NONLINEAR LIOUVILLE THEOREMS IN THE HEISENBERG GROUP VIA THE MOVING PLANE METHOD

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1 Introduction

The "moving plane" method, that goes back to Alexandrov and J. Serrin [21] , has known a great development since the symmetry results by B. Gidas, W.M. Ni and L. Nirenberg [12] for positive solutions of elliptic semi-linear equations in symmetric domains of $\mathbb{R}^n$. The method relies in particular on two features of the Laplacian viz., the *maximum principle* and the *invariance by reflection with respect to a hyperplane.*

An important feature of this paper is that we develop the analogue of the "moving plane" method for the Heisenberg group and this allows us to obtain some new non existence results for a class of positive solutions of the following semi-linear equation

$$\Delta_H u + u^p = 0 \quad \text{in} \quad H^n, \tag{1.1}$$

where Δ_H is the Heisenberg laplacian, $H^n = (\mathbb{R}^{2n+1}, \circ)$ is the Heisenberg group and p is subcritical (see section 2).We would like to mention that although the Heisenberg laplacian satisfies the Maximum principle (see Bony [5]) it is **not** invariant by the usual reflection with respect to a hyperplane. This means that a new notion of reflection needs to be introduced in order to apply the moving plane method.

We will say that u is cylindrical in H^n if for any $(x, y, t) \in H^n$ where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t \in \mathbb{R}$ is the anisotropic direction, we have $u(x, y, t) = u(r, t)$ with $r = \sqrt{x^2 + y^2}$. Let $Q = 2n + 2$ denote the homogenous dimension of H^n.

Our main result is the following

Theorem 1.1 *If $p < \frac{Q+2}{Q-2}$ the only non negative, C^2, cylindrical solution of (1.1) is $u \equiv 0$.*

Let us recall that $\frac{Q+2}{Q-2} = 1 + \frac{2}{n}$ is the "critical exponent" for the Sobolev type embedding in the Stein spaces S_1^2 (see e.g. [15], [16]). Theorem 1.1 infact allows us to prove non-existence results for larger class of functions viz;

Corollary 1.1 *Let u be a non negative, C^2 solution of (1.1) with the property* *that $u \circ \Phi$ is cylindrical, where $\Phi : H^n \to H^n$ is a map which leaves Δ_H invariant. Then $u \equiv 0$.*

Clearly the most natural action Φ is the Heisenberg group action but it may also be any other map that leaves Δ_H invariant; for example in H^1 we may consider $\Phi(x, y, t) = (2\lambda - x, y, -t - 4\lambda y)$ for any $\lambda \in \mathbb{R}$.

Theorem 1.1 extends the Liouville theorems of [2]. Indeed one of the results of [2] was that for $1 < p \leq \frac{Q}{Q-2}$, the only non negative solutions of

$$\Delta_H u + u^p \leq 0 \quad \text{in} \quad H^n$$

are the trivial ones. There, as well as here, no conditions at infinity for u are required. Let us emphasize that although the exponent $\frac{Q}{Q-2}$ is optimal for inequalities, as it was shown in [2], non existence results for the equation (1.1) were stated as an open problem for $\frac{Q}{Q-2} < p < \frac{Q+2}{Q-2}$.

In the euclidean case, non existence results in $\mathbb{R}^n$ up to the critical case, have first been proved by Gidas and Spruck in [13] with a very difficult proof and then by Chen and Li in [8] using the method of moving plane. For other Liouville type results for semilinear inequalities involving more general sub-elliptic operators see the work of I. Capuzzo Dolcetta and A. Cutri [7] and the works there mentioned.

N. Garofalo and E. Lanconelli, in [11], prove some non existence theorems for S_1^2 positive solutions of (1.1) when p is subcritical, but their results differ in nature from ours (and those cited above) since they require that the solutions decay at infinity. In a framework similar to [11], though with different techniques, E. Lanconelli and F. Uguzzoni in [18] and F. Uguzzoni in [22] prove non-existence results for S_o^1 solutions of (1.1) in a half-space with critical exponent, see also ([1]). The results of L. Brandolini, Rigoli and Setti [6] and Wei Zu [19], concern non existence results in the Heisenberg group with different non-linear terms.

