskind approach the problem of continuous chiral symmetry and associated Goldstone boson does remain. Drell *et al.*[5] proposed a model, known as SLAC fermions, where the condition of locality has been abandoned. In this scheme an infinite jump propagator has been suggested to solve the fermion doubling problem. This implies a nonlocal lattice action which contains products of fields arbitrarily far apart. As a consequence, in a gauge theory with SLAC fermions one gets nonlocal and noncovariant contributions in the continuum limit. Nielsen and Ninomiya[6] have constructed a model with only one two-component fermion on a lattice, dropping the assumption of the existence of a conserved charge. In this scheme the fermion field is taken to be real. One can assign to the field components charges that are not conserved at the scale of the fundamental lattice and only approximately conserved in the low-energy regime.

Karsten and Smit[7] have pointed out that the presence of doublers is related to the fact that the axial currents are necessarily nonanomalous on a lattice and one has more fermions canceling the anomaly. Recently Creutz and Tytgat[8] have noted that the problem of species doubling is not merely a particular property of the lattice gauge theory. Rather, it is more general in the sense that a similar phenomenon occurs when we have gauge fields coupled to the chiral currents from an effective Lagrangian for pseudoscalar mesons. The problem is intricately related to the axial anomaly. Thus the issue reduces to nonperturbatively removing the extra species when the original theory is made anomaly free. In this note we shall argue that the very geometrical aspect of a lattice space structure does not allow the anomaly to exist on a lattice as the anomaly is cancelled when the topological aspect of chiral anomaly is considered in the background of this lattice geometry. This bears the seed to remove the problem of fermion doubling on a lattice nonperturbatively.

II. FERMIONS ON A LATTICE

In this section we follow Karsten and Smit[7] to show degeneracy of fermions on a lattice. Let us consider a four-dimensional Euclidean hypercubic lattice. The lattice spacing is a and the lattice points are labeled with

$$x_\mu = n_\mu a, \quad n_\mu = 0, \pm 1, \pm 2, \ldots, \quad \mu = 1,2,3,4. \tag{1}$$

The range of the momenta is restricted to the interval

$$-\pi/a < k_\mu < \pi/a. \tag{2}$$

To find a lattice version of the continuum action for a free spin $\frac{1}{2}$ fermion field ψ,

$$S = \int d^4x \left[i \sum_\mu \frac{1}{2} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{\partial}_\mu \psi(x) - m \bar{\psi}(x) \psi(x) \right], \tag{3}$$

we have to replace the differentials by differences. Thus the lattice fermion action takes the form

$$S = \sum \left(\sum_\mu \frac{i}{2a} [\bar{\psi}(x) \gamma_\mu \psi(x + a_\mu) - \bar{\psi}(x + a_\mu) \gamma_\mu \psi(x)] - m \bar{\psi}(x) \psi(x) \right). \tag{4}$$

Here $\Sigma = a^4 \Sigma_n$ and a_μ is a vector along the μ direction with length a. The action (4) has a global $U(1)$ invariance $\hat{\psi}(x) = V\psi(x)$ and $\hat{\bar{\psi}}(x) = \bar{\psi}(x) V^{-1}$, $V \in U(1)$, which is made local by introducing a gauge field $U_\mu(x)$ defined on links $(x, x + a_\mu)$. Thus we write

$$S = \sum \left\{ \sum \frac{i}{2a} [\bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + a_\mu) - \bar{\psi}(x + a_\mu) \gamma_\mu U_\mu(x) \psi(x)] - m \bar{\psi}(x) \psi(x) \right\} + S(U). \tag{5}$$

Here U transforms as follows:

$$\hat{U}_\mu(x) = V(x) U_\mu(x) V^{-1}(x + a_\mu). \tag{6}$$

For weak fields we can introduce a vector potential $v_\mu(x)$ by defining

$$U_\mu(x) = \exp[igav_\mu(x)]. \tag{7}$$

Inserting this definition in (5) we can derive Feynman rules by Fourier transformation. The fermion propagator is

$$S(p)=\left(\sum_{\mu}\gamma_{\mu}\frac{1}{a}\sin p_{\mu}a+m\right)^{-1}. \tag{8}$$

