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Numerical solution of a singular integro-differential equation

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Abstract

This paper presents a method based on polynomial approximation using Bernstein polynomial basis to obtain approximate numerical solution of a singular integro-differential equation with Cauchy kernel. The numerical results obtained by the present method compares favorably with those obtained by various Galerkin methods earlier in the literature. Also the convergence of the method is established rigorously for the studied integro-differential equation.

Keywords: Singular integro-differential equation; Cauchy kernel; Bernstein polynomials

1. Introduction

The singular integro-differential equation

$$2\frac{d\phi}{dx} + \lambda \int_{-1}^{1} \frac{\phi(t)}{(t-x)} dt = f(x), \quad -1 < x < 1, \ \lambda > 0 \tag{1.1}$$

with Cauchy type kernel, the integral being understood in the sense of Cauchy principal value, with specified end conditions, $\phi(\pm 1) = 0$, and a special forcing function $f(x) = -\frac{x}{2}$, was solved earlier by Frankel [1], Chakrabarti and Hamsapriye [2], and recently by Mandal and Bera [3].

Frankel [1] solved it by Galerkin's method after utilizing the various properties of Chebychev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ respectively for $n \in N$, while Chakrabarti and Hamsapriye [2] gave three separate ways of solution, which were all essentially based on Galerkin's method after recasting the equation into another where the derivative occurs inside the integral. Recently, Mandal and Bera [3] employed a simple method based on polynomial approximation of a function to obtain approximate numerical solutions.

The forcing function $f(x) = -\frac{x}{2}$, is of some special importance because it arises in the study of problems concerning heat conduction and radiation (cf. [1]). Also singular integro-differential equations arise in connection with solving some special type of mixed boundary value problems involving the two dimensional Laplace's equation in the quarter plane (cf. [2]).

Here, we have employed the method of polynomial approximation in the Bernstein polynomial basis. Bernstein polynomials are defined in an interval $[a,b]$ as

$$B_{i,n}(x) = \binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \quad i = 0, 1, 2, \ldots, n. \tag{1.2}$$

These polynomials have recently been used to solve some linear as well as non-linear differential equations, ordinary and partial, approximately by Bhatta and Bhatti [4] and Bhatti and Bracken [5] and some integral equations, by Mandal and Bhattacharya [6]. These polynomials defined over an interval forms a complete basis there, and each of them are positive and their sum is unity. Description of the properties of Bernstein polynomials can be found in the papers of Bhatta and Bhatti [4] and Bhatti and Bracken [5].

Using these polynomials, we have obtained an approximate numerical solution of (1.1), which are found to be in good agreement with the results obtained by Frankel [1], Chakrabarti and Hamsapriye [2] and Mandal and Bera [3].

2. The general method

The unknown function $\phi(x)$ of (1.1) with $\phi(\pm 1) = 0$, can be represented in the form

$$\phi(x) = (1-x^2)^{\frac{1}{2}}\psi(x), \quad -1 \leqslant x \leqslant 1, \tag{2.1}$$

where $\psi(x)$ is a well behaved function of x in the interval $-1 \leqslant x \leqslant 1$. To find an approximate solution of (1.1), $\psi(t)$ is approximated using the Bernstein polynomials in $[-1,1]$ as

$$\psi(t) = \sum_{i=0}^{n} a_i B_{i,n}(t), \tag{2.2}$$

where $B_{i,n}(x)$, $(i = 0, \ldots, n)$ are defined on $[-1,1]$ as

$$B_{i,n}(x) = \binom{n}{i}\frac{(1+x)^i(1-x)^{n-i}}{2^n}, \quad i = 0, 1, 2, \ldots, n \tag{2.3}$$

and a_i $(i = 0, \ldots, n)$ are unknown constants to be determined. Substituting (2.3) in (1.1), we get

$$\sum_{i=0}^{n} a_i\left[-\frac{2x}{(1-x^2)^{\frac{1}{2}}}B_{i,n}(x) + 2(1-x^2)^{\frac{1}{2}}B'_{i,n}(x) + \lambda \,⨍_{-1}^{1}(1-t^2)^{\frac{1}{2}}\frac{B_{i,n}(t)}{t-x}\mathrm{d}t\right] = f(x), \quad -1 \leqslant x \leqslant 1. \tag{2.4}$$