Let us go back to the "moving plane" technique which is at the base of theorem 1.1. The reflection that we introduce to replace the reflection w.r.t. a plane is the following "H-reflection".

Definition 1.1 *Let* $\xi = (x, y, t)$*, let* $T_\lambda = \{\xi \in H^1; t = \lambda\}$*. We define*

$$\xi_\lambda = (y, x, 2\lambda - t)$$

to be the H-reflection of ξ *with respect to the plane* T_λ*.*

It is easy to see that Δ_H is invariant with respect to this H-reflections i.e. if $v(\xi) = u(\xi_\lambda)$ and $\Delta_H u(\xi) = f(\xi)$ then $\Delta_H v(\xi) = f(\xi_\lambda)$.

Let us remark that the H-reflection with respect to the plane T_λ leaves the plane T_λ invariant but not fixed (only the line $\{x = y, t = \lambda\}$ is fixed); this feature is the reason why we work with cylindrical functions. Symmetry results, similar to [12], for positive solutions of semilinear equations in bounded or unbounded symmetric domains of H^n is an open problem; in [4] a partial result is obtained for a class of solutions in bounded domains.

The other ingredient we use is the map defined by D. Jerison and J.M. Lee in [15], which we shall refer to as the CR inversion. It plays the role of the "Kelvin transform". For simplicity we will illustrate the case $n = 1$. Precisely, let $\rho = |(x,y,t)|_H$ be the Folland norm in H^n (see section 2), then we define the CR inversion of u to be the function v given by

$$v(x,y,t) = \frac{1}{\rho^{Q-2}} u(\tilde{x}, \tilde{y}, \tilde{t}), \tag{1.2}$$

where $\tilde{x} = \frac{xt+yr^2}{\rho^4}$, $\tilde{y} = \frac{yt-xr^2}{\rho^4}$ and $\tilde{t} = \frac{-t}{\rho^4}$. We will prove that if u is a solution of (1.1), then v satisfies

$$\Delta_H v + \frac{1}{\rho^{(Q+2)-p(Q-2)}} v^p = 0 \quad \text{in} \quad H^n.$$

Clearly, here since $n = 1$ we have that $Q = 4$.

We should also add that the basic steps in the proof of Theorem 1.1 are similar to the one developed by W. Chen and C. Li in [8]. They combine the moving plane with the Kelvin transform to obtain in particular non existence of positive solutions of

$$\Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N$$

when $p < \frac{N+2}{N-2}$ i.e. p is subcritical, while for p critical they prove that any solution is radial with respect to a point and therefore it is of the form $u(x) = [N(N-2)\lambda^2]^{(N-2)/4}(\lambda^2 + |x - x_o|^2)^{-(N-2)/2}$.

Let us also recall that the moving plane method has already been adapted to non-euclidean frame works, see [17] and [20].

It is quite natural to expect that the solutions of (1.1) will be cylindrical, hence we expect the result of Theorem 1.1 to hold for any non negative solution of (1.1). Similarly, we expect to be able to give a simpler proof of the result due to Jerison and Lee [15] that the only solutions of (1.1) for p critical i.e. $p = 1 + \frac{2}{n}$ are functions of the following type $u(r,t) = C(t^2 + (r^2 + \mu)^2)^{-\frac{n}{2}}$.

The paper is organized as follows. In section 2 we introduce some well known facts about H^n and Δ_H; in section 3 we define the CR inversion and its properties; finally in section 4 we prove theorem 1.1.

2 Preliminary facts

For the sake of completeness, in this section we collect a few basic properties concerning the Heisenberg group and the operator Δ_H. For proofs and more information we refer for example to [9, 10, 11, 14].