The $\bar{\psi}\psi v_{\mu}$ vertex function is

$$g\gamma_{\mu}\cos\tfrac{1}{2}(p-q)_{\mu}a,\quad p+q+k=0. \tag{9}$$

It can be shown that Eq. (5) effectively describes 16 fermions with degenerate mass m and charge g. It is noted that we have a 2^d degeneracy of fermions in d-dimensions. We can shift the range of mementa to $\pi/a<p_{\mu}<3\pi/a$ using periodicity $2\pi/a$. The fermion propagator $S(p)$ does not vanish in the limit $a\to 0$ in 16 regions in momentum space about the points $p_{\mu}=0$ or π/a. Let us denote such points by $\bar{p}$. The action (5) is invariant under a group of 16 symmetry transformations,

$$\hat{\psi}(x)=T\psi(x),\quad \hat{\bar{\psi}}(x)=\bar{\psi}(x)T^{-1}, \tag{10}$$

where $T=1$, and $\gamma_{\mu}\gamma_5(-1)^{x_{\mu}/a}$. The transformation $T=\gamma_1\gamma_5(-1)^{x_1/a}$ shifts p_1 to $p_1+\pi/a$ (modulo $2\pi/a$), and $\bar{p}=(0,0,0,0)$ is transformed into $\bar{p}=(\pi/a,0,0,0)$ and vice versa, $\bar{p}=(0,\pi/a,0,0)$ into $\bar{p}=(\pi/a,\pi/a,0,0)$ and so on. Now from a study of the propagator $S(p)$ about a point p we have around each point $\bar{p}$ all the states of a free Dirac particle with mass m, 16 particles in total. Wilson removed these extra particles by giving them a mass of the order of the cutoff. He adds an extra term

$$\frac{1}{2a}\sum\left[\bar{\psi}(x)U_{\mu}(x)\psi(x+a_{\mu})+\bar{\psi}(x+a_{\mu})U^{\dagger}_{\mu}(x)\psi(x)-2\bar{\psi}(x)\psi(x)\right] \tag{11}$$

to the action (5). The fermion propagator now becomes

$$S(p)=\left[\sum_{\mu}\gamma_{\mu}\frac{1}{a}\sin p_{\mu}a+m+\frac{1}{a}\sum_{\mu}(1-\cos p_{\mu}a)\right]^{-1}. \tag{12}$$

For small p_{μ}, the extra term is $O(a)$. This extra term is a momentum-dependent mass term and gives 15 fermions a mass $m+k2/a$ ($k=1$, 2, 3, or 4). We have just one fermion with a relativistic spectrum in the continuum limit $a\to 0$. However, this extra term breaks chiral symmetry.

In the foregoing it is tacitly assumed that all four components of ψ are defined in a lattice. Susskind has proposed a lattice fermion formulation where $\psi(x)$ is a one-component field. This solves part of the naive degeneracy. However, in this case one has a discrete γ_5 invariance as a continuous chiral transformation cannot be defined. An infinite jump propagator $[O(1/a)]$ has been proposed by Drell *et al.*,[5] known as SLAC propagator, which is characterized by the property that p_{μ} is defined in the range $-\pi/a<p_{\mu}<\pi/a$ with a period of $2\pi/a$ and has a gap at $p_{\mu}=\pi/a$ with width $2\pi/a$. This gap implies a nonlocal lattice action and in the continuum limit $a\to 0$, one has a nonlocal and Lorentz noncovariant contribution.

Karsten and Smit[7] have shown that these 16 fermions have the interesting properties that we have 8 particles with chiral charge 1 and another 8 particles with chiral charge -1 so that the fermion content of the theory is anomaly free. Thus we find that in a lattice formulation one has to give up something: explicit chiral symmetry (Wilson) or locality (SLAC fermions) or one has extra fermions canceling the anomaly.

