Sankhyā : The Indian Journal of Statistics
1984, Volume 46, Series A, Pt. 3, pp. 319-325.

RENEWAL TYPE EQUATIONS ON Z

By B. RAMACHANDRAN

Indian Statistical Institute

SUMMARY. An "elementary" (non-functional-analytic, "real variables") proof is obtained for the fact that the solutions of equation (1) below ($w_n \geqslant 0$ given, $v_n \geqslant 0$ to be solved for) are given by the theorem below as supplemented by the Remark following its statement.

This paper is a continuation of the work, mainly of Shanbag (1977), and also of Lau-Rao (1982) and Ramachandran (1982), as regards both the type of problem considered and the method of solution. The result proved is a special case of a general result due to Deny (1961), who, however, uses the Choquet theory. Lau and Rao (1984) deal with the corresponding result pertaining to the whole real line, but they, again, appeal to the Krein-Milman theorem instead of Choquet's. The present paper is an outcome of the search for an "elementary" (non-functional-analytic) proof for the real line case; the discrete case dealt with here should be of independent interest and a sequel* will deal with the real line case.

Let $\boldsymbol{Z}$ denote as usual the set of all integers, let the w_n, for $n \in \boldsymbol{Z}$, be given non-negative real numbers, and let v_m, for $m \in \boldsymbol{Z}$, be non-negative real numbers satisfying the equations

$$v_m = \sum_{n=-\infty}^{\infty} v_{m+n} w_n \text{ for all } m \in \boldsymbol{Z}. \qquad \ldots \quad (1)$$

(We may refer to such equations as of "renewal type", in analogy with such equations arising in probability theory, with the set of non-negative integers or the half-line $[0, \infty)$ as the domain of reference rather than $\boldsymbol{Z}$ or R^1). We then have the

Theorem : *Let $A = \{n \in \boldsymbol{Z} : w_n > 0\}$. If the (additive) group generated by A is $\boldsymbol{Z}$ itself, then we have either $v_m = 0$ for all $m \in \boldsymbol{Z}$, or $v_m = Bb^m$ for all $m \in \boldsymbol{Z}$, or $v_m = Bb^m + Cc^m$ for all $m \in \boldsymbol{Z}$, for some $B \geqslant 0$ and $C \geqslant 0$, respectively according as the equation $\Sigma w_n x^n = 1$ has no positive solution, or has exactly one positive solution b, or has two distinct positive solutions b and c (note that there can not be more than two distinct positive solutions).*

AMS (1980) *subject classification :* Primary 45E10, 39B99 ; Secondary 62E10, 60K05.

Key words and phrases : Renewal type (convolution type) equations ; Characterization problems.

*Vide this issue, pp. 326–338.

Remark : If the group generated by A is not $\mathbf{Z}$ itself but the subgroup $k\mathbf{Z}$ for some integer $k > 1$, then, as the preamble portion of the following proof will show, (1) is not really one equation but can be broken up into k equations, each having a coset of $k\mathbf{Z}$ in $\mathbf{Z}$ as its domain of definition, and our theorem can be applied separately (after suitable trivial modifications) to each of these k equations : the solution may be stated as :

$$v_m = p_1(m)b^m + p_2(m) \cdot c^m,$$

where p_1, p_2 are periodic on $\mathbf{Z}$ with k as period.

Proof : We shall consider below only the case where A contains at least one positive element and one negative element : the complementary cases are covered by the papers listed as References.

Since A generates $\mathbf{Z}$, we may assume without loss of generality that $w_n > 0$ for every $n \in \mathbf{Z}$, the reason being as follows :

Let $(2)\, A = A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$ and, generally, $(n)A = (n-1)A + A$ for positive integers $n \geqslant 2$; let $S = \bigcup_{n \geqslant 1} (n)A$, so that S is a semi-group under addition. Then our assumption implies that $S = \mathbf{Z}$. For, let $d_1 > 0$ and $d_2 < 0$ (by our choice) be the g.c.d.'s of the positive and of the negative elements of A respectively, so that d_1 and d_2 are mutually prime and so there exist integers p_1 and p_2 such that $p_1 d_1 + p_2 d_2 = 1$. Since $d_1 > 0$ and $d_2 < 0$, it follows that p_1 and p_2 are either both non-negative or both non-positive (and at least one of them is non-zero). For all sufficiently large positive integers, say for $n \geqslant N$, nd_1 and $nd_2 \in S$ (see, for instance, Lau-Rao, 1982, Lemma 2.3). If now both p_1 and p_2 are non-negative, then, for $n \geqslant N$, $n = (np_1)d_1 + (np_2)d_2 \in S$, S being closed under addition; and if p_1 and p_2 are non-positive, then, for $n \geqslant N$, $-n = (-np_1)d_1 + (-np_2)d_2 \in S$. Since S also contains arbitrarily large negative integers and arbitrarily large positive integers respectively, it follows that in both the cases (p_1 and p_2 both $\geqslant 0$, p_1 and p_2 both $\leqslant 0$), S contains all the integers (being closed under addition).