To denote the elements of H^n we shall either use the notation $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ or $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ where $z = x+iy$, $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$. The Heisenberg group H^n is the space $\mathbb{R}^{2n+1}$ (or $\mathbb{C}^n \times \mathbb{R}$) endowed with the group action $\circ$ defined by

$$\xi_0 \circ \xi = (x+x_0, y+y_0, t+t_0+2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})). \tag{2.1}$$

Let us denote by δ_λ the parabolic dilation in $\mathbb{R}^{2n+1}$ i.e.

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t), \tag{2.2}$$

which satisfies $\delta_\lambda(\xi_o \circ \xi) = \delta_\lambda(\xi_o) \circ \delta_\lambda(\xi)$. The following norm in H^n

$$|\xi|_H := \left[\left(\sum_{i=1}^{n}(x_i^2+y_i^2)\right)^2 + t^2\right]^{\frac{1}{4}} \equiv |t+i|z|^2|^{\frac{1}{2}} \tag{2.3}$$

is homogeneous of degree one with respect to the dilation δ_λ (see [9], [10]).

The associated distance between two points ξ, η of H^n is defined accordingly by

$$d_H(\xi,\eta) = |\eta^{-1} \circ \xi|_H$$

where η^{-1} denotes the inverse of η with respect to $\circ$ that is $\eta^{-1} = -\eta$.

The open ball of radius R centered at ξ_o is the set:

$$B_H(\xi_o, R) = \{\eta \in H^n \ : \ d_H(\eta, \xi_o) < R\}.$$

It is important to note that

$$|B_H(\xi_o, R)| = |B_H(0, R)| = |B_H(0, 1)|R^Q$$

where $Q = 2n+2$ and $|.|$ denotes the Lebesgue measure. The even integer Q is called the homogeneous dimension of H^n. Observe that for $R > 1$, if $B(0, R)$ is the euclidean ball of radius R centered at the origin, then

$$B(0, R) \subset B_H(0, R) \subset B(0, R^2). \tag{2.4}$$

The vector fields $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ defined by

$$\begin{aligned}
&X_i = \tfrac{\partial}{\partial x_i} + 2y_i \tfrac{\partial}{\partial t}, \ \text{ for } \ \ i = 1, \ldots, n,\\
&Y_i = \tfrac{\partial}{\partial y_i} - 2x_i \tfrac{\partial}{\partial t}, \ \text{ for } \ \ i = 1, \ldots, n,\\
&T = \tfrac{\partial}{\partial t},
\end{aligned}$$

form a base of the Lie Algebra of vector fields which are left invariant with respect to the Heisenberg group action $\circ$. The Heisenberg gradient of a function ϕ is defined as

$$\nabla_H \phi = (X_1\phi, \ldots, X_n\phi, Y_1\phi, \ldots, Y_n\phi).$$

We can now state a few properties concerning the Heisenberg Laplacian Δ_H: The operator Δ_H is defined by

$$\Delta_H := \sum_{i=1}^{n} X_i^2 + Y_i^2 =$$

$$= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2}.$$

Δ_H is a degenerate elliptic operator, but it is easy to check that X_i and Y_i satisfy

$$[X_i, Y_j] = -4T\delta_{i,j}, \quad [X_i, X_j] = [Y_i, Y_j] = 0$$

for any $i, j \in \{1, \dots, n\}$. Therefore, the vector fields X_i, Y_i $(i = 1, \dots, n)$ and their first order commutators span the whole Lie Algebra. Hence, Δ_H satisfies the Hormander rank condition, see [14]. In particular, this implies that Δ_H is hypoelliptic (i.e. if $\Delta_H u \in C^\infty$ then $u \in C^\infty$) and it satisfies Bony's maximum principle (see [5]).

Furthermore, since X_i and Y_i are homogeneous of degree minus one with respect to δ_λ i.e.

$$X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i)$$

then Δ_H is homogeneous of degree minus two and of course it is left invariant with respect to $\circ$.