III. LATTICE FERMIONS AND TOPOLOGICAL ASPECTS OF CHIRAL ANOMALY

We here observe that the discretization of space in a lattice can be achieved from Minkowski space–time by a Lorentz symmetry breaking condition as well as by the q-deformation of the Lorentz group.[9] The Lorentz symmetry is broken when we consider the motion of a particle in an anisotropic space. Indeed, if we consider that in three-space dimension the components of the linear momentum satisfy a commutation relation of the form

$$[p_i, p_j]=i\mu\epsilon_{ijk}\frac{x^k}{r^3}, \tag{13}$$

this corresponds to an axisymmetric system where the anisotropy is introduced along a particular direction. The motion of a particle in such a space is analogous to the motion of a charged particle in the field of a magnetic monopole where μ corresponds to the monopole strength. The angular momentum operator $\mathbf{J}$ is given by

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}-\mu\mathbf{r}, \quad \mu=0,\pm 1/2,\pm 1,\pm 3/2,\ldots, \tag{14}$$

and the eigenvalue of J^2 is a conserved quantity in this space instead of the eigenvalue of L^2. In Minkowski space–time the effect of anisotropy can be incorporated, retaining relativistic covariance if we consider that a ''direction vector'' ξ_μ is introduced at each space–time point x_μ so that in a complexified space–time the coordinate is given by $z_\mu=x_\mu+i\xi_\mu$. In an earlier paper[10] it has been shown that the quantization of a fermion can be achieved when we introduce an anisotropy in the internal space so that the internal variable appears as a ''direction vector'' attached to a space–time point. The opposite orientations of the ''direction vector'' correspond to a particle and antiparticle. In the complexified space–time having the coordinate $z_\mu=x_\mu+i\xi_\mu$, a fermion (antifermion) is characterized by the domain such that ξ_μ belongs to the interior of the forward (backward) light cone and as such represents the upper (lower) half-plane.[11] In such a space one should take into account the polar coordinates r,θ,ϕ along with the angle χ specifying the rotational orientation around the ''direction vector'' ξ_μ. The eigenvalue of the operator $i\partial/\partial\chi$ given by μ just corresponds to the ''internal helicity.'' This disconnected and anisotropic nature of space indicates that the behavior of the angular momentum operator in such a region is similar to that of a charged particle moving in the field of a magnetic monopole. The spherical harmonics incorporating the term μ have been extensively studied by Fierz[12] and Hurst.[13] In fact, we have

$$Y_l^{m,\mu}=(1+x)^{-(m-\mu)/2}(1-x)^{-(m+\mu)/2}\frac{d^{l-m}}{d^{l-m}x}[(1+x)^{l-m}(1-x)^{l+\mu}]e^{im\phi}e^{-i\mu\chi}, \tag{15}$$

where $x=\cos\theta$. The quantities m and μ just represent the eigenvalues of the operators $i\partial/\partial\phi$ and $i\partial/\partial\chi$, respectively. It is noted that in such a space we can have half-orbital angular momentum ($l=1/2$) with $m=\pm 1/2$ and $\mu=\pm 1/2$. This is found to be analogous to the result that a monopole-charged particle composite representing a dyon satisfying the condition $gq=\frac{1}{2}$ has its angular momentum shifted by $\frac{1}{2}$ unit and its statistics shift accordingly.[14] This suggests that a fermion can be viewed as a scalar particle moving with $l=\frac{1}{2}$ in such a space.

Now we point out that the overall space where a ''direction vector'' (vortex line) is attached to a space–time point effectively leads to the discretization of space depicting a lattice structure as any two space points cannot come close together within an infinitesimal distance due to the vortex lines. Indeed the ''direction vector'' (vortex line) may be associated with the l_z-value of a particle moving in such a space with $l=1/2$, and the specification of l_z-value for particle and antiparticle states may be viewed as to represent chiral spinors. Thus we can associate a spin system with this ''direction vector'' (vortex line) when fermions are polarized in one or other direction. This can be generalized to an n-dimensional Euclidean space when we consider that the latticization of such a space can be achieved when we take that this is given by the surface of an $(n+1)$-dimensional sphere of infinitely large radius having a fictitious monopole at the center. The continuum limit is attained when the monopole charge vanishes.

