Quasi-classical investigation of *nonpolynomial central potentials* with broken supersymmetry

G. Junker, P. Roy, and Y.P. Varshni

Abstract: We evaluate energy eigenvalues corresponding to different central potentials with nonpolynomial character and exhibiting broken supersymmetry. The calculations were carried out using numerical integration, the WKB rule, and a quasi-classical quantization rule for broken supersymmetry. It is shown that in most cases the quasi-classical rule gives far better estimates than the WKB rule applied separately to the supersymmetry partners.

Résumé: Nous calculons les valeurs propres d'un Hamiltonien avec potentiel central de type non-polynomial et capable de briser la supersymétrie. Trois méthodes sont utilisées: l'intégration numérique, la méthode WKB et une règle quasi-classique de quantification adaptée aux cas de bris de supersymétrie. Nous trouvons que dans la plupart des cas, la méthode semi-classique donne de bien meilleurs résultats que la méthode WKB appliquée séparément aux partenaires supersymétriques.
[Traduit par la rédaction]

1. Introduction

Ever since the supersymmetric (SUSY) modification of the WKB quantization rule (henceforth called CBC rule) was introduced by Comtet et al. the [1], it has been used in various potential models. Surprisingly, when applied to shape-invariant potentials the CBC rule produces the exact spectrum [1–4]. In contrast, the WKB rule does not produce the exact spectrum unless Langer-type modifications are introduced. Also, in the case of nonshape-invariant potentials, the CBC rule produces better results compared to the WKB rule [5–7]. For some exceptional cases we refer to ref. 8.

However, a major shortcoming of the CBC rule is that it is applicable to only those SUSY systems where SUSY is unbroken. Recently, Inomata et al. [9–11], following the path-integral approach, derived a modification of the CBC rule that takes into account systems with broken SUSY. Subsequently, the modified CBC rule was applied to a number of power-law as well as shape-invariant potentials with broken SUSY [10–13]. Surprisingly, this modified CBC rule also generates the exact spectrum for those shape-invariant potentials. For the power-law potentials, it has been observed that the modified CBC rule always overestimates the exact energy eigenvalue whereas the usual WKB rule leads to an underestimation [11, 14].

In this work we shall present an application of the modified CBC rule to radial potential problems. The first one will be the nonpolynomial oscillator with broken supersymmetry. The nonpolynomial oscillator is important in a number of areas [15–17] and has been studied for exact as well as

approximate solutions by a number of authors [18–24]. Since this interaction is markedly different from the polynomial ones (as also nonshape invariant), the effectiveness of the modified CBC rule can be best verified when applied to this sort of potential. Also, in a sense, the present work is complimentary to an earlier one [25] where the CBC rule was applied to the nonpolynomial oscillator with unbroken SUSY. We would like to note that when a potential is supersymmetrized, the coupling constants are constrained by some relations and, using the broken SUSY for the nonpolynomial oscillator, the range of the dependent coupling constant is enlarged. In a second example, we shall apply the CBC rule to a system that is relevant to planar SUSY models with magnetic fields [26] and the Dirac equation with anomalous magnetic moment interaction [27].

In Sect. 2, we will briefly review the basics of SUSY quantum mechanics and its quasi-classical approximation. In Sect. 3 we discuss the SUSY aspects of the two systems under investigation and in Sect. 4 our numerical results are presented and discussed.

2. SUSY quantum mechanics and its quasi-classical approximation

Since supersymmetric quantum mechanics was introduced by Nicolai [28], it has become an important algebraic tool in various branches of theoretical physics [14,26]. In particular, the model introduced by Witten [29] has attracted much attention in the last 10 years. This model consists of a pair of Hamiltonians (we use units such that the mass m and Planck's constant $\hbar$ are given by $2m = \hbar = 1$)

$$H_{\pm} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_{\pm}(r) \tag{1}$$

where the so-called partner potentials

$$V_{\pm}(r) = W^2(r) \pm W'(r) \tag{2}$$

are defined via the SUSY potential W and its derivative $W' = \mathrm{d}W/\mathrm{d}r$. As we are considering only radial problems, the variable r takes values on the positive half-line. Hilbert space is chosen to be the vector space of square integrable wave functions on the half-line that vanish at the origin, that is, $\psi^{(\pm)}(0) = 0$.