Now, (1) implies that

$$\begin{aligned} v_m &= \Sigma v_{m+n} w_n, \quad \text{where } w_n > 0 \text{ for } n \in A, \\ &= \sum_{k,l} v_{m+k+l} w_k w_l \\ &= \Sigma v_{m+n} w_n^{(2)} \end{aligned}$$

where $w_n^{(2)} = \sum_{k+l=n} w_k w_l > 0$ for every $n \in (2)A$; the $w_n^{(2)}$ are finite, for any $n \in (2)A$, as seen by taking an m for which $v_{m+n} > 0$ and using the fact that $v_m \geqslant v_{m+n}w_n^{(2)}$; repeating the argument, we see that for every positive integer k,

$$v_m = \Sigma v_{m+n} w_n^{(r)}$$

where $w_n^{(r)} > 0$ for every $n \epsilon (r)A$; hence, with $w_n^{(1)} = w_n$ and $w_n^* = \sum_{r=1}^{\infty} 2^{-r} w_n^{(r)}$ we have

$$v_m = \Sigma v_{m+n} w_n^*.$$

Note that $w_n^* > 0$ for all $n \epsilon S = \mathbf{Z}$.

Thus, henceforth we may and shall assume that, in (1), $w_n > 0$ for every $n \epsilon \mathbf{Z}$. Then $v_m = 0$ for *some* m implies that $v_n = 0$ for every n, and the theorem is true. We shall consider below the cases where $v_m > 0$ for *every* m. Then (1) yields the obvious inequalities

$$v_m > w_1 v_{m+1}, \; v_{m+1} > w_{-1} v_m \qquad \ldots \quad (2)$$

so that $\left\{\frac{v_{m+1}}{v_m}\right\}$ and $\left\{\frac{v_m}{v_{m+1}}\right\}$ are both bounded sequences. Let then

$$b = \sup_m \frac{v_{m+1}}{v_m}, \quad c = \inf_m \frac{v_{m+1}}{v_m} = 1/\left(\sup_m \left(\frac{v_m}{v_{m+1}}\right)\right) \qquad \ldots \quad (3)$$

If $b = c$, then $v_m = v_0 b^m$ for all m and the theorem hold . Hence, again, we need only consider the case where $b > c$. Then $\tau_m = bv_m - v_{m+1}$ and $\tau_m' = v_{m+1} - cv_m$ define sequences of non-negative numbers, both sequences satisfying the (same) basic equation (1). If τ_m were to be zero for some m, then it follows that $\tau_n = 0$ for all n, so that $v_m = v_0 b^m$ for all m, and we have $b = c$, ruled out by assumption. Similarly, $\tau_m' > 0$ for all m as well. We then have the analogues of (2), namely,

$$\tau_m > w_1 \tau_{m+1} \text{ as well as } > w_{-1} \tau_{m-1}$$

$$\tau_m' > w_1 \tau_{m+1}' \text{ as well as } > w_{-1} \tau_{m-1}'$$

or
$$bv_m - v_{m+1} > w_1(bv_{m+1} - v_{m+2}) \text{ as well as } > w_{-1}(bv_{m-1} - v_m) \qquad \ldots \quad (4)$$

$$v_m - cv_{m+1} > w_1(v_{m+1} - cv_{m+2}) \text{ as well as } > w_{-1}(v_{m-1} - cv_m) \qquad \ldots \quad (5)$$

(4) implies that

$$b - \frac{v_{m+2}}{v_{m+1}} \leqslant \frac{v_m}{w_1 v_{m+1}}\left(b - \frac{v_{m+1}}{v_m}\right) \leqslant \frac{1}{cw_1}\left(b - \frac{v_{m+1}}{v_m}\right).$$

On iteration, and setting $\theta = (cw_1)^{-1}$ so that $\theta > 1$ ($\because v_m > w_1 v_{m+1}$), we get for every positive integer n,

$$b - \frac{v_{m+n}}{v_{m+n-1}} \leqslant \theta^{n-1}\left(b - \frac{v_{m+1}}{v_m}\right)$$

so that, if $b-\frac{v_{m+1}}{v_m} < \varepsilon_1$, then

$$\frac{v_{m+n}}{v_m} > b^n\left(1-\frac{\varepsilon_1}{b}\right) \dots \left(1-\frac{\varepsilon_1\theta^{n-1}}{b}\right)$$

$$> b^n\left[1-\frac{\varepsilon_1(\theta^n-1)}{b(\theta-1)}\right].$$

If follows that, given any $\varepsilon > 0$ and an arbitrary, fixed positive integer k, there exists an integer $m = m(k, \varepsilon)$ such that

$$\frac{v_{m+n}}{v_m} > b^n-\varepsilon \text{ for } n = 0, 1, \dots, k. \qquad \dots (6)$$