Now to study the problem of anomaly on a lattice we note that chiral anomaly arises as a consequence of the quantization procedure. In an earlier paper[10] it has been shown that Nelson's stochastic quantization procedure can be generalized to have a relativistic framework and the quantization of a Fermi field can be achieved when we take into account Brownian motion processes in the internal space also apart from that in the external space. For the quantization of a Fermi field, we have to introduce an anisotropy in the internal space so that the internal variable appears as a "direction vector." The opposite orientations of the "direction vector" correspond to particle and antiparticle. To be equivalent to the Feynman path integral we have to take into account complexified space–time when the coordinate is given by $z_\mu = x_\mu + i\xi_\mu$ where ξ_μ corresponds to the "direction vector" attached to the space–time point x_μ.[11] Since for quantization we have to introduce Brownian motion process both in the external and internal space, after quantization, for an observational procedure, we can think of the mean position of the particle in the external observable space with a stochastic extension as determined by the internal stochastic variable. The nonrelativistic quantum mechanics is obtained in the sharp point limit.[11] This analysis of the quantization procedure suggests that we can write the position and momentum variable of this extended body as

$$
\begin{aligned}
Q_\mu &= q_\mu + i\hat{Q}_\mu , \\
P_\mu &= p_\mu + i\hat{P}_\mu ,
\end{aligned}
\tag{16}
$$

where $q_\mu(p_\mu)$ denotes the mean position (momentum) in the external observable space and $\hat{Q}_\mu(\hat{P}_\mu)$ is given by the internal variable denoting the stochastic extension. Introducing a new constant $\omega = \hbar/lmc$, where m is the mass of the particle, the quantum uncertainty relations can now be written in terms of the dimensionless variables where we replace Q_μ by Q_μ/l and P_μ by P_μ/mc:

$$
[Q_\mu , P_\nu] = i\omega g_{\mu\nu}, \quad [\hat{Q}_\mu , \hat{P}_\nu] = i\omega g_{\mu\nu}. \tag{17}
$$

As has been shown by Brooke and Prugovecki,[15] these relations admit the following representation of Q_μ/ω and P_μ/ω:

$$
\begin{aligned}
\frac{Q_\mu}{\omega} &= -i\left(\frac{\partial}{\partial p_\mu} + \phi_\mu\right), \\
\frac{P_\mu}{\omega} &= i\left(\frac{\partial}{\partial q_\mu} + \psi_\mu\right),
\end{aligned}
\tag{18}
$$

where ϕ_μ and ψ_μ are complex-valued functions. However, when we introduce an anisotropy in the internal space giving rise to the internal helicity to quantize a fermion, ϕ_μ and ψ_μ become matrix-valued functions due to the noncommutativity character of the components $\phi_\mu(\psi_\mu)$. To interpret the "direction vector" as an internal helicity we can choose the chiral coordinate as

$$
z^\mu = x^\mu + i\xi^\mu = x^\mu + \frac{i}{2}\lambda^\mu_\alpha \theta^\alpha, \tag{19}
$$

where we identify the coordinate in the complex manifold with

$$
\xi^\mu = \tfrac{1}{2}\lambda^\mu_\alpha \theta^\alpha, \quad \alpha = 1,2, \tag{20}
$$

θ being a two-component spinor. We now replace the chiral coordinate by the matrices

$$z^{AA'} = x^{AA'} + \tfrac{1}{2}\lambda_\alpha^{AA'}\theta^\alpha, \tag{21}$$

where

$$x^{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix} x^0 - x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + x^1 \end{bmatrix}$$

and

$$\lambda_\alpha^{AA'} \in \mathrm{SL}(2,C).$$

This helps us to associate the internal helicity with the spinorial variable θ^α as we can now construct the helicity operator[10]

$$S = -\lambda_\alpha^{AA'}\theta^\alpha \bar{\pi}_A \pi_{A'}, \tag{22}$$

where $\bar{\pi}_A(\pi_{A'})$ denotes the spinorial variable corresponding to the four-momentum p_μ (the canonical conjugate of x_μ) and is given by the matrix representation

$$p^{AA'} = \bar{\pi}^A \pi^{A'}. \tag{23}$$

The internal helicity can now be identified with the fermion number. It may be noted that since we have taken the matrix representation of p_μ as $p^{AA'} = \bar{\pi}^A \pi^{A'}$ necessarily implying $p_\mu^2 = 0$, the particle will have its mass due to the nonvanishing character of the quantity ξ_μ^2. It is observed that the complex conjugate of the chiral coordinate given by (19) will give rise to a massive particle with opposite internal helicity corresponding to an antifermion.