SUSY is said to be unbroken if the ground state of either H_+ or H_- belongs to a vanishing energy eigenvalue. Denoting such a state by $\psi_0^{(\pm)}$ is given via the SUSY potential by

$$\psi_0^{(\pm)}(r) = N \exp\left\{ \pm \int_{r_0}^{r} \mathrm{d}r'\, W(r') \right\}, \qquad r_0 > 0 \tag{3}$$

where N is a normalization constant. If the eigenvalues of both Hamiltonians (1) are strictly positive then SUSY is said to be broken. In any case, the strictly positive eigenvalues of H_+ and H_- are identical. That is, for broken SUSY the eigenvalues $E_n^{(\pm)}$ of $H_{\pm}$ (for simplicity a purely discrete spectrum is assumed) are identical

$$E_n^{(+)} = E_n^{(-)} > 0, \qquad n = 0, 1, 2, \ldots \tag{4}$$

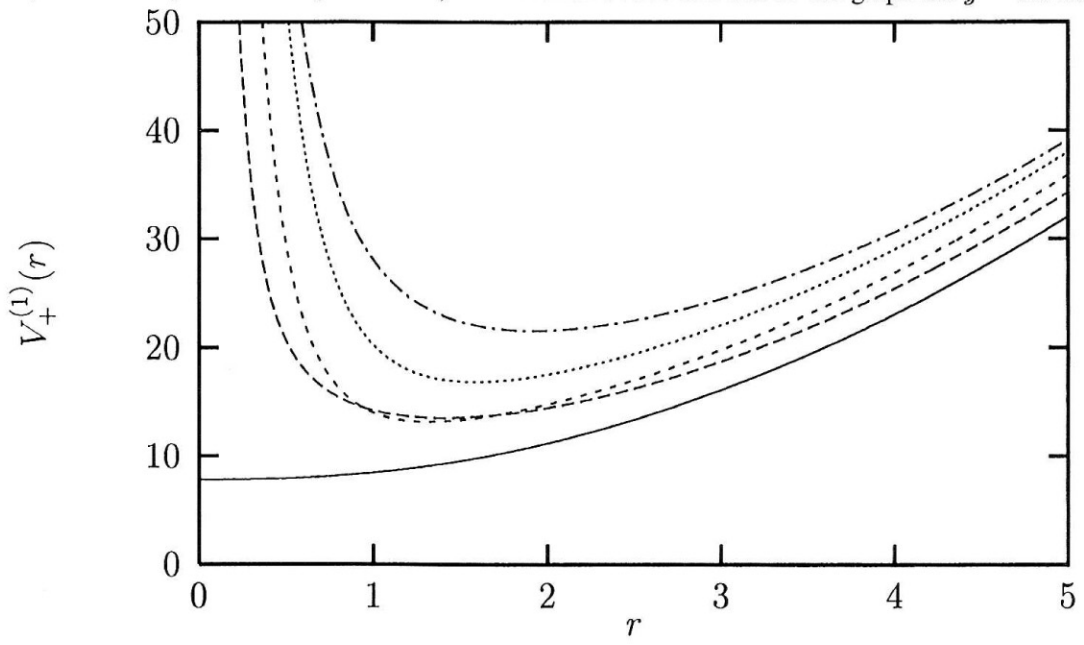
Now we briefly describe the modified CBC rule that has been suggested by Inomata et al. [9–11] for the case of broken SUSY. In this case, the quasi-classical quantization condition is given by

$$\int_{r_a}^{r_b} \mathrm{d}r\, \sqrt{E_n^{(\pm)} - W^2(r)} = \pi\hbar \left(n + \frac{1}{2} \right) \tag{5}$$

where r_a and r_b are the quasi-classical turning points:

$$W^2(r_a) = W^2(r_b) = E_n^{(\pm)} \tag{6}$$

Fig. 1. Graphs for the potential $V_+^{(1)}$ with parameters $g = 0.8$ and $c = 1$ (unbroken line), $c = 2$ (long dashes), $c = 3$ (short dashes) and $c = 4$ (dotted line). The broken-dotted line shows the graph for $g = 0.1$ and $c = 4$.



