

ON ASYMPTOTIC REPRESENTATION AND
APPROXIMATION TO NORMALITY OF
L-STATISTICS-II

By KESAR SINGH

Indian Statistical Institute and Stanford University

SUMMARY. An asymptotic representation and some convergence rates to normality have been studied for linear functions of order statistics based on mixing processes.

1. INTRODUCTION

An earlier paper of the author, Singh (1979), presented a study on an asymptotic representation and convergence rates to normality of *L*-statistics based on i.i.d. random variables. The present paper intends to extend the domain of this work to some weakly dependent processes. Singh (1979) is referred to as S1 hereafter in this paper. Our main attempt is to give some extensions of the basic bounds used in S1. These bounds are mostly about empirical distribution functions and are of independent importance. To avoid undesirable duplication, no serious attempt is made to make this article self-contained. The reader is referred to S1 for a detailed description of the problem tackled. Technical terms are defined at their first appearances in the paper.

The dependent processes considered here are the two most popular mixing processes— ϕ -mixing and strong mixing. A process $\{X_n\}$ is called ϕ -mixing if there exists a sequence of non-negative numbers $\{\phi_i\}$ decreasing to zero such that

$$\sup\{|P(B|A) - P(B)|; P(A) > 0, A \in \mathcal{B}_{-\infty}^a, B \in \mathcal{B}_{a+n}^{\infty}\} \leq \phi_n$$

where \mathcal{B}_a^b stands for $\sigma(X_a, \dots, X_b)$, $b \geq a$. $\{X_n\}$ is called strong mixing if there exists a sequence of non-negative reals $\{\alpha_i\}$ decreasing to zero such that

$$\sup\{|P(AB) - P(A)P(B)|; A \in \mathcal{B}_{-\infty}^a, B \in \mathcal{B}_{a+n}^{\infty}\} \leq \alpha_n.$$

The class of ϕ -mixing processes includes *m*-dependent r.v.s. and some Markov processes. Most commonly given examples of strong mixing processes come

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from the classes of dependent gaussian, moving average and autoregressive processes. The condition of ϕ -mixing is much more restrictive than that of strong mixing but the asymptotic analysis is generally tougher in the later case and some times the bounds are less good.

The proofs of the key bounds about o.d.f.s are collected in Section 2. This section is mostly self-contained and a reader who is basically interested in e.d.f. may confine to this section. Also, this section contains two very general exponential bounds, one for each of the above mentioned mixing processes, which are usable in several contexts. In Section 3, we combine the bounds of Section 2 and some of S1, which do not require independence, to produce a.s. expansion of L -statistics based on mixing processes. Finally, we aim at the convergence rates in Section 4 where we borrow some results of Statulevičius on Berry Esseen type bounds for mixing processes.

Throughout l_n^* and ll_n^* denote $(\log n)^*$ and $(\log \log n)^*$ respectively and c_i 's stand for positive constants.

2. SOME ASYMPTOTIC BOUNDS FOR E.D.F.

Let $\{X, X_1, X_2, \dots\}$ be a stationary sequence of r.v.s with $E|X| < \infty$ and X having a continuous d.f.F, so that the transformed r.v.s $U = F(X)$, $U_i = F(X_i)$, $i = 1, 2, \dots$ have $U[0, 1]$ distribution. Let $F_n(\cdot)$ and $V_n(\cdot)$ denote the o.d.f.s (right continuous versions) of $\{X_1, X_2, \dots, X_n\}$ and $\{U_1, U_2, \dots, U_n\}$ respectively. For some $r \in [0, 1/2]$, we let

$$V_n(t, r) = (t(1-t))^{-r} |V_n(t) - t|$$

and

$$E_r = \sup\{V_n(t, r); t \in (0, 1)\}.$$

As mentioned in S1, if $\{X_i\}$ are independent, $E_r \ll n^{-1/l_n^*}$ a.s. for all $r \in [0, 1/2]$. Our first plan is to present the extensions of this bound that have been possible for mixing r.v.s. The results are stated below.