Let us recall that it is easy to see that if u is cylindrical then

$$\Delta_H u(r,t) = \frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + 4r^2 \frac{\partial^2 u}{\partial t^2} \tag{2.5}$$

3 CR inversion

As in [15], we define the CR inversion of a regular function $u(z,t)$ in H^n as

$$v(z,t) = \frac{1}{\rho^{(Q-2)}} u(\frac{z}{\omega}, \frac{-t}{|\omega|^2})$$

where $\omega = t + i|\ z\ |^2$ or equivalently

$$v(x,y,t) = \frac{1}{\rho^{(Q-2)}} u(\tilde{x},\tilde{y},\tilde{t})$$

with $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ and $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_n)$ where

$$\tilde{x}_i = \frac{x_i t + y_i|z|^2}{\rho^4},\ \tilde{y}_i = \frac{y_i t - x_i|z|^2}{\rho^4} \text{ and } \tilde{t} = \frac{-t}{\rho^4}.$$

Observe that v is a regular function in $H^n \setminus \{0\}$.

Now we want to prove that

$$\text{if } \Delta_H u(x,y,t) = f(x,y,t) \text{ then } \Delta_H v(x,y,t) = \frac{1}{\rho^{(Q+2)}} f(\tilde{x},\tilde{y},\tilde{t}). \tag{3.1}$$

It is immediate to see that

$$\tilde{r} := \sqrt{\tilde{x}^2 + \tilde{y}^2} = \frac{r}{\rho^2} \text{ and that } \tilde{\rho} = |(\tilde{x},\tilde{y},\tilde{t})|_H = \frac{1}{\rho}.$$

Therefore if u is cylindrical then so is v. Since the proof of (3.1) is a very long and tedious computation, we will sketch it only for cylindrical solutions. It is easy to see that the following equalities hold

$$\begin{aligned} \frac{\partial \tilde{r}}{\partial r} &= \frac{t^2 - r^4}{\rho^6} \qquad & \frac{\partial \tilde{r}}{\partial t} &= \frac{-rt}{\rho^6} \\ \frac{\partial \tilde{t}}{\partial r} &= \frac{4r^3 t}{\rho^8} \qquad & \frac{\partial \tilde{r}}{\partial t} &= \frac{t^2 - r^4}{\rho^8}. \end{aligned}$$

Therefore $v(r,t) = \frac{1}{\rho^{Q-2}} u(\tilde{r},\tilde{t})$ satisfies

$$\frac{\partial v}{\partial r} = \frac{(2-Q)r^3}{\rho^{Q+2}} u + \frac{1}{\rho^{Q-2}} \left[\frac{\partial u}{\partial \tilde{r}} \left(\frac{t^2 - r^4}{\rho^6} \right) + \frac{\partial u}{\partial \tilde{t}} \left(\frac{4r^3 t}{\rho^8} \right) \right],$$

and hence

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial}{\partial r} \left(\frac{(2-Q)r^3}{\rho^{Q+2}} \right) u + \frac{2(2-Q)r^3}{\rho^{Q+2}} \left[\frac{\partial u}{\partial \tilde{r}} \left(\frac{t^2 - r^4}{\rho^6} \right) + \frac{\partial u}{\partial \tilde{t}} \left(\frac{4r^3 t}{\rho^8} \right) \right]$$

$$+\frac{1}{\rho^{Q-2}}\left[\frac{\partial u}{\partial\tilde{r}}\frac{\partial}{\partial r}\left(\frac{t^2-r^4}{\rho^6}\right)+\frac{\partial u}{\partial\tilde{t}}\frac{\partial}{\partial r}\left(\frac{4r^3t}{\rho^8}\right)\right]$$

$$+\frac{t^2-r^4}{\rho^{Q+4}}\left[\frac{\partial^2 u}{\partial\tilde{r}^2}\left(\frac{t^2-r^4}{\rho^6}\right)+\frac{\partial^2 u}{\partial\tilde{r}\partial\tilde{t}}\left(\frac{4r^3t}{\rho^8}\right)\right]+\frac{4r^3t}{\rho^{Q+6}}\left[\frac{\partial^2 u}{\partial\tilde{t}\partial\tilde{r}}\left(\frac{t^2-r^4}{\rho^6}\right)+\frac{\partial^2 u}{\partial\tilde{t}^2}\left(\frac{4r^3t}{\rho^8}\right)\right].$$