It follows that both relations in (4) are satisfied by (5) so that H_+ and H_- are strictly isospectral and have strictly positive ground-state energies even within this quasi-classical approximation. The modified CBC rule (5) differs from the usual WKB rule,

$$\int_{\tilde{r}_a}^{\tilde{r}_b} \mathrm{d}r \sqrt{E_n^{(\pm)} - V_\pm(r)} = \pi\hbar\left(n + \frac{1}{2}\right), \qquad V_\pm(\tilde{r}_a) = V_\pm(\tilde{r}_b) = E_n^{(\pm)} \tag{7}$$

as the left-hand side of (7) contains the full potential. For this reason the WKB rule does not in general obey the isospectral condition (4) and may even lead to negative ground-state energies. In the next section we shall use the modified quantization condition (5) to evaluate the energy eigenvalues for two classes of central potentials and compare them with the exact values as well as the values obtained from the WKB condition (7).

3. Two models with broken SUSY

The nonpolynomial oscillator potential, which we will investigate first, is given by [15–24]

$$V^{(1)}(r) = r^2 + \frac{\lambda r^2}{1 + gr^2} = r^2 + \frac{\lambda}{g} - \frac{\lambda/g}{1 + gr^2}, \qquad g > 0,\ \lambda \in \mathbb{R} \tag{8}$$

and since we shall study this system in three dimensions the effective radial potential reads

$$V_{\mathrm{eff}}^{(1)}(r) = r^2 + \frac{\lambda}{g} - \frac{\lambda/g}{1 + gr^2} + \frac{l(l+1)}{r^2}, \qquad l = 0, 1, 2, \ldots \tag{9}$$

To apply the modified CBC rule to obtain energy eigenvalues corresponding to (9) it is necessary to cast it in a supersymmetric form, that is, the potential (9) has to be identified with either of the supersymmetric partner potentials $V_\pm(r)$. To achieve this we choose the following SUSY potential:

$$W^{(1)}(r) = r + \frac{2gr}{1 + gr^2} + \frac{c}{r}, \qquad c \geq 1 \tag{10}$$

Then from (2) it follows that

$$V_+^{(1)}(r) = r^2 + \frac{4cg + 2g - 4}{1 + gr^2} + \frac{c(c-1)}{r^2} + (2c + 5) \tag{11}$$

$$V_-^{(1)}(r) = r^2 + \frac{4cg - 2g - 4}{1 + gr^2} + \frac{8g^2r^2}{(1 + gr^2)^2} + \frac{c(c+1)}{r^2} + (2c + 3) \tag{12}$$

Clearly, $V_+^{(1)}$ in (11) resembles $V_{\text{eff}}^{(1)}$ in (9). Graphs of $V_+^{(1)}$ for various values of the parameters g and c are shown in Fig. 1. If we identify the stationary radial Schrödinger equations corresponding to (9) and (11) we get

$$-\frac{\lambda}{g} = 4cg + 2g - 4 \tag{13}$$

$$E_n^{(\pm)} = E_n^{(1)} - \frac{\lambda}{g} + (2c + 5) \tag{14}$$

$$c = l + 1 \tag{15}$$

Note that the effective potential depends on the three parameters λ, g, and l, whereas the SUSY potential has only the two parameters g and c. Hence, for a given l and g the potential (9) can be put into the supersymmetric form $V_+^{(1)}$ only if we set $\lambda = 4g - 4lg^2 - 6g^2$. Note that in the realization with unbroken SUSY [25] the constraint was $\lambda = 4g + 4lg^2 + 2g^2$. Hence, using the broken-SUSY approach enlarges the range of parameters considerably. Let us also remark that for $g = 0$ the potentials $V_\pm^{(1)}$ are shape-invariant, a case for which the modified CBC rule (5) is known [9, 10] to reproduce the exact energy spectrum.

The zero-energy wave functions in the $(+)$ and $(-)$ sectors are given by (3) and we find that both

$$\psi_0^{(\pm)}(r) \propto \left[r^c\,(1 + gr^2)\,\mathrm{e}^{r^2/2}\right]^{\pm 1} \tag{16}$$

are not normalizable. Thus, the ground-state energy $E_0^{(\pm)} > 0$ and SUSY is broken.