In this complexified space–time exhibiting the internal helicity state we can write the metric

$$g_{\mu\nu}(x,\theta,\bar{\theta}) = g_{\mu\nu}^{AA'}(x)\,\bar{\theta}_A \theta_{A'}. \tag{24}$$

It has been shown elsewhere[16] that this metric structure gives rise to the SL(2,C) gauge theory of gravitation and generates the field strength tensor $F_{\mu\nu}$ given in terms of the gauge fields B_μ, which are matrix-valued having the SL(2,C) group structure, and is given by

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \tag{25}$$

This suggests that we can consider a gauge theoretic extension for a fermion and we can identify the matrix-valued functions $\phi_\mu(\psi_\mu)$ in Eq. (18) with the gauge field $B_\mu(C_\mu)$ where $B_\mu(C_\mu) \in \mathrm{SL}(2,C)$. Now we note that if we demand $F_{\mu\nu}=0$ at all points on the boundary S^3 of a certain volume V^4 inside which $F_{\mu\nu} \neq 0$, then the gauge potentials tend to a pure gauge in the limiting case towards the boundary, i.e., we have

$$B_\mu = U^{-1}\partial_\mu U. \tag{26}$$

This helps us to write the Lagrangian in the limiting case

$$L = M^2\ \mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) + \mathrm{Tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2, \tag{27}$$

where M is a suitable constant having the dimension of mass. It is noted that the Skyrme term $\mathrm{Tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2$ arises here from the term $F_{\mu\nu}F^{\mu\nu}$ where the first term is related to the gauge noninvariant term $M^2 B_\mu B^\mu$ in the Lagrangian. In view of this we note that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity corresponds to the realization of a nonlinear σ-model where the Skyrme term automatically arises, stabilizing the soliton. Indeed this is no surprise as the anisotropic feature of the internal space prevents it from shrinking to zero size. The simplest Lagrangian density which is invariant under $SL(2,C)$ transformation in spinor affine space is given by[17]

$$L=\tfrac{-1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{28}$$

which is reflection noninvariant. Indeed, it is to be noted that the Skyrme term does not manifestly express the internal anisotropy as it is invariant under P and T. So to incorporate this anisotropic feature in the Lagrangian (27) we should add the Wess–Zumino term where the action is given by

$$S_{\mathrm{WZ}}=\frac{iN}{240\pi^2}\int \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}[U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\lambda UU^{-1}\partial_\sigma UU^{-1}\partial_\rho U]\,d^5x, \tag{29}$$

where $x=\mathbf{x},t,x^5$. Here the physical space–time is the boundary of the five-dimensional domain. Witten[18] has shown that the constant N has to be an integer for the existence of a consistent quantum description of the Skyrmion. The quantization of N is analogous to the Dirac quantization of the product eg of electric and magnetic charges. It is noted that the Lagrangian (28) is associated with the Wess–Zumino term in the nonlinear σ-model. It may be pointed out here that the expression (29) vanishes unless $U\in \mathrm{SU}(n)$ with $n\geqslant 3$.[18] From this analysis, it appears that massive fermions appear as solitons and the fermion number is of topological origin. Indeed, for the Hermitian representation we can take the group manifold SU(2) and this leads to a mapping from the space three-sphere S^3 to the group space $S^3[\mathrm{SU}(2)=S^3]$ and the corresponding winding number is given by

$$q=\frac{1}{24\pi^2}\int_{S^3} ds_\mu \epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[U^{-1}\partial_\nu UU^{-1}\partial_\alpha UU^{-1}\partial_\beta U]. \tag{30}$$

Evidently q can be taken to represent the fermion number.