While

$$\frac{\partial v}{\partial t}=\frac{(2-Q)t}{\rho^{Q+2}}u+\frac{1}{\rho^{Q-2}}\left[-\frac{\partial u}{\partial\tilde{r}}\left(\frac{rt}{\rho^6}\right)+\frac{\partial u}{\partial\tilde{t}}\left(\frac{t^2-r^4}{\rho^8}\right)\right]$$

and then

$$\frac{\partial^2 v}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{(2-Q)t}{\rho^{Q+2}}\right)u+\frac{2(2-Q)t}{\rho^{Q+2}}\left[-\frac{\partial u}{\partial\tilde{r}}\left(\frac{rt}{\rho^6}\right)+\frac{\partial u}{\partial\tilde{t}}\left(\frac{t^2-r^4}{\rho^8}\right)\right]$$

$$+\frac{1}{\rho^{Q-2}}\left[-\frac{\partial u}{\partial\tilde{r}}\frac{\partial}{\partial t}\left(\frac{rt}{\rho^6}\right)+\frac{\partial u}{\partial\tilde{t}}\frac{\partial}{\partial t}\left(\frac{t^2-r^4}{\rho^8}\right)\right]$$

$$+\frac{-rt}{\rho^{Q+4}}\left[-\frac{\partial^2 u}{\partial\tilde{r}^2}\frac{rt}{\rho^6}+\frac{\partial^2 u}{\partial\tilde{r}\partial\tilde{t}}\frac{t^2-r^4}{\rho^8}\right]+\frac{t^2-r^4}{\rho^{Q+4}}\left[-\frac{\partial^2 u}{\partial\tilde{r}\partial\tilde{t}}\frac{rt}{\rho^6}+\frac{\partial^2 u}{\partial\tilde{t}^2}\frac{t^2-r^4}{\rho^8}\right].$$

Now observe that $\Delta_H(\rho^{2-Q})=0$ for $\rho\neq 0$, therefore

$$\Delta_H v(r,t)=a_1\frac{\partial^2 u}{\partial\tilde{r}^2}+a_2\frac{\partial^2 u}{\partial\tilde{r}\partial\tilde{t}}+a_3\frac{\partial^2 u}{\partial\tilde{t}^2}+b_1\frac{\partial u}{\partial\tilde{r}}+b_2\frac{\partial u}{\partial\tilde{t}},$$

where a_1, a_2, a_3 and b_1, b_2 are the coefficients to be determined.

We obtain using the previous computations and formula (2.5)

$$a_1=\frac{1}{\rho^{Q+10}}[(t^2-r^4)^2+4r^2(r^2t^2)]=\frac{1}{\rho^{Q+10}}(t^4+r^8+2r^4t^2)=\frac{1}{\rho^{Q+2}}$$

$$a_2=\frac{1}{\rho^{Q+12}}[2(t^2-r^4)(4r^3t)-8r^2(rt)(t^2-r^4)]=0$$

$$a_3=\frac{1}{\rho^{Q+12}}[16(r^6t^2+4r^2(t^2-r^4)^2]=\frac{1}{\rho^{Q+2}}\cdot\frac{4r^2}{\rho^4}$$

and

$$b_1=\frac{1}{\rho^{Q+2}}\frac{(Q-3)\rho^2}{r},\quad b_2=0.$$

Recalling that $\tilde{r}=\frac{r}{\rho^2}$, we conclude $\Delta_H v(r,t)=\frac{1}{\rho^{Q+2}}\Delta_H u(\tilde{r},\tilde{t})$.

