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NOTES

PROBABILITY CONTENT OF A RECTANGLE UNDER NORMAL DISTRIBUTION : SOME INEQUALITIES*

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SUMMARY. The paper deals with the problem of finding the optimum location of a rectangle R of specified volume for which $P(X \epsilon R)$ is maximized, where $X \sim N_p$. Some special cases are considered for maximizing the above probability when the lengths of the sides of R are only specified.

1. Introduction

Let X be a $p \times 1$ random vector distributed as $N_p(\mu, \Sigma)$. Consider the problem of selecting the region of largest confidence level for μ from all regions of fixed Lebesgue measure, based on a single observation X, Σ being a known positive-definite matrix. It follows easily from Neyman-Pearson lemma that such an optimal region is given by the corresponding concentration ellipsoid, if we restrict our attention to the class of translation-invariant regions. However, in this paper, we focus our attention only to a class of rectangular regions of fixed volume. The problem is of theoretical interest, besides the fact that rectangular confidence regions are easier to apply.

2. Optimal rectangular confidence region

Consider a class of rectangular regions given by

$$R(\Gamma, a) = \{x \epsilon R^p : |y_i| \leqslant a_i, \; i = 1, \ldots, p;$$

$$Y = (y_1, \ldots, y_p)' = \Gamma X, \; \Gamma \text{ is orthogonal}\}.$$

The confidence region based on $R(\Gamma, a)$, where $a = (a_1, \ldots, a_p)'$, is defined to be

$$X - \mu \; \epsilon \; R + c,$$

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$c \in R^p$. Our problem reduces to the following : To maximize

$$G(\Gamma, \mathbf{a}, \mathbf{c}) \equiv P[X \in R+\mathbf{c} : \mathbf{X} \sim N_p(0, \Sigma)]$$

subject to $a_1 a_2 \ldots a_p = k$, a given constant, by varying Γ, $\mathbf{a}$ and $\mathbf{c}$.

Our first theorem is a direct consequence of Anderson's (1955) probability inequality.

Theorem 1 : *$G(\Gamma, \mathbf{a}, \mathbf{c}) \leqslant G(\Gamma, \mathbf{a}, 0) \equiv G_1(\Gamma, \mathbf{a})$, for all $\mathbf{c} \in R^p$, $\mathbf{a} \in R_+^p$ and all orthogonal $p \times p$ matrices Γ.*

Without any loss of generality, we may assume $\Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2) = \Delta$, say. First, let us consider the case $\Delta = \sigma^2 I_p$. Then

$$G_1(\Gamma, \mathbf{a}) = \prod_{i=1}^{p} H(a_i/\sigma),$$

where
$$H(c) = \Phi(c) - \Phi(-c),$$

Φ being the c.d.f. of $N(0, 1)$.

Lemma 1 : *For $c > 0$, log $H(c)$ is a concave function of log c.*

Proof : Note that

$$H(c) = P[\chi^2 \leqslant c_1^2] = P[\log \chi_1^2 \leqslant 2 \log c].$$

It is easy to check that the p.d.f of log χ_1^2 (χ_1^2 stands for the chi-square variate with 1 degree of freedom) is log-concave. Thus it follows from Prekopa's inequality (Das Gupta, 1976) that the c.d.f. of log χ_1^2 is log-concave.

Theorem 2 : *For $\Sigma = \sigma^2 I_p$, $G_1(\Gamma, \mathbf{a})$ equals $G_1(I_p, \mathbf{a})$ which is maximized at $a_1 = \ldots = a_p = k^{1/p}$ when $a_1 \ldots a_p = k$.*

Proof : The first part of this theorem is trivial. Now write

$$H(a_i/\sigma) = H_1(b_i),$$

where $b_i = \log(a_i/\sigma)$. Then

$$G_1(\Gamma, \mathbf{a}) = \prod_{i=1}^{p} H_1(b_i)$$

is a (permutational) symmetric, log-concave function of $(b_1, \ldots, b_p)$. Hence

$$\prod_{i=1}^{p} H_1(b_i) \leqslant H_1^p(b_i^*),$$

where
$$b^* = \sum_{i=1}^{p} b_i/p = \log(k^{1/p}/\sigma).$$

Now let us consider the general case. Denote the covariance matrix of $Y = \Gamma X$ by ψ and note that $\psi = \Gamma\Delta\Gamma'$. Denote the conditional distribution of Y_p, given $Y_1, \ldots, Y_{p-1}$, by

$$N\left(\sum_{i=1}^{p-1} \beta_{pi} Y_i,\ \tau_p^2\right).$$

Then
$$G_1(\Gamma, \boldsymbol{a}) = E\Big[I(|Y_i| \leqslant a_i,\ \ i = 1, \ldots, p-1) \times \int_{-a_p}^{a_p} \phi\left(y_p; \sum_{i=1}^{p-1} \beta_{pi} Y_i,\ \tau_p^2\right) dy_p \Big],$$

where I is the indicator function and the p.d.f. of $N(\mu, \sigma^2)$ is denoted by $\phi(\cdot, \mu, \sigma^2)$. It follows from Anderson's (1955) probability inequality that

$$G_1(\Gamma, \boldsymbol{a}) \leqslant P[|Y_i| \leqslant a_i,\ i = 1, \ldots, p]\cdot H(a_p/\tau_p).$$