Proposition 2.1: If $\{X_i\}$ is a ϕ -mixing process with $\Sigma \phi_i^{1/s} < \infty$ for some $s \geq 1$, then for all $0 \leq r < 1/2 - 1/2(s+1)$, $E_r \ll n^{-1/l_n^*}$ a.s.

Proposition 2.2: If $\{X_i\}$ is a strong-mixing process with $\Sigma x_i^{1/s} < \infty$ for some $s > 0$, then, for all $0 \leq r < 1/2 - 4/(s+2)$, $E_r \ll n^{-1/l_n^*}$ a.s.

Let us see first the proof of Proposition 2.1. The proof is based on probability inequalities provided by Lemma 2.1 given below. In fact, Lemma 2.1 is the only property of ϕ -mixing processes to be used throughout this section. Let us define

$$x_i(s, t) = I(\min(s, t) \leq U_i \leq \max(s, t)) - |s - t|$$

for some $0 < s, t < 1$ and let ψ_n be the smallest positive integer i such that $n\phi_i/i \leq 1$. Further, let $S_{u,\beta}$ stand for the sum $\sum_{i=H+1, H+u}^{\infty} x_i(\alpha, \alpha+\beta)$, suppressing H and α which are irrelevant for probability bounds.

Lemma 2.1: Let D, Z, B be positive numbers and N some positive integer obeying the relationship $ZNB \leq D^2$. Let $\{X_i\}$ be a ϕ -mixing process with $\sum \phi_i < \infty$. Then, for all $\beta \leq B, u \leq N$ and $c > 0$, there exist constants c_1, c_2, c_3 such that

$$P(|S_{u,\beta}| > 2c_1 D) \leq c_2 e^{-Z} + c_3 N \psi_N^{-1} (D/Z)^{-2c} \quad \dots (2.1)$$

The constants c_1, c_2, c_3 are independent of D, Z, B and N ; however c_3 may depend upon c .

Proof: W.l.g. let us assume $S_{u,\beta} = \sum_{i=1, u}^{\infty} x_i(0, \beta)$ for some $\beta \in (0, 1)$ and let $k = [u/2\psi_N]$. Let us write

$$S_{u,\beta} = \sum_{i=1, k}^{\infty} (\xi_i + \xi'_i) + \xi_{k+1}$$

where

$$\xi_i = \sum_{j=1, \psi_N}^{\infty} x_{(2i-2)\psi_N+j}^{(0, \beta)}$$

and

$$\xi'_i = \sum_{j=1, \psi_N}^{\infty} x_{(2i-1)\psi_N+j}^{(0, \beta)}$$

provided $k > 1$. If $k = 0$, we understand that $\xi_{k+1} = S_{u,\beta}$ and $\sum_{i=1, k} \xi_i = \sum_{i=1, k} \xi'_i = 0$. We show now that $P(\sum_{i=1, k+1} \xi_i > c_1 D)$ has an upper bound like the r.h.s. of (2.1). The bound is obtained similarly for $-\sum_{i=1, k+1} \xi_i$ and $\pm \sum_{i=1, k} \xi'_i$ to conclude (2.1).

Let ξ_i^* denote $\xi_i I\{|\xi_i| \leq D/Z\}$. Now using the moment inequality given by Lemma 2 of Ghosh and Babu (1977), which clearly remains valid under the condition $\sum \phi_i < \infty$ for bounded r.v.s., we see that

$$\begin{aligned} & |P(\sum_{i=1, k+1} \xi_i > c_1 D) - P(\sum_{i=1, k+1} \xi_i^* > c_1 D)| \\ & \leq \sum_{i=1, k+1} P(|\xi_i| > D/Z) \leq c_4 (N/\psi_N) \psi_N^c (D/Z)^{-2c} \end{aligned}$$