As a second example, we consider a model for which the SUSY potential is given by [18–25]

$$W^{(2)}(r) = ar + b + \frac{c}{r}, \qquad a > 0, \qquad b \geq -\sqrt{ac},\ c \geq 1 \tag{17}$$

Then from (2) we find

$$V_\pm^{(2)}(r) = a^2r^2 + 2abr + \frac{2bc}{r} + \frac{c(c \mp 1)}{r^2} + (b^2 + 2ac \pm a) \tag{18}$$

This potential, in essence, characterizes the radial harmonic oscillator with an additional linear and Coulomb potential. Note that for $b = 0$ this potential is shape-invariant and thus the modified CBC rule (5) produces the exact energy spectrum [9, 10]. As before we can obtain the zero-energy wave functions,

$$\psi_0^{(\pm)}(r) \propto \left[r^c \exp\{ar^2/2 + br\}\right]^{\pm 1} \tag{19}$$

none of which is normalizable and thus SUSY is broken in this case, too.

Next we use the modified quantization condition to evaluate the energy eigenvalues corresponding to (11), (12), and (18) and compare them with WKB and exact numerical values.

Table 1. Numerical results for the potential $V_{\pm}^{(1)}$ for parameter values $g = 0.1, 0.8$, and $c = 1, 2, 3, 4$. Given are the exact as well as the approximate energy eigenvalues derived via the WKB and modified CBC rule. The table also shows the relative errors of the approximation. From the data energy values corresponding to (9) can be obtained using the relations (13)–(15).

			$g = 0.1,$		$c = 1$		
n	0	1	2	3	4	5	10
Exact	6.99579	11.3717	15.6377	19.8374	23.9941	28.1212	48.5211
WKB $V_{-}^{(1)}$	6.80696	11.1872	15.4562	19.6580	23.8163	27.9445	48.3470
%error	0.699	−1.623	−1.161	−0.904	−0.741	−0.629	−0.359
WKB $V_{+}^{(1)}$	6.99976	11.3730	15.6380	19.8374	23.9941	28.1211	48.5210
%error	0.057	0.011	0.002	0.000	0.000	0.000	0.000
Mod. CBC	6.99141	11.3682	15.6350	19.8354	23.9926	28.1200	48.5206
%error	−0.063	−0.031	−0.017	−0.010	−0.007	−0.004	−0.001
			$g = 0.8,$		$c = 1$		
n	0	1	2	3	4	5	10
Exact	10.4197	14.3094	18.2551	22.2218	26.1988	30.1818	50.1346
WKB $V_{-}^{(1)}$	10.1890	14.1167	18.0735	22.0446	26.0237	30.0078	49.9625
%error	0.214	−1.346	−0.995	−0.798	−0.668	−0.576	−0.343
WKB $V_{+}^{(1)}$	10.4311	14.3111	18.2556	22.2220	26.1989	30.1818	50.1346
%error	0.109	0.012	0.003	0.001	0.000	0.000	0.000
Mod. CBC	10.3058	14.2568	18.2235	22.2003	26.1829	30.1694	50.1294
%error	−1.093	−0.368	−0.173	−0.097	−0.061	−0.041	−0.010
			$g = 0.1,$		$c = 2$		
n	0	1	2	3	4	5	10
Exact	11.5473	15.8197	20.0189	24.1723	28.2950	32.3960	52.7217
WKB $V_{-}^{(1)}$	11.4383	15.7125	19.9131	24.0675	28.1909	32.2924	52.6194
%error	−0.944	−0.677	−0.528	−0.434	−0.368	−0.320	−0.194
WKB $V_{+}^{(1)}$	11.3634	15.6386	19.8399	23.9948	28.1186	32.2203	52.5480
%error	−1.592	−1.144	−0.894	−0.734	−0.624	−0.542	−0.329
Mod. CBC	11.5418	15.8157	20.0159	24.1701	28.2933	32.3946	52.7211
%error	−0.048	−0.025	−0.015	−0.009	−0.006	−0.004	−0.001
			$g = 0.8,$		$c = 2$		
n	0	1	2	3	4	5	10
Exact	15.4806	19.2593	23.1076	26.9975	30.9137	34.8474	54.6484
WKB $V_{-}^{(1)}$	15.3902	19.1583	23.0045	26.8942	30.8107	34.7448	54.5469
%error	−0.584	−0.524	−0.446	−0.383	−0.333	−0.294	−0.186
WKB $V_{+}^{(1)}$	15.3622	19.1138	22.9505	26.8349	30.7482	34.6803	54.4784
%error	−0.765	−0.756	−0.680	−0.602	−0.535	−0.480	−0.311
Mod. CBC	15.4486	19.2261	23.0793	26.9741	30.8942	34.8311	54.6402
%error	−0.207	−0.172	−0.123	−0.087	−0.063	−0.047	−0.015
			$g = 0.1,$		$c = 3$		
n	0	1	2	3	4	5	10
Exact	16.0253	20.2200	24.3669	28.4827	32.5770	36.6556	56.9156
WKB $V_{-}^{(1)}$	15.9487	20.1445	24.2922	28.4086	32.5033	36.5822	56.8430
%error	−0.478	−0.373	−0.306	−0.260	−0.226	−0.200	−0.128
WKB $V_{+}^{(1)}$	15.9189	20.1147	24.2625	28.3789	32.4736	36.5526	56.8136
%error	-0.664	-0.521	-0.429	-0.364	-0.317	-0.281	-0.179
Mod. CBC	16.0196	20.2159	24.3639	28.4804	32.5751	36.6541	56.9149
%error	−0.036	−0.020	−0.013	−0.008	−0.006	−0.004	−0.001