The Lagrangian density in spinor affine space given by Eq. (28) gives rise to a conserved current

$$\mathbf{J}_\mu^\theta=\epsilon^{\mu\nu\alpha\beta}\mathbf{B}_\nu\times\mathbf{F}_{\alpha\beta}, \tag{31}$$

where the gauge field $B_\mu=\mathbf{B}_\mu\cdot\mathbf{g}$ and the field strength $F_{\mu\nu}=\mathbf{F}_{\mu\nu}\cdot\mathbf{g}$, $\mathbf{g}$ being the infinitesimal generators of the group $SL(2,C)$ in tangent space

$$g_1=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad g_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad g_3=\begin{bmatrix}0&1\\0&0\end{bmatrix}. \tag{32}$$

Indeed from the properties of the above Lagrangian, we find[17]

$$\epsilon^{\mu\nu\alpha\beta}(\partial_\nu\mathbf{F}_{\alpha\beta}-\mathbf{B}_\nu\times\mathbf{F}_{\alpha\beta})=0, \tag{33}$$

which suggests that

$$\mathbf{J}_\theta^\mu=\epsilon^{\mu\nu\alpha\beta}\mathbf{B}_\nu\times\mathbf{F}_{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\partial_\nu\mathbf{F}_{\alpha\beta}. \tag{34}$$

Then, using the antisymmetric property of the Levi-Civita tensor density $\epsilon^{\mu\nu\alpha\beta}$, we get

$$\partial_\mu\mathbf{J}_\theta^\mu=\epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\mathbf{F}_{\alpha\beta}=0. \tag{35}$$

Since in this formalism SL(2,C) gauge fields act as background fields for a Dirac spinor giving rise to its topological properties, to describe matter field in this formalism, the Lagrangian will be modified by the introduction of the SL(2,C) invariant Lagrangian density. Hence for a Dirac field neglecting the mass term, we write

$$L=-\bar{\psi}\gamma_\mu D_\mu\psi-\tfrac{1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{36}$$

where D_μ is the SL(2,C) gauge covariant derivative defined by $D_\mu=\partial_\mu-igB_\mu$ where g is some coupling strength. It is to be observed that by the introduction of the SL(2,C) gauge field Lagrangian we are effectively taking into account the effect of the extension of the fermionic part giving rise to the internal helicity in terms of the gauge fields. Now if we split the Dirac massless spinor in chiral forms and identify the internal helicity $+1/2(-1/2)$ with left (right) chirality corresponding to $\theta(\bar{\theta})$, we can write

$$\begin{aligned}\bar{\psi}\gamma_\mu D_\mu\psi&=\bar{\psi}\gamma_\mu\partial_\mu\psi-ig\bar{\psi}\gamma_\mu B^a_\mu g^a\psi\\&=\bar{\psi}\gamma_\mu\partial_\mu\psi-\frac{ig}{2}\{\bar{\psi}_R\gamma_\mu B^1_\mu\psi_R-\bar{\psi}_R\gamma_\mu B^2_\mu\psi_R+\bar{\psi}_L\gamma_\mu B^2_\mu\psi_L+\bar{\psi}_L\gamma_\mu B^3_\mu\psi_L\}.\end{aligned} \tag{37}$$

Then the three SL(2,C) gauge field equations give rise to the following three conservation laws:[19]

$$\begin{gathered}\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_\mu\psi_R)+J^1_\mu]=0,\\ \partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L)+ig\bar{\psi}_R\gamma_\mu\psi_R)+J^2_\mu]=0,\\ \partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L)+J^3_\mu]=0.\end{gathered} \tag{38}$$

These three equations represent a consistent set of equations if we choose

$$J^1_\mu=-J^2_\mu/2,\quad J^3_\mu=+J^2_\mu/2, \tag{39}$$

which evidently guarantees the vector current conservation. Then we can write

$$\begin{gathered}\partial_\mu(\bar{\psi}_R\gamma_\mu\psi_R+J^2_\mu)=0,\\ \partial_\mu(\bar{\psi}_L\gamma_\mu\psi_L-J^2_\mu)=0.\end{gathered} \tag{40}$$

From these we have

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=\partial_\mu J^5_\mu=-2\,\partial_\mu J^2_\mu. \tag{41}$$

Thus the anomaly is expressed here in terms of the gauge field current J^2_μ. However, since in this formalism the chiral currents are modified by the introduction of J^2_μ, we note from Eq. (40) that the anomaly vanishes. The charge corresponding to the gauge field part is

$$q=\int J^2_0\,d^3x=\int_{\text{surface}}\epsilon^{ijk}\,d\sigma_i F^2_{jk},\quad i,j,k=1,2,3. \tag{42}$$