Now, using Markov's inequality, the fact that $\phi_{\psi_N} \leq \psi_N/N$ and Lemma 1 on page 170 of Billingsley (1968) repeatedly, we find that

$$\begin{aligned} P(\sum_{i=1, k+1} \xi_i^* > c_1 D) & \leq \exp(-c_1 Z) E \exp(ZD^{-1} \sum_{i=1, k+1} \xi_i^*) \\ & \leq 3 \exp(-c_1 Z) [E \exp(ZD^{-1} \xi_1^*) + 2r \phi_{\psi_N}]^k \end{aligned}$$

$$\begin{aligned} &< 3 \exp(-c_1 Z + k \log(1 + |E Z D^{-1} \xi_{1,0} + E(Z D^{-1} \xi_{1,0})^2 + 2c_2 \psi_N N^{-1})|) \\ &< 3 \exp(-c_1 Z + k(2 E(Z D^{-1} \xi_{1,0})^2 + 2c_2 \psi_N N^{-1})) \\ &< c_3 e^{-Z} \end{aligned}$$

if c_1 is appropriately large. In the above derivation, we also used the condition that $ZNB \leq D^2$ and the easily verifiable facts that

$$|E Z D^{-1} \xi_{1,0}| \leq E(Z D^{-1} \xi_{1,0})^2 \text{ and } E \xi_{1,0}^2 \leq c_4 \psi_N B.$$

Proof of Proposition. 2.1: Let us fix some $0 < \delta < 1/2 - 1/2(s+1)$ and show that

$$\sup\{V_n(t, 1/2 - 1/2(s+1) - \delta); t \in (0, 1/2)\} \ll n^{-1} l_n^s \text{ a.s.}$$

which will then imply by symmetry that

$\sup\{V_n(t, 1/2 - 1/2(s+1) - \delta); t \in [1/2, 1)\} \ll n^{-1/2} l_n^{s/2}$ a.s. and these two together will complete the proof of the proposition.

We write a for $1/\delta$ in rest of this proof. Let us divide the interval $(0, 1/2]$ into three parts namely I_{n1} , I_{n2} and I_{n3} where $I_{n1} = (0, n^{-1} l_n^a]$, $I_{n2} = (n^{-1} l_n^a, l_n^a)$ and $I_{n3} = (l_n^a, 1/2]$. Since $\sum n^{-1} l_n^{-a} < \infty$, the event $\{U_i \in I_{n1}\}$ happens only for finitely many i 's a.s. This along with the fact that $n^{-1} l_n^a \downarrow 0$ implies that $V_n(n^{-1} l_n^a) = 0$ for all sufficiently large n a.s. Using this, it is trivial to see that

$$\sup\{V_n(t, 1/2 - 1/2(s+1) - \delta); t \in I_{n1}\} \ll n^{-1/2} l_n^{s/2} \text{ a.s.}$$

Thus, it suffices to see that

$$\sup\{V_n(t, 1/2 - 1/2(s+1) - \delta); t \in I_{n2}\} \ll n^{-1/2} l_n^{s/2} \text{ a.s.} \quad \dots (2.2)$$

and

$$\sup\{V_n(t, 1/2 - 1/2(s+1) - \delta); t \in I_{n3}\} \ll n^{-1/2} l_n^{s/2} \text{ a.s.} \quad \dots (2.3)$$

To prove (2.2), we divide the interval I_{n2} into subintervals of length $n^{-1} l_n^{-a}$ and observe that

$$\begin{aligned} &\sup\{|V_n(t) - t| t^{-1/2 + 1/2(s+1) + \delta}; t \in I_{n2}\} \\ &< 2 \max\{|V_n(t) - t| t^{-1/2 + 1/2(s+1) + \delta}; t = n^{-1} l_n^a, 2n^{-1} l_n^a \\ &\dots ((l_n^a/n^{-1} l_n^a) + 1)n^{-1} l_n^a\} + O(n^{-1/2}). \quad \dots (2.4) \end{aligned}$$