Table 1 continued on next page.

Table 1*(concluded)*.

			$g = 0.8,$		$c = 3$		
n	0	1	2	3	4	5	10
Exact	20.0411	23.8521	27.7007	31.5776	35.4758	39.3902	59.1071
WKB $V_-^{(1)}$	19.9885	23.7919	27.6359	31.5100	35.4066	39.3201	59.0356
%error	−0.262	−0.252	−0.234	−0.214	−0.195	−0.178	−0.121
WKB $V_+^{(1)}$	19.9773	23.7772	27.6181	31.4898	35.3845	39.2966	59.0086
%error	−0.318	−0.314	−0.298	−0.278	−0.257	−0.238	−0.166
Mod. CBC	20.0335	23.8392	27.6859	31.5627	35.4615	39.3769	59.0983
%error	−0.038	−0.054	−0.053	−0.047	−0.040	−0.034	−0.015
			$g = 0.1,$		$c = 4$		
n	0	1	2	3	4	5	10
Exact	20.4418	24.5784	28.6845	32.7699	36.8406	40.9003	61.1026
WKB $V_-^{(1)}$	20.3828	24.5202	28.6268	32.7126	36.7836	40.8435	61.0464
%error	−0.289	−0.237	−0.201	−0.175	−0.155	−0.139	−0.092
WKB $V_+^{(1)}$	20.3670	24.5042	28.6107	32.6965	36.7675	40.8274	61.0302
%error	−0.366	−0.302	−0.257	−0.224	−0.199	−0.178	−0.118
Mod. CBC	20.4364	24.5744	28.6814	32.7675	36.8386	40.8987	61.1018
%error	−0.027	0.016	−0.011	−0.007	−0.005	−0.004	−0.001
			$g = 0.8,$		$c = 4$		
n	0	1	2	3	4	5	10
Exact	24.3877	28.2378	32.1083	35.9959	39.8977	43.8113	63.4993
WKB $V_-^{(1)}$	24.3483	28.1949	32.0625	35.9478	39.8478	43.7601	63.4452
%error	−0.161	−0.152	−0.143	−0.134	−0.125	−0.117	−0.085
WKB $V_+^{(1)}$	24.3413	28.1870	32.0536	35.9379	39.8371	43.7486	63.4314
%error	−0.190	−0.180	−0.171	−0.161	−0.152	−0.143	−0.107
Mod. CBC	24.3861	28.2333	32.1018	35.9882	39.8894	43.8027	63.4919
%error	−0.007	−0.016	−0.020	−0.021	−0.021	−0.020	−0.012

4. Discussion of the results

We have obtained energy levels corresponding to the potentials in (11), (12), and (18) by numerical integration, by the WKB rule (7), as well as by the modified CBC rule (5) for various values of the parameters. The results for $V_\pm^{(1)}$ with parameters $g = 0.1, 0.8$ and $c = 1, 2, 3, 4$ are presented in Table 1 and those for $V_\pm^{(2)}$ with $(a, b, c) = (1, \frac{1}{2}, 2), (1, \frac{1}{2}, 3), (2, 1, 2), (2, 1, 3)$ are given in Table 2. In both tables, we give the exact and approximate energy eigenvalues (the latter with their relative errors) for the ground state, the first five states, and the tenth excited state.[2] Relative errors of the WKB and the modified CBC results for potentials $V_\pm^{(2)}$ with parameter sets $(a, b, c) = (1.5, \frac{1}{2}, 2), (1, 1, 2)$ — the latter set gave the worst data for the modified CBC rule — have been presented graphically in Fig. 2.