Visualizing F^2_{jk} to be the magnetic-field-like components for the vector potential B^2_i, we see that q is actually associated with the magnetic pole strength for the corresponding field distribution. Thus we find that the quantization of Fermi field associates a background magnetic field and the charge corresponding to the gauge field effectively represents a magnetic charge. The term $\epsilon^{\mu\nu\alpha\beta}$ Tr $F_{\mu\nu}F_{\alpha\beta}$ in the Lagrangian can be actually expressed as a four divergence of the form $\partial_\mu\Omega^\mu$ where

$$\Omega^\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}\left[B_\alpha F_{\beta\gamma} - \frac{2}{3}(B_\alpha B_\beta B_\gamma)\right]. \tag{43}$$

We recognize that the gauge field Lagrangian is related to the Pontryagin density

$$P = -\frac{1}{16\pi^2}\,\mathrm{Tr}^*\, F_{\mu\nu}F_{\mu\nu} = \partial_\mu\Omega^\mu, \tag{44}$$

where Ω^μ is the Chern–Simons secondary characteristic class. The Pontryagin index

$$q = \int P\, d^4x \tag{45}$$

is a topological invariant. As we know, the introduction of the Chern–Simons characteristic class modifies the axial vector current as

$$\widetilde{J}^5_\mu = J^5_\mu + 2\hbar\Omega_\mu, \tag{46}$$

where $\partial_\mu\widetilde{J}^5_\mu = 0$ though $\partial_\mu J^5_\mu \neq 0$. We find from Eq. (40) that the Chern–Simons secondary characteristic class is effectively represented by the current J^2_μ constructed from the SL(2,C) gauge field. Thus we have the Chern–Simons topology in-built in the system and it is associated with the topological aspects of a fermion.

This analysis suggests that anomaly vanishes when we take into account the quantum geometry where the microlocal space–time is characterized by an anisotropic feature such that a ''direction vector'' (vortex line), ξ_μ is attached to a space–time point x_μ so that in the complexified space–time the coordinate is given by $z_\mu = x_\mu + i\xi_\mu$. However, in the naive Minkowski space–time where the coordinate is just represented by x_μ, we come across anomaly as a consequence of the quantum mechanical symmetry breaking. We have pointed out earlier that the geometrical feature of lattice structure of space effectively incorporates a similar geometry as the latticization can be viewed as the introduction of a ''vortex line'' at a space–time point. Also, this may be caused by the introduction of a fictitious magnetic monopole at the center of a sphere so that the lattice space is given by the surface. Now as the anomaly is associated with a magnetic charge q given by the Pontryagin index where the monopole strength μ is related to this by the relation $q = 2\mu$, we note that the very lattice structure suggests that there cannot be any anomaly on a lattice. Indeed the anomaly which is associated with the current J^2_μ effectively may be taken to arise from the geometrical feature of a lattice space characterized by the ''vortex line'' attached to a space–time point or a ''fictitious'' monopole at the center of a sphere where the surface represents a lattice structure. Thus we have a chiral current associated with a lattice site when the gauge field lies in the link. This chiral current may be related to a chiral fermion as we have the relation

$$\partial_\mu J^2_\mu = -\tfrac{1}{2}\partial_\mu J^5_\mu,$$

so that we have a solution of the form

$$J^2_\mu = -\tfrac{1}{2}\bar{\psi}\gamma_\mu(1+\gamma_5)\psi \tag{47}$$

as the vector current is conserved. In view of this we note that the very geometrical feature of a lattice structure is as if there are ''fictitious'' chiral fermions at a lattice site. These are responsible for the cancelation of anomaly on a lattice. This unveils the root cause of fermion doubling and the extra spinors just represent these ''fictitious'' chiral fermions.