4 Proof

Let u be a cylindrical function satisfying the equation

$$\Delta_H u + u^p = 0 \quad \text{in} \quad H^n.$$

Then v, the CR inverse of u, is given by

$$v(r,t) = \frac{1}{\rho^{Q-2}} u\left(\frac{r}{\rho^2}, \frac{-t}{\rho^4}\right).$$

We have seen in the previous section that v satisfies the equation

$$\Delta_H v + \frac{1}{\rho^{Q+2-p(Q-2)}} v^p = 0 \ \text{ in } \ H^n \setminus \{0\}. \tag{4.1}$$

Note that the function v might be singular at the origin and that

$$\lim_{\rho \to \infty} \rho^{Q-2} v(r,t) = u(0). \tag{4.2}$$

We want to use the moving plane method adapted to this setting. We will shift the hyperplane $T_\lambda = \{(x,y,t) \in H^n : t = \lambda\}$, which is orthogonal to the t-direction and use the H-reflection given in definition 1.1. Let

$$\Sigma_\lambda := \{(x,y,t) \in H^n : t \leq \lambda\}$$

and consider the function v_λ defined on Σ_λ as follows

$$v_\lambda(x,y,t) := v_\lambda(r,t) = v(r, 2\lambda - t) := v(y, x, 2\lambda - t),$$

for any (x,y) such that $(x^2+y^2)^{1/2} = r$. Since v may be singular at 0, v_λ might be singular at the point $0_\lambda = (0, 2\lambda)$. From (4.1) and the invariance with respect to the H-reflection, it follows that v_λ satisfies

$$\Delta_H v_\lambda + \frac{1}{{\rho_\lambda}^{Q+2-p(Q-2)}} {v_\lambda}^p = 0 \ \text{ in } \ \tilde{\Sigma}_\lambda; \tag{4.3}$$

here and in the following $\rho_\lambda := |\xi_\lambda|_H$ and $\tilde{\Sigma}_\lambda := \Sigma_\lambda \setminus \{0_\lambda\}$. Consider the function $w_\lambda = v_\lambda - v$ defined on $\tilde{\Sigma}_\lambda$. Then w_λ satisfies

$$\Delta_H w_\lambda + c(\xi) w_\lambda \leq 0 \quad \text{in} \quad \tilde{\Sigma}_\lambda \tag{4.4}$$

where

$$c(\xi) = \frac{p}{\rho^{Q+2-p(Q-2)}} h(\xi)^{p-1} \tag{4.5}$$

and $h(\xi)$ is a real number between $v(\xi)$ and $v_\lambda(\xi)$.

We claim that *there exists* R_1 *and* $C_1 > 0$, *independent of* λ, *such that*

$$|c(\xi)| \leq \frac{C_1}{\rho^4}, \text{ for } \rho \geq R_1.$$

Indeed, observe that by (4.2), v_λ and v are $O\left(\rho^{-(Q-2)}\right)$; while clearly v^p and v_λ^p are $O(\rho^{-p(Q-2)})$. Since $\rho_\lambda \leq \rho$, from equation (4.5) there exists a constant $C > 0$ such that for ρ large

$$| \, c(\xi) \, | \leq \frac{C}{\rho^{(Q+2)-p(Q-2)}(\rho+1)^{(Q-2)(p-1)}}. \tag{4.6}$$

The claim follows from (4.6).

Now let $g(r,t) := (t^2 + (r^2+\mu)^2)^{\frac{-n}{2}}$ where $\mu = \frac{\sqrt{C_1}}{n}$; C_1 is the constant above. It is easy to check that

$$\Delta_H g = -4n^2\mu^2 (t^2 + (r^2+\mu)^2)^{\frac{-n}{2}-1}.$$

Hence with this choice of μ we get

$$\frac{\Delta_H g}{g} = \frac{-4C_1}{(t^2 + (r^2+\mu)^2)}. \tag{4.7}$$

We define $\tilde{w}_\lambda = \frac{w_\lambda}{g}$ on $\tilde{\Sigma}_\lambda$.