From the tables we find that the modified CBC values (as well as the WKB values) underestimate the exact values. This is in contrast to the power-law potentials where the broken SUSY rule overestimates the true value [11, 14]. However, in the present case, relative errors corresponding to the energy values obtained using the modified CBC rule are always smaller than the corresponding WKB values for $V_+^{(1,2)}$ and $V_-^{(1,2)}$. The accuracy of the modified CBC rule (5) is also reflected by the

[2] Numerical results for the sixth to ninth excited states as well as for other sets of parameters may be requested from the authors.

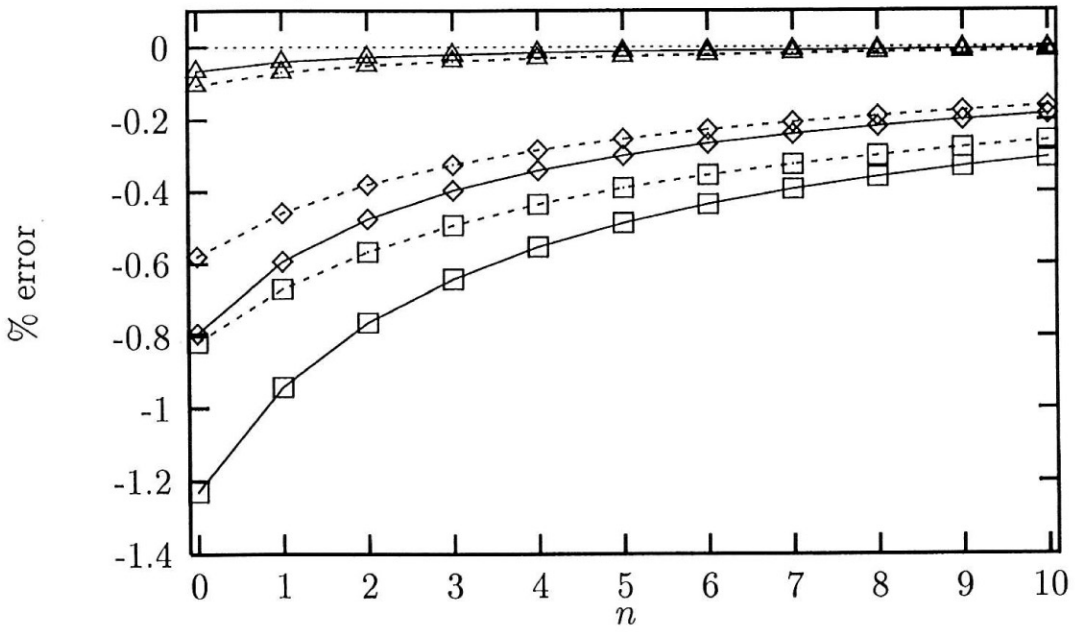
Table 2. Numerical results for the potential $V_{\pm}^{(2)}$ for parameter values $a = 1, 2$, $b = \frac{1}{2}, 1$ and $c = 2, 3$. Given are the exact as well as the approximate energy eigenvalues derived via the WKB and modified CBC rule. The table also shows the relative errors of the approximations.