For some fixed value of t as in the r.h.s. of (2.4) it follows using Lemma 2.1, with $N = n$, $D = (4nl_n)^{1/2} t^{1/2 - 1/2(s+1) - \delta/2}$, $Z = 4l_n$, $B = t$ and c large enough, that

$$P(|V_n(t) - t| > 2c_1(4nl_n)^{1/2} t^{1/2 - 1/2(s+1) - \delta/2}) \ll n^{-\delta}$$

for some $\epsilon > 0$. In the above application of Lemma 2.1, we also require the fact that $\Sigma \phi_i^{1/\epsilon} < \infty \implies \psi_n \ll n^{1/(\epsilon+1)}$. Now, in view of this probability bound, we have

$$\sup\{V_n(t, 1/2 - 1/2(\epsilon+1) - \delta/2 : t \in I_{n3}\} \ll n^{-1/2} n^{1/(\epsilon+1)} \text{ a.s.} \quad \dots (2.5)$$

But, since $t^{1/2} \ll I_n^{-1/2}$ for all $t \in I_{n3}$, (2.5) yields (2.2).

Coming to (2.3), we divide the interval I_{n3} into subintervals of length n^{-1} and see that

$$\begin{aligned} & \sup\{|V_n(t) - t| | t^{-1/2} + 1/2(\epsilon+1) + \delta : t \in I_{n3}\} \\ & \leq 2 \max\{|V_n(t) - t| | t^{-1/2} + 1/2(\epsilon+1) + \delta : t = I_n^{-\epsilon}, I_n^{-\epsilon} + n^{-1}, \\ & \quad I_n^{-\epsilon} + 2n^{-1}, \dots, I_n^{-\epsilon} + ([n/2] + 1)n^{-1}\} + O(n^{-1/2}). \quad \dots (2.6) \end{aligned}$$

Now, we define three sequences of events and see that all the three happen only finitely often which then implies that the r.h.s. of (2.6) $\ll n^{-1/2} n^{1/(\epsilon+1)}$ a.s. Let $n_r = \exp(r^{1/2})$ (so that $n_r(r+1)^{-1/2} < n_{r+1} - n_r < n_r/r^{1/2}$) and $S_r = \{n : n_r < n < n_{r+1}\}$. For $n \in S_r$, let us define

$$\begin{aligned} A_n &= \{\max\{|\Sigma_{t=n_r+1}^n x_i(0, t)| | t^{-1/2} + 1/2(\epsilon+1) + \delta; \\ & \quad t = I_n^{-\epsilon} + jr^{-4\epsilon}, 1 \leq j \leq [r^{4\epsilon}/2] + 1\} > 4c_1 n^{1/2}\} \\ I_r &= \{\max\{|V_n(t) - t| | t^{-1/2} + 1/2(\epsilon+1) + \delta; t = I_n^{-\epsilon} + jr^{-4\epsilon}, \\ & \quad 1 \leq j \leq [r^{4\epsilon}/2] + 1\} > (8\epsilon+4)^{1/2} 2c_1 n_r^{-1/2} n^{1/2}\} \end{aligned}$$

and

$$\begin{aligned} O_n &= \{\max\{|V_n(x(j), 1/2 - 1/2(\epsilon+1) - \delta) - V_n(x'(j), 1/2 - 1/2(\epsilon+1) - \delta)|; \\ & \quad 1 \leq j \leq [n/2] + 1\} > 2c_2 n_r^{-1/2}\} \end{aligned}$$

where $x(j) = I_n^{-\epsilon} + j/n$ and $x'(j) = I_n^{-\epsilon} + [jr^{4\epsilon}/n]r^{-4\epsilon}$