Before starting the proof let us give the following key

Lemma 4.1

(i) For $\lambda < 0$ and large enough in norm, if $\inf_{\tilde{\Sigma}_\lambda} \tilde{w}_\lambda(\xi) < 0$, then the infimum is achieved.

(ii) There exists a $R_0 > 0$ independent of λ such that if ξ_o is a minimum point for $\tilde{w}_\lambda$ in $\tilde{\Sigma}_\lambda$, satisfying $\tilde{w}_\lambda(\xi_o) < 0$, then $|\xi_o|_H < R_0$.

Proof Observe that since v is cylindrical, $\tilde{w}_\lambda \equiv 0$ in T_λ, hence the proof of (i) is as in [8]. We shall give below proof of (ii).

Recall that w_λ satisfies the equation

$$\Delta_H w_\lambda + c(\xi) w_\lambda \leq 0 \ \text{ in } \ \tilde{\Sigma}_\lambda, \tag{4.8}$$

where $c(\xi)$ was given in (4.5). Hence $\tilde{w_\lambda}$ satisfies

$$\Delta_H \tilde{w_\lambda} + \left(\frac{2}{g}\right) \nabla_H g \cdot \nabla_H \tilde{w_\lambda} + \left(c(\xi) + \frac{\Delta_H g}{g}\right) \tilde{w_\lambda}(\xi) \leq 0 \tag{4.9}$$

Using equation (4.7) we see that there exists $R_o > 0$, independent of λ such that for $\xi \in \Sigma_\lambda \cap \{|\xi|_H > R_o\}$

$$c(\xi) + \frac{\Delta_H g}{g} \leq \frac{C_1}{\rho^4} - \frac{4C_1}{(t^2 + (r^2+\mu)^2)} \leq \frac{-3C_1\rho^4 + 2\rho^2\mu C_1 + C_1\mu^2}{\rho^4(t^2 + (r^2+\mu)^2)} \leq 0. \tag{4.10}$$

Indeed, just choose $R_o^2 = \max\{R_1^2, \frac{\sqrt{C_1}}{n}\}$.

The inequality (4.10) implies that $|\xi_o|_H \leq R_o$. For, if $|\xi_o|_H > R_o$ then

$$\Delta_H \tilde{w}_\lambda(\xi_o) = -\left(c(\xi) + \Delta_H g/g\right) \tilde{w}_\lambda(\xi_o) < 0$$

which is absurd since ξ_o is a minimum. $\square$

Proof of Theorem 1.1 The proof consists of three steps. To start the moving plane procedure we will prove

First step: For large negative λ, $\tilde{w}_\lambda \geq 0$ in $\tilde{\Sigma}_\lambda$

This is clearly just a corollary of the lemma. Suppose by contradiction that for large negative λ, $\inf_{\tilde{\Sigma}_\lambda} \tilde{w}_\lambda(\xi) < 0$. Then by (i) of the lemma the minimum is achieved, say in $\xi_o \in \tilde{\Sigma}_\lambda$. But then, by (ii), $|\xi_o|_H < R_0$ while $|\xi_o|_H \geq |\lambda|$. We have reached a contradiction for $|\lambda|$ sufficiently large.