			$a=1,\ b=\frac{1}{2},\ c=2$				
n	0	1	2	3	4	5	10
Exact	13.2516	17.5448	21.8085	26.0503	30.2751	34.4861	55.3992
WKB $V_{-}^{(2)}$	13.1518	17.4449	21.7084	25.9501	30.1748	34.3858	55.2987
%error	−0.753	−0.570	−0.459	−0.385	−0.331	−0.291	−0.181
WKB $V_{+}^{(2)}$	13.0993	17.3886	21.6499	25.8901	30.1137	34.3239	55.2345
%error	−1.149	−0.891	−0.727	−0.615	−0.533	−0.470	−0.297
Mod. CBC	13.2415	17.5366	21.8014	26.0440	30.2693	34.4808	55.3952
%error	−0.076	−0.047	−0.033	−0.024	−0.019	−0.015	−0.007
			$a=1,\ b=\frac{1}{2},\ c=3$				
n	0	1	2	3	4	5	10
Exact	17.8558	22.1074	26.3393	30.5557	34.7594	38.9526	59.8039
WKB $V_{-}^{(2)}$	17.7854	22.0368	26.2686	30.4849	34.6886	38.8817	59.7327
%error	−0.394	−0.319	−0.268	−0.232	−0.204	−0.182	−0.119
WKB $V_{+}^{(2)}$	17.7625	22.0129	26.2440	30.4597	34.6630	38.8558	59.7060
%error	−0.522	−0.428	−0.362	−0.314	−0.277	−0.248	−0.164
Mod. CBC	17.8499	22.1024	26.3348	30.5517	34.7557	38.9492	59.8012
%error	−0.033	−0.023	−0.017	−0.013	−0.011	−0.009	−0.004
			$a=2,\ b=1,\ c=2$				
n	0	1	2	3	4	5	10
Exact	29.4819	38.3029	47.0429	55.7226	64.3553	72.9497	115.5248
WKB $V_{-}^{(2)}$	29.2833	38.1038	46.8435	55.5230	64.1554	72.7496	115.3242
%error	−0.674	−0.520	−0.424	−0.358	−0.311	−0.274	−0.174
WKB $V_{+}^{(2)}$	29.1895	38.0009	46.7350	55.4107	64.0403	72.6324	115.2009
%error	−0.992	−0.788	−0.654	−0.560	−0.489	−0.435	−0.280
Mod. CBC	29.4548	38.2804	47.0233	55.7052	64.3394	72.9350	115.5137
%error	−0.092	−0.059	−0.042	−0.031	−0.025	−0.020	−0.010
			$a=2,\ b=1,\ c=3$				
n	0	1	2	3	4	5	10
Exact	39.1932	47.8981	56.5490	65.1570	73.7302	82.2740	124.6736
WKB $V_{-}^{(2)}$	39.0535	47.7579	56.4084	65.0162	73.5891	82.1327	124.5318
%error	−0.357	−0.293	−0.249	−0.216	−0.191	−0.172	−0.114
WKB $V_{+}^{(2)}$	39.0119	47.7137	56.3624	64.9689	73.5408	82.0837	124.4804
%error	−0.463	−0.385	−0.330	−0.289	−0.257	−0.231	−0.155
Mod. CBC	39.1772	47.8843	56.5366	65.1458	73.7199	82.2644	124.6663
%error	−0.041	−0.029	−0.022	−0.017	−0.014	−0.012	−0.006

ground-state energies. For both SUSY potentials (10) and (17) with $c > 1$, $E_0^{(\pm)}$ obtained using (5) lies within 0.016% – 0.207% of the exact value. Furthermore, for fixed c, the relative error decreases with increasing n. This trend can also be observed for fixed n and increasing c. In fact, for very high levels ($n = 10$) the maximum relative error being 0.015% while the minimum is 0.001%. It is also interesting to note that the level ordering $E_{n,l} < E_{n,l+1}$ holds within all the three methods for both SUSY models.

Up to now we have discussed essentially three-dimensional scenarios where both supersymmetric partners $V_{+}^{(1,2)}$ and $V_{-}^{(1,2)}$ have the angular-momentum barriers. We now turn to the s-wave, that is, $c = 1$, $l = 0$. In this case $V_{+}^{(1,2)}$ does not have the angular-momentum barrier, while $V_{-}^{(1,2)}$ has

Fig. 2. Graphs of the relative errors for the modified CBC rule (△), the WKB rule for $V_+^{(2)}$(◇) and $V_-^{(2)}$(□). The datasets are $a = 1.5, b = \frac{1}{2}, c = 2$ (unbroken line) and $a = 1, b = 1, c = 2$ (broken line).



this term. From Table 1, it can be observed that for $c = 1$, the application of the WKB rule to $V_+^{(1)}$ in (11) produces better approximations than the modified CBC rule (5). In fact, the former rule leads to an overestimation of the exact value, whereas the CBC rule as well as the WKB rule for $V_-^{(1)}$ underestimate the exact energy eigenvalue. This effect seems to be due to the missing angular–momentum barrier in $V_+^{(1)}$. Actually, in this case the left turning point in (7) has been set to zero, $\tilde{r}_a = 0$, representing an infinite repulsive wall. As is well known, for such a barrier (7) is not valid. On the right-hand side in (7) the factor $\frac{1}{2}$ was replaced by $\frac{3}{4}$ for this case. In any case, the parameter value $c = 1$ is rather exceptional and not qualified to test the CBC rule against the WKB rule.