Multiplying both sides by $B_{j,n}(x)$, $(j = 0, 1, \ldots, n)$ and integrating from -1 to 1, we get a linear system given by

$$\sum_{i=0}^{n} a_i C_{ij} = b_j, \quad j = 0, 1, \ldots, n, \tag{2.5}$$

where

$$C_{ij} = -2\int_{-1}^{1}\frac{x}{(1-x^2)^{\frac{1}{2}}}B_{i,n}(x)B_{j,n}(x)\mathrm{d}x + 2\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}B'_{i,n}(x)B_{j,n}(x)\mathrm{d}x$$
$$+ \lambda\int_{-1}^{1}\left\{⨍_{-1}^{1}(1-t^2)^{\frac{1}{2}}\frac{B_{i,n}(t)}{t-x}\mathrm{d}t\right\}B_{j,n}(x)\mathrm{d}x \tag{2.6}$$

and

$$b_j = \int_{-1}^{1} f(x)B_{j,n}(x)\mathrm{d}x. \tag{2.7}$$

For $\lambda = 1$, we can write C_{ij} as

$$C_{ij} = D_{ij} + E_{ij} + F_{ij}, \tag{2.8}$$

where

$$D_{ij} = -n\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}B_{i,n}(x)B_{j-1,n-1}(x)\mathrm{d}x$$
$$= -4j\binom{n}{i}\binom{n}{j}\frac{\Gamma(\frac{2i+2j+1}{2})\Gamma(\frac{4n-2i-2j+3}{2})}{\Gamma(2n+2)}, \quad j=1,2,\ldots,n, \quad i=0,1,\ldots,n, \tag{2.9}$$

$$E_{ij} = n\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}B_{i,n}(x)B_{j,n-1}(x)\mathrm{d}x$$
$$= 4(n-j)\binom{n}{i}\binom{n}{j}\frac{\Gamma(\frac{2i+2j+3}{2})\Gamma(\frac{4n-2i-2j+1}{2})}{\Gamma(2n+2)}, \quad j=0,2,\ldots,n-1, \quad i=0,1,\ldots,n \tag{2.10}$$

and

$$F_{ij} = \int_{-1}^{1}A_i(x)B_{j,n}(x)\mathrm{d}x, \tag{2.11}$$

where

$$A_i(x) = \frac{1}{2^n}\binom{n}{i}\sum_{k=0}^{n}d_k^{i,n}\left[-\pi x^{k+1} + \sum_{m=0}^{k-1}\frac{1+(-1)^m}{4}\frac{\Gamma[\frac{m+1}{2}]\Gamma[\frac{1}{2}]}{\Gamma[\frac{m+4}{2}]}x^{k-m-1}\right], \tag{2.12}$$

so that

$$F_{ij} = \frac{1}{2^{2n}}\binom{n}{i}\binom{n}{j}\sum_{k=0}^{n}\sum_{r=0}^{n}d_k^{i,n}d_r^{j,n}\left[-\pi\frac{1-(-1)^{k+r}}{k+r+2} + \sum\nolimits_{m=0}^{k-1}\frac{1-(-1)^{k+r-m}}{k+r-m}\frac{1+(-1)^m}{4}\frac{\Gamma[\frac{m+1}{2}]\Gamma[\frac{1}{2}]}{\Gamma[\frac{m+4}{2}]}\right],$$
$$j=0,1,\ldots,n, \quad i=0,1,\ldots,n \tag{2.13}$$

with

$$d_k^{i,n} = \sum_s(-1)^{k-s}\binom{i}{s}\binom{n-i}{k-s}, \quad k=0,1,\ldots,n, \quad i=0,1,\ldots,n \tag{2.14}$$

the summation over s being taken as follows: for $i<n<n-i$, (i) $s=0$ to k for $k \leqslant i$, (ii) $s=0$ to i for $i<k \leqslant n-i$, (iii) $s=k-(n-i)$ to $n-i$ for $n-i<k \leqslant n$, while for $i=n-i$ (n being an even integer) (i) $s=0$ to k for $k \leqslant i$, (ii) $s=k-i$ to i for $i<k \leqslant n$; for $i>n-i$, i and $n-i$ above are to be interchanged. Also, we find that for the choice of $f(x) = -\frac{x}{2}$,

$$b_j = \frac{1}{2^n}\binom{n}{j}(n-2j)\frac{\Gamma(n-j+1)\Gamma(j+1)}{\Gamma(n+3)}. \tag{2.15}$$

The system (2.5) is solved for unknown a_i ($i = 0, 1, \ldots, n$) by standard numerical method and numerical values of $\phi(x)$ for different values of x are obtained approximately.

In our numerical calculations, we have chosen $n = 7$, 10, 13 and $a_0, a_1, \ldots, a_n$ are obtained numerically. Using these coefficients the values of $\phi(x)$ at $x = (0.2)k$, $k = 0, 1, \ldots, 5$ are presented in Table 1, for a comparison between the present method and that of the method used in [1], values of $\phi(x)$ at these points obtained by Frankel [1] are also given. It is obvious that the result compares favorably with the results of Frankel and

Table 1
Numerical values of $\phi(x)$

x		0	0.2	0.4	0.6	0.8	1.0
$\phi(x)$ (present method)	$n = 07$	.06973	.06711	.05964	.04736	.02811	0
	$n = 10$	.06948	.06714	.05988	.04711	.02821	0
	$n = 13$	.06950	.06717	.05981	.04723	.02805	0
$\phi(x)$ (Frankel's method)		.06950	.06712	.05984	.04718	.02891	0

also those obtained by Chakrabarti and Hamsapriye [2] and Mandal and Bera [3]. It was further observed that on increasing n the accuracy of the result increases.