$(\tau_p > 0)$. Repeating this argument we get

$$G_1(\Gamma, \boldsymbol{a}) \leqslant \prod_{i=1}^{p} H(a_i/\tau_i).$$

Following the arguments given in the proof of Lemma 1, we get

$$\prod_{i=1}^{p} H(a_i/\tau_i) \leqslant H^p(a^*),$$

where
$$a^* = (k/\tau_1 \ldots \tau_p)^{1/p}.$$

However
$$\sigma_1^2 \ldots \sigma_p^2 = \det \Delta = \det \psi = \tau_1^2 \ldots \tau_p^2.$$

Hence
$$a^* = (k/\sigma_1 \ldots \sigma_p)^{1/p}.$$

This leads to our main result given below.

Theorem 3 : *$G(\Gamma, \boldsymbol{a}, \mathbf{c})$ is maximized for $\mathbf{c} = \mathbf{0}$, $\Gamma = I_p$, and $a_i/\sigma_i = a^*(i = 1, \ldots, p)$, where $a^* = (k/\sigma_1 \ldots \sigma_p)^{1/p}$, given that $\Sigma = diag(\sigma_1^2 \ldots \sigma_p^2)$ and $\prod_{i=1}^{p} a_i = k$.*

3. Some partial results

The problem of maximizing $G_1(\Gamma, \boldsymbol{a})$ for fixed $\boldsymbol{a}$ is quite involved. We consider some special cases of this problem.

Suppose $p = 2$, and write

$$\Gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad G_1(\Gamma, \boldsymbol{a}) \equiv g(\theta, \boldsymbol{a}).$$

We shall show that

$$g(\theta, \boldsymbol{a}) \leqslant g(0, \boldsymbol{a})$$

when
$$\sigma_2 < \sigma_1,\ a_2/\sigma_2 \leqslant a_1/\sigma_1.$$

To see this, recall that

$$G_1(\Gamma, a) \leqslant H(a_1/\tau_1)H(a_2/\tau_2).$$

Note that

$$\tau_1^2 \leqslant \sigma_1^2,\ \tau_1\tau_2 = \sigma_1\sigma_2.$$

Thus

$$a_2/\tau_2 \leqslant a_2/\sigma_2 \leqslant a_1/\sigma_1 \leqslant a_1/\tau_1.$$

Using Lemma 1, we get

$$\begin{aligned} G_1(\Gamma, a) &\leqslant H(a_1/\tau_1)H(a_2/\tau_2) \\ &\leqslant H(a_1/\sigma_1)H(a_2/\sigma_2) = G_1(I_p, a). \end{aligned}$$

Next note that

$$g(\theta, a_1, a_2) = g(\pi/2-\theta, a_2, a_1).$$

Thus $g(\theta, a)$ is maximized at $\theta = \pi/2$ when

$$a_1/\sigma_2 \leqslant a_2/\sigma_1,\ \sigma_2 < \sigma_1.$$

However, we have not been able to find the optimum θ when

$$\sigma_2/\sigma_1 < a_1/a_2 < \sigma_1/\sigma_2\ (\sigma_1 > \sigma_2).$$

At best, we can say that $\theta = 0$ cannot be an optimum solution for all $a_1, a_2, \sigma_1, \sigma_2$ satisfying $1 < a_1/a_2 < \sigma_1/\sigma_2$. To see this, suppose θ_0 is the limiting value of a sequence of optimum values of θ, as $\sigma_2 \to 0$.

$$\min(a_1/\cos\theta,\ a_2/\sin\theta) \leqslant \min(a_1/\cos\theta_0,\ a_2/\sin\theta_0).$$

for all θ. This implies $\tan\theta_0 = a_2/a_1$.

Consider the case $a_1 = a_2 = a$ for $\sigma_1^2 > \sigma_2^2$. Then

$$g(\theta, a, a) = E\left[\int_{-a}^{a}\phi(x_1;\ Z\cos\theta, \sigma_2^2)dx_1 \int_{-a}^{a}\phi(x_2;\ Z\sin\theta, \sigma_2^2)dx_2\right],$$

where $Z \sim N(0, \sigma_1^2-\sigma_2^2)$. It follows from the result of Hall *et al.* (1980) that the above expression within square-brackets is maximized at $\theta = \pi/4$. Hence for $a_1 = a_2$ the optimum θ is $\pi/4$. It seems that the explicit expression of the optimum θ in the general case is quite involved.

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