We appeal to Lemma (2.1) to estimate the probabilities of all the three events above. In Lemma (2.1) we take $N = n_{r+1} - n_r$, $B = t$, $D = (4n_r)^{1/2} t^{1/2} - 1/2(\epsilon+1) - \delta$ and $Z = 4I_n$ for A_n and $N = n_r$, $B = t$, $D = ((8\epsilon+4)n_r)^{1/2} t^{1/2} - 1/2(\epsilon+1) - \delta$ and $Z = (8\epsilon+4)I_n$ for I_r to conclude that $\sum_{n=1, \infty} P(A_n) < \infty$ and $\sum_{r=1, \infty} P(I_r) < \infty$. Finally, coming to O_n , we note that, for all $1 \leq j \leq [n/2] + 1$.

$$\begin{aligned}
& |V_n(z(j), 1/2 - 1/2(\delta + 1) - \delta) - V_n(z'(j), 1/2 - 1/2(\delta + 1) - \delta)| \\
& \leq 2 |V_n(z(j)) - z(j) - V_n(z'(j)) + z'(j)| \{(z'(j))^{1/2} - 1/2(\delta + 1) - \delta\} \\
& + 2 |V_n(z(j)) - z(j)| \{(z(j))^{-1/2} + 1/2(\delta + 1) + \delta\} - (z'(j))^{-1/2} + 1/2(\delta + 1) + \delta | \\
& \leq c_7 |V_n(z(j)) - z(j) - V_n(z'(j)) + z'(j)| r^{a/2} + c_8 |V_n(z(j)) - z(j)| / r
\end{aligned}$$

where the last inequality follows from the facts that $|z(j) - z'(j)| \leq r^{-2a}$, $z(j) \geq r^{-a}$, $z'(j) \geq r^{-a}$ and $a = 1/\delta \geq 2$. Thus,

$$\begin{aligned}
P(C_n) & \leq n \max_{1 \leq j \leq [n/2] + 1} P(|V_n(z(j)) - z(j) - V_n(z'(j)) + z'(j)| \\
& \geq 2c_7 c_7^{-1} r^{-a/2} n^{1/2} + P(|V_n(z(j)) - z(j)| \geq 2c_8 c_8^{-1} r n^{1/2}).
\end{aligned}$$

The first term in the r.h.s. above is shown to be summable taking $N = n$, $B = r^{-2a}$, $Z = 4l_n$ and $D = c_7^{-1} r^{-a/2} n^{1/2}$ in Lemma 2.1 and the second term is shown to be summable taking $N = n$, $B = 1$, $Z = 4l_n$ and $D = c_8^{-1} r n^{1/2}$. Now the assertion (2.3) follows using Borel-Cantelli Lemma.

Proposition (2.2) is also proved using exactly the same argument but we need an exponential bound for strong mixing random variables to replace Lemma 2.1. The bound is given by

Lemma 2.3.: Let $\{X_i\}$ be a strong mixing process with $\sum \alpha_i^{1/8} < \infty$ for some $s > 1$, s_0 be defined by $s^{-1} + s_0^{-1} = 1$ and $s_{u,s}$ be as defined earlier. Let D, Z, B be positive numbers and N some positive integer satisfying $ZNB^{1/s_0} \leq D^2$ and $D \geq 4Z$. Then for all $u \leq N$ and $\beta \leq B$, there exist constants, c_9, c_{10} such that

$$P(|s_{u,s}| \geq 2c_9 D) \leq 2e^{-Z} + c_{10} N^2 D^{-2} \alpha_{[D]Z} \beta. \quad \dots (2.7)$$

The constants c_9, c_{10} do not depend upon Z, N, B and D .

Proof: The basic idea of martingale approximation used in this proof is borrowed from Philipp (1977), but the technique has been modified considerably. Let us write $s_{u,s} = \sum_{i=1}^u \xi_i + \sum_{i=1}^k \xi'_i$ just as in the proof of Lemma 2.1, except that we replace ψ_u by $p = [D/4Z]$. Thus $k = [u/2p]$. We get below a bound like (2.7) for $P(\sum_{i=1}^u \xi_i \geq c_9 D)$ and the procedure is to be repeated for $-\sum_{i=1}^u \xi_i$ and $\pm \sum_{i=1}^k \xi'_i$ to conclude the lemma.