3. Error analysis

Substitution of $\phi(x)$ in terms of $\psi(x)$ in Eq. (1.1), produces an equation for $\psi(x)$ which can be written in the operator form as

$$\left(D+\frac{\lambda\pi}{2}C\right)\psi=f,\quad -1<x<1, \tag{3.1}$$

where C and D respectively denote the operators defined by

$$Cu(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{(1-t^2)^{\frac{1}{2}}}{t-x}u(t)\mathrm{d}t,\quad -1<x<1 \tag{3.2}$$

and

$$Du(x)=\sqrt{(1-x^2)}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{x}{\sqrt{(1-x^2)}}u,\quad -1<x<1. \tag{3.3}$$

Then

$$CU_n(x)=-T_{n+1}(x),\quad n\geqslant 0, \tag{3.4}$$

where $T_n(x)=\cos n\theta$ with $x=\cos\theta$ are the Chebyshev polynomials of first kind and $U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}$ are the Chebyshev polynomials of second kind.

Thus (3.4) shows that C can be extended as a bounded linear operator from $L_1(w)$ to $L(w)$ (cf. [7, p. 306]), where $L_1(w)$ is the space of functions square integrable with respect to $w(x)=(1-x^2)^{1/2}$ in $[-1,1]$ and $L(w)$ is the subspace of functions $u\in L(w)$ satisfying

$$\|u\|_1^2=\sum_{k=0}^{\infty}(k+1)^2\langle u,\psi_k\rangle_w^2<\infty, \tag{3.5}$$

where

$$\langle u,v\rangle_w=\int_{-1}^{1}u(t)v(t)(1-t^2)^{1/2}\mathrm{d}t \tag{3.6}$$

and

$$\psi_k=\left(\frac{2}{\pi}\right)^{1/2}T_k. \tag{3.7}$$

Again

$$DU_n(x)=-\frac{n+1}{\sqrt{(1-x^2)}}T_{n+1}(x),\quad n\geqslant 0, \tag{3.8}$$

which shows that D can also be extended as a bounded linear operator from $L_1(w)$ to $L(w)$. Also assuming $f\in L(w)$, we find that Eq. (3.1) possess an unique solution $\psi\in L_1(w)$ for each $f\in L(w)$.

Now, we have approximated the function $\psi(x)$ in terms of the Bernstein polynomials $B_{i,n}(x)$ as

$$\psi(x)\simeq p_n(x), \tag{3.9}$$

where $p_n(x)$ is given by

$$p_n(x)=\sum_{i=0}^{n}a_iB_{i,n}(x)=\sum_{j=0}^{n}b_jU_j(x), \tag{3.10}$$

where b_j ($j = 0, 1, \ldots, n$) can be expressed in terms of a_i ($i = 0, 1, \ldots, n$) and vice-versa by following the transformation

$$B_{i,n}(x) = \binom{n}{i}\frac{1}{2^n}\sum_{s=0}^{n} d_s^{i,n}\frac{1}{2^s}\sum_{m=0}^{[\frac{s}{2}]}\left\{\binom{s}{m} - \binom{s}{m+1}\right\}U_{s-2m}(x) \tag{3.11}$$

(c.f. [8]).

If we denote the right side of (3.10) by $u_n(x)$, where

$$u_n(x) = \sum_{j=0}^{n} c_j\phi_j(x) \tag{3.12}$$

with

$$c_j = \left(\frac{\pi}{2}\right)^{1/2} b_j,\ \phi_j(x) = \left(\frac{2}{\pi}\right)^{1/2} U_j(x). \tag{3.13}$$

The functions $\phi_j(x)$ ($j = 0, 1, \ldots, n$) form a set of orthonormal polynomial basis in $[-1,1]$ with respect to the weight function $(1 - x^2)^{1/2}$. On p 306 of their book, Golberg and Chen [7] proved that if $f \in C^r[-1,1]$, then $u_n \to \psi$ as $n \to \infty$ in $L_1(w)$ and

$$\|\psi - u_n\|_1 < C_1 n^{-r}, \tag{3.14}$$

where C_1 is a constant. Thus convergence is fast for large r. In our example f is $-\frac{x}{2}$, thus $f \in C^\infty[-1,1]$. Hence convergence is good as is seen in the numerical computations.

4. Conclusion

A simple method of approximating the unknown function in terms of the truncated series involving the Bernstein polynomials is presented here, for solving a special Cauchy singular integro-differential equation arising in various fields of applied mathematics. The method illustrated here gives a simple way of getting the approximate solution avoiding the appearance of ill-conditioned matrices or complicated integrations. The method, tested, for numerical verification is proved to give favorable result. The convergence of the method is also discussed. Although the numerical computations have been carried out for $f(x) = -x/2$, the method can be employed for other forms of $f(x)$.

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