IV. SPECIES DOUBLING AND FERMIONIC CHARGE

Nielson and Ninomiya[6] have constructed a model with only one two-component fermion on a lattice, dropping the assumption of the existence of a conserved charge, e.g., fermion number. Thus the corresponding fermion field is taken to be real. However, we can assign locally defined but only approximately conserved charges. It may be noted that if we do not require that the charge be locally defined, it is possible to define a conserved charge for a real field on a lattice since we can assign the charge $\bar{Q}(p)$ the value $+1$ for some values of the momentum p and -1 for the opposite p-value. One might then either leave the charge undefined outside such regions in p space or let $\bar{Q}(p)$ have discontinuities as a function of p, meaning the charge not being locally defined. If, however, we want the charge to be locally defined, then there must exist for all values of the momentum p in the Brillouin Zone many eigenstates with a given charge eigenvalue. To a model with only one Weyl particle, we can then only assign approximately conserved charge. In fact, it is possible to formulate a model which looks like that of the complex field formulation with nonconserved charges as the Hamiltonian does not satisfy the condition for the complex-valued formulation with conserved charges. This suggests that charges are not conserved but are approximately conserved in the low-energy regime only. It may be pointed out here that as the lattice fermions are found to be nonanomalous there cannot be any locally defined conserved charge. This follows from the fact that the chiral anomaly is associated with the Pontryagin index which is a topological index related to the fermion number. Indeed, it has been shown in an earlier paper[20] that we have the Pontryagin index q which satisfies the relation

$$q=2\mu=\int \partial_\mu J_\mu^2\, d^4x=-\frac{1}{2}\int \partial_\mu J_\mu^5\, d^4x. \tag{48}$$

Here μ (as well as q) corresponds to a monopole strength satisfying the Dirac quantization condition $e\mu=\frac{1}{2}$. It is noted that q is an integer where μ can take the value $0,\pm 1/2,\pm 1,\pm 3/2,\ldots$. Thus the quantization condition $e\mu=1/2$ is equivalent to the relation $eq=1$ and for $\mu=\pm 1/2$, i.e., $q=\pm 1$, we have $e=\pm 1$ depicting the fermionic charge. Now from the relation (48) we note that when there is no anomaly, the Pontryagin index vanishes and hence we cannot define a conserved charge like fermion number. It may be added here that chiral anomaly is related with the Berry phase[20] where the phase factor is given by $e^{i\phi_{\mathrm{B}}}$ with

$$\phi_{\mathrm{B}}=2\pi\mu. \tag{49}$$

Evidently from the above relation (48) we note that when the theory is nonanomalous, the Berry Phase loses its topological character and can be removed. Thus for lattice fermions we have the specific property that the Hamiltonian for a time-dependent closed path evolution allows an eigenstate with only a dynamical phase factor when the quantum phase of Berry is removed to the dynamical phase.

V. DISCUSSION

In this note we have argued that the problem of fermion doubling on a lattice is not an isolated event as it may occur when the original theory of fermion is anomaly free. This is consistent with the observation of Creutz and Tytgat.[8] Indeed, it has been pointed out that the very geometrical aspect of the discretization of space in a lattice can be viewed as if there are ''fictitious'' chiral fermions on a site which cancels the anomaly. In view of this, we can interpret that the unwanted fermions just reflect the geometrical properties of a lattice and can be identified with these ''fictitious'' chiral fermions associated with the lattice structure. In reality, these fermions need not exist. It may be added here that in a recent paper[21] we have shown that chiral anomaly gives rise to the mass of a particle. This suggests that anomaly effectively generates a length scale. Now for a discrete space in a lattice, the length scale of the order of lattice spacing effectively can be viewed as if generated by the ''hidden'' anomaly which may be taken to be generated by the ''fictitious'' chiral fermions associated with the latticization of Minkowski space–time. It may be noted that the lattice theory does not just provide one extra particle which is enough to cancel the anomaly as in d-dimensions, we have in total 2^d fermions. This follows from the fact that the ''direction vector'' ξ_μ attached to a lattice site x_μ also has the same dimension as x_μ and this is responsible for the discretization in every direction. This leads to the generation of all other ''fictitious'' spinors such that the theory becomes nonanomalous. Indeed this picture suggests that one can generate a one-dimensional lattice theory taking the continuum limit in the remaining directions, e.g., in three dimensions of a four-dimensional lattice theory.

Finally, we point out here that though we generally take that the lattice spacing $a \to 0$ to attain the continuum limit, it is not so naive as it involves the loss of specific geometrical features solely related to the lattice structure where the continuum space–time is devoid of these properties.

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