In particular, since $v_{m+k}/v_m \leqslant b^k$ for positive integers k, it follows from (6) that

$$\sup_m \frac{v_{m+k}}{v_m} = b^k, \text{ similarly from (5), } \inf_m \frac{v_{m+k}}{v_m} = c^k. \qquad \dots (7)$$

Next, we note that, also from (4),

$$b-\frac{v_{m+1}}{v_m} \geqslant w_{-1}\frac{v_{m-1}}{v_m}\left(b-\frac{v_m}{v_{m-1}}\right)$$

$$\geqslant w_{-1}b^{-1}\left(b-\frac{v_m}{v_{m-1}}\right)$$

whence, by iteration, it follows (as above) that for any $\varepsilon > 0$ and an arbitrary but fixed positive integer k, there exists an integer $m = m(k, \varepsilon)$ such that

$$\frac{v_{m-n}}{v_m} < b^{-n}+\varepsilon \text{ for } n = 0, 1, \dots, k. \qquad \dots (8)$$

We shall now show that $\Sigma w_n b^n = 1$, a dual argument will show that $\Sigma w_n c^n = 1$ as well.

Let k be an arbitrary fixed positive integer; and $\varepsilon > 0$ be arbitrary. By (6), there exists an integer m such that

$$\frac{v_{m+n}}{v_m} > b^n-\varepsilon \text{ for } n = 0, 1, \dots, 2k.$$

Hence
$$1 = \Sigma \frac{v_{m+n+k}}{v_{m+k}} w_n \geqslant \sum_{-k}^{k} \dots$$

$$\geqslant \left(\sum_{-k}^{k} \frac{v_{m+n+k}}{v_m} w_n\right)\frac{v_m}{v_{m+k}}$$

$$\geqslant \left(\sum_{-k}^{k} \frac{v_{m+n+k}}{v_m} w_n\right) b^{-k}, \quad \text{by (7)},$$

$$> \sum_{-k}^{k} (b^n-\varepsilon b^{-k})w_n = \sum_{-k}^{k} b^n w_n - \varepsilon b^{-k}\left(\sum_{-k}^{k} w_n\right)$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\sum_{-k}^{k} b^n w_n \leqslant 1$. k being an arbitrary positive integer, we then have that $\Sigma b^n w_n \leqslant 1$. A dual argument shows that $\Sigma c^n w_n \leqslant 1$. To obtain the reverse inequalities, we proceed as follows :

Given any $\varepsilon > 0$ and a fixed positive integer k, there exists an $m = m(k, \varepsilon)$, according to (8), such that

$$\frac{v_{m-n}}{v_m} < b^{-n} + \varepsilon \text{ for } n = 1, 2, \ldots, k.$$

Hence, also using (7), we have

$$1 = \Sigma \frac{v_{m+n}}{v_m} w_n \leqslant \sum_{0}^{\infty} b^n w_n + \sum_{-\infty}^{-(k+1)} c^n w_n + \sum_{-k}^{-1} (b^n + \varepsilon) w_n.$$

Since $\varepsilon > 0$ is arbitrary, it follows that for every fixed positive integer k,

$$\sum_{0}^{\infty} b^n w_n + \sum_{-k}^{-1} b^n w_n + \sum_{-\infty}^{-(k+1)} c^n w_n \geqslant 1.$$

Letting $k \to \infty$, and noting that $\Sigma c^n w_n$ is a convergent series, we conclude that $\sum_{-\infty}^{\infty} b^n w_n \geqslant 1$.

Hence we must have $\Sigma b^n w_n = 1$. A dual argument, using (5) instead of (4), shows that $\Sigma c^n w_n = 1$.