In summary, the modified CBC rule (5) has been found to be an extremely good approximation for the systems under consideration. This seems to be due to the fact that upon proper tuning of parameters ($g = 0$ and $b = 0$) these potentials become shape-invariant and in turn the modified CBC rule leads to the exact energy eigenvalues.

Acknowledgement

One of the authors (P.R.) thanks the DFG and INSA for support. This work was also supported in part by a research grant from the Natural Sciences and Engineering Council of Canada to YPV.

References

1. A. Comtet, A.D. Bandrauk, and D.K. Campbell. Phys. Lett. **150B**, 159 (1985).
2. R. Dutt, A. Khare, and U.P. Sukhatme. Phys. Lett. **181B**, 295 (1986).
3. R. Adhikari, R. Dutt, A. Khare, and U.P. Sukhatme. Phys. Rev. A: Gen. Phys. **38**, 1679 (1988).
4. K. Raghunathan, M. Seetharaman, and S.S. Vasan. Phys. Lett. **188B**, 351 (1987).
5. A. Khare. Phys. Lett. **161B**, 131 (1985).
6. R. Dutt, A. Khare, and Y.P. Varshni. Phys. Lett. **123A**, 375 (1987); Y.P. Varshni. J. Phys. A: Math. Gen. **25**, 5761 (1992).
7. A. Khare and Y.P. Varshni. Phys. Lett. **142A**, 1 (1989).
8. D. DeLaney and M.M. Nieto. Phys. Lett. **247B**, 301 (1990).

9. A. Inomata and G. Junker. *In* Lectures on path integration: Trieste, 1991. *Edited by* H.A. Cerdeira et al. World Scientific, Singapore. 1993. p. 460.
10. A. Inomata, G. Junker, and A. Suparmi. J. Phys. A: Math. Gen. **26**, 2261 (1993).
11. A. Inomata and G. Junker. Phys. Rev. A, **50**, 3638 (1994).
12. R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta, and U. Sukhatme. Phys. Lett. **174A**, 363 (1993).
13. R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta, and U. Sukhatme.Phys. Rev. A, **48**, 1845 (1993).
14. G. Junker. Supersymmetric methods in quantum and statistical physics. Springer-Verlag, Berlin. 1996.
15. H. Risken and H.D. Vollmar. Z. Phys. **201**, 323 (1967).
16. H. Haken. Laser theory. Springer-Verlag, Berlin. 1983.
17. S.N. Biswas, K. Datta, R.P. Saxena, P.K. Srivastava, and V.S. Varma. J. Math. Phys. **14**, 1190 (1973).
18. A.K. Mitra. J. Math. Phys. **19**, 2018 (1978).
19. N. Bessis and G. Bessis. J. Math. Phys. **21**, 2780 (1980).
20. G.P. Flessas. Phys. Lett. **83A**, 121 (1981).
21. C.S. Lai and H.E. Lin. J. Phys. A: Math. Gen. **15**, 1495 (1982).
22. M. Znojil. J. Phys. A: Math. Gen. **16**, 293 (1983).
23. Y.P. Varshni. Phys. Rev. A: Gen. Phys. **36**, 3009 (1987); H. Scherrer, H. Risken, and T. Leiber. Phys. Rev. A: Gen. Phys. **38**, 3949 (1988).
24. R. Dutt, R. Adhikari, and Y.P. Varshni. J. Math. Phys. **32**, 447 (1991).
25. P. Roy, R. Roychoudhury,and Y.P. Varshni. J. Phys. A: Math. Gen. **21**, 1589 (1988).
26. F. Cooper, A. Khare, and U.P. Sukhatme. Phys. Rep. **251**, 267 (1995).
27. V.V. Semenov. J. Phys. A: Math. Gen. **21**, L721 (1990).
28. H. Nicolai.J. Phys. A: Math. Gen. **9**, 1497 (1976).
29. E. Witten. Nucl. Phys. **B188**, 513 (1981).