Second step: There exists λ_1 such that $\tilde{w}_{\lambda_1}(\xi) \equiv 0$

Define $\tilde{\lambda}_1 \leq 0$ to be the largest possible value of $\lambda \leq 0$ with the property that $\tilde{w}_\mu(\xi) \geq 0$ for all $\mu < \lambda$. We first prove that if $\tilde{\lambda}_1 < 0$ then $\tilde{w}_{\tilde{\lambda}_1}(\xi) \equiv 0$. By continuity, clearly $\tilde{w}_{\tilde{\lambda}_1}(\xi) \geq 0$ and hence $w_{\lambda_1} \geq 0$. Further, by the maximum principle, since $c(\xi)$ is bounded, we can conclude that either $w_{\lambda_1} > 0$ or $w_{\lambda_1} \equiv 0$. This is just an application of the Hopf lemma for Δ_H, see [3] for details. Hence we have that either $\tilde{w}_{\lambda_1} > 0$ or $\tilde{w}_{\lambda_1} \equiv 0$.

Suppose by contradiction that $\tilde{w}_{\lambda_1}$ is not identically zero. Hence $\tilde{w}_{\lambda_1} > 0$. By the definition of λ_1, there is a sequence of λ_k converging from above to λ_1 and a sequence of $\xi_k \in \tilde{\Sigma}_{\lambda_k}$, such that $\tilde{w}_{\lambda_k}(\xi_k) \leq 0$.

We first check that the ξ_k stays away from the singular value 0_{λ_k}. Since in a neighborhood of 0_{λ_1} we have $c(\xi) > 0$, $w_{\lambda_1} > 0$ and hence $\Delta_H w_{\lambda_1} < 0$ it is immediate that

$$\text{there exist } \ \delta > 0, \varepsilon > 0 \ \text{ such that } \tilde{w}_{\lambda_1} \geq \varepsilon \ \text{ in } B_H(0_{\lambda_1}, 2\delta). \tag{4.11}$$

By continuity, (4.11) implies that the ξ_k are not in $B_H(0_{\lambda_k}, \delta)$.

The above argument also implies that the $\tilde{w}_{\lambda_k}$ reach a negative minimum at ξ_k^o. We can apply lemma 4.1 to conclude that $|\xi_k^o|_H \leq R_o$. Hence we can extract a subsequence still denoted ξ_k^o, that converges to ξ_o.

Continuity implies that $\tilde{w}_{\tilde{\lambda}_1}(\xi_o) \leq 0$. But since by hypothesis $\tilde{w}_{\lambda_1} > 0$ in Σ_{λ_1}, it follows that $\xi_o \in T_{\lambda_1}$ and $\tilde{w}_{\lambda_1}(\xi_o) = 0$. The Hopf lemma would then imply that $X_1 \tilde{w}_{\lambda_1} > 0$. We have reached a contradiction since $\nabla \tilde{w}_{\lambda_k}(\xi_k) = 0$. Hence we can conclude that $w_{\tilde{\lambda}_1} \equiv 0$.

Consider now the case when $\lambda_1 = 0$. Then $\tilde{w}_0(\xi) \geq 0$. We repeat the procedure starting with λ large positive. Hence, either one finds a positive value $\lambda_2 > 0$ such that $\tilde{w}_{\lambda_2}(\xi) \equiv 0$ or the procedure continues up to 0. But then, together with the previous result, we get $\lambda_2 = 0$ and $\tilde{w}_0(\xi) \equiv 0$.

Third step. Conclusions

Since $p < \frac{Q+2}{Q-2}$, equations (4.1), (4.3) and $v_{\lambda_1} = v$ imply that $|\xi_{\lambda_1}|_H \equiv |\xi|_H$, but this holds true if and only if $\lambda_1 = 0$. Hence we have proved that v is even in t. Therefore u is even in t. But since we can chose the origin on the t-axis arbitrarily, it follows that u is independent of t. However, this implies that u satisfies the equation

$$\Delta u + u^p = 0 \quad \text{in} \ \ \mathbb{R}^{2n},$$

here Δ indicates the Laplacian in the variables x and y. Now, by standard non linear Liouville theorems for the Laplacian (see [8] and [13]) if $p < \frac{2n+2}{2n-2}$ then $u \equiv 0$. This concludes the proof since $\frac{Q+2}{Q-2} = \frac{2n+4}{2n} < \frac{2n+2}{2n-2}$.

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