Now we set for convenience of writing

$$l_m = v_m b^{-m}, \; p_m = w_m b^m, \; \lambda = c/b \; (< 1)$$

so that

$$l_m = \Sigma l_{m+n} p_n \text{ for all } m, \quad \Sigma p_n = \Sigma p_n \lambda^n = 1,$$

$$\sup \frac{l_{m+1}}{l_m} = 1, \quad \inf \frac{l_{m+1}}{l_m} = \lambda.$$

Also, let $u_m = l_m - l_{m+1}$ so that, in particular, $u_m \geqslant 0$ for all m. Also, u_m satisfies the same equation as does l_m, i.e., $u_m = \Sigma u_{m+n} p_n$ for all m. If $u_m = 0$ for some m, then $u_n = 0$ for all n, so that $l_m = l_0$ or $v_m = v_0 b^m$, ruled out by our hypothesis about $\{v_m\}$. Hence $u_m > 0$ for every m and we then necessarily have $\sup \frac{u_{m+1}}{u_m} = 1$ or λ while $\inf \frac{u_{m+1}}{u_m} = \lambda$ since 1 and λ are the only two (positive) solutions of the equation $\Sigma p_n x^n = 1$. (If $\inf \frac{u_{m+1}}{u_m} = 1$ then again $u_m = \text{constant} = \lim_{\to\infty} u_n = 0$ since $\{l_m\} \downarrow$ a finite limit as $m \to \infty$, and we have a contradiction, which rules out this possibility). We shall show below that $\sup \frac{u_{m+1}}{u_m}$ is indeed $= \lambda$.

The relations $t_m = \Sigma t_{m+n} p_n$, $\Sigma p_n = 1$, imply that

$$\Sigma(t_m - t_{m+n})p_n = 0$$

or

$$\sum_{n \geqslant 1} (t_m - t_{m+n})p_n = \sum_{n \leqslant -1} (t_{m+n} - t_m)p_n. \quad \ldots (9)$$

In view of the analogue of (7) for $\{u_m\}$, we have that for all positive integers k,

$$\frac{u_{m+k}}{u_m} \geqslant \lambda^k \text{ while } \frac{u_{m+k}}{u_m} \leqslant \lambda^{-k}. \quad \ldots (10)$$

Using (10) below, we get that the LHS of (9) is

$$= u_m p_1 + (u_m + u_{m+1})p_2 + (u_m + u_{m+1} + u_{m+2})p_3 + \ldots$$

$$\geqslant u_m\{p_1 + p_2(1+\lambda) + p_3(1+\lambda+\lambda^2) + \ldots\}$$

$$= \frac{u_m}{1-\lambda}\{p_1(1-\lambda) + p_2(1-\lambda^2) + p_3(1-\lambda^3) + \ldots\}$$

$$= \frac{u_m}{1-\lambda}\left\{\sum_1^\infty p_i - \sum_1^\infty p_i\lambda^i\right\} \quad \ldots (11)$$

whereas the RHS of (9) is

$$= u_{m-1}p_{-1} + (u_{m-1} + u_{m-2})p_{-2} + (u_{m-1} + u_{m-2} + u_{m-3})p_{-3} + \ldots$$

$$\leqslant u_{m-1}\{p_{-1} + p_{-2}(1+\lambda^{-1}) + p_{-3}(1+\lambda^{-1}+\lambda^{-2}) + \ldots\}$$

$$= \frac{\lambda u_{m-1}}{1-\lambda}\left\{\sum_{i=-1}^{-\infty} \lambda^i p_i - \sum_{i=-1}^{-\infty} p_i\right\} \text{ as easily verified.} \quad \ldots (12)$$

Since $\sum_{-\infty}^{\infty} p_i = \sum_{-\infty}^{\infty} p_i\lambda^i = 1$, (11) and (12) imply that $u_m \leqslant \lambda u_{m-1}$ but the reverse inequality is already true. Hence $u_m = \lambda u_{m-1}$ for all m, whence we conclude that

$$t_m = B + C\lambda^m \text{ where } B = \downarrow \lim_{m\to\infty} t_m > 0,\ C > 0$$

or

$$v_m = Bb^m + Cc^m, \text{ as claimed.}$$

The cases B and/or $C = 0$ have already been covered earlier. The theorem stands proved.

REFERENCES

DENY, J. (1961): Sur l'equation de convolution : $\mu = \mu * \sigma$. *Semin. Theor. Potentiel M. Brelot*, Fac. Sci. Paris, 1959-60, No. 5.

LAU, K. S. and RAO, C. R. (1982): Integrated Cauchy functional equation and characterization of the exponential law. *Sankhyā*, A, **44**, 72-90.

——— (1984): Solution to the integrated Cauchy functional equation on the whole line. *Sankhyā*, A, **46**, 311-318.

RAMACHANDRAN, B. (1982): On the equation $f(x) = \int_{[0,\infty)} f(x+y)d\mu(y)$. *Sankhyā* A, **44**, 364-371.

SHANBAG, D. N. (1977): An extension of the Rao-Rubin characterization, *J. Appl. Prob.* **14**, 640-646.

Paper received : September, 1983.