Let $\xi_0 = 0 = \xi_j$ for all $j > k+1$, $\mathcal{L}_i = \sigma(\xi_0, \dots, \xi_i)$ and $\eta_i = \xi_i - E(\xi_i/\mathcal{L}_{i-1})$ for all $i \geq 1$. Thus $\{\eta_1, \eta_2, \dots\}$ is a martingale difference sequence bounded by D/Z . Now, we use the result on page 209 of Stout (1974) (also stated as Lemma 3.2.5. in Philipp (1977)) to conclude that the sequence $\{T_i\}$ defined as $T_0 = 1$ and for all $i \geq 1$,

$$T_i = \exp(ZD^{-1} \sum_{j=1, i} \eta_j - (3/4)(ZD^{-1})^2 \sum_{j=1, i} E(\eta_j^2/\mathcal{L}_{j-1}))$$

is a non-negative super-martingale and that

$$\begin{aligned} P(ZD^{-1} \sum_{i=1, k+1} \eta_i - (3/4)(ZD^{-1})^2 \sum_{i=1, k+1} E(\eta_i^2/\mathcal{L}_{i-1}) > Z) \\ = P(T_{k+1} > Z) < e^{-2}. \end{aligned}$$

Thus, it only remains to be shown that

$$\begin{aligned} P(|\sum_{i=1, k+1} E(\xi_i/\mathcal{L}_{i-1}) + (3/4)(ZD^{-1}) \sum_{i=1, k+1} E(\eta_i^2/\mathcal{L}_{i-1})| \\ > c_{11}D) < c_{12}N^2 D^{-3} \alpha_{[D]/4\pi}. \end{aligned} \quad \dots (2.8)$$

To this end, we require a few estimates based on Davydov's inequality (see Lemma 1 of Deo, 1973) which we establish now. The first estimate is as follows :

$$\begin{aligned} \|\sum_{i=1, k+1} E(\xi_i/\mathcal{L}_{i-1})\|_2 &\leq \sum_{i=1, k+1} \|E(\xi_i/\mathcal{L}_{i-1})\|_2 \\ &= \sum_{i=1, k+1} (E(E(\xi_i/\mathcal{L}_{i-1})^2))^{1/2} \leq \sum_{i=1, k+1} (4\alpha_p \|E(\eta_i^2/\mathcal{L}_{i-1})\|_2)^{1/2} \leq 2\alpha_p^{1/2} N. \end{aligned} \quad \dots (2.9)$$

Due to the same reasons, we also have

$$\|\sum_{i=1, k+1} E(\eta_i^2/\mathcal{L}_{i-1} - E\eta_i^2)\|_2 \leq c_{13} \alpha_p^{1/2} p N. \quad \dots (2.10)$$

Finally, for any $u \geq 2$

$$\begin{aligned} E\sigma_{u, \beta}^2 &= E(\sum_{i=1, u} x_i^2(0, |\beta|) + 2\sum_{i=1, u-1} (u-i)x_i(0, |\beta|)x_{i+1}(0, |\beta|)) \\ &\leq u|\beta|(1-|\beta|) + 2u\sum_{i=1, u-1} (1-i/u)\|x_i(0, |\beta|)\|_{\infty} \alpha_1^{1/2} \\ &\leq uB + 2B^{1/2} \alpha_1^{1/2} \sum_{i=1, u} \alpha_1^{1/2} u \leq (1 + 2\sum_{i=1, u} \alpha_1^{1/2}) u B^{1/2} \end{aligned} \quad \dots (2.11)$$

so that

$$\begin{aligned} \sum_{i=1, k+1} E\eta_i^2 &= \sum_{i=1, k+1} E(\xi_i - E(\xi_i/\mathcal{L}_{i-1}))^2 \\ &= \sum_{i=1, k+1} (E\xi_i^2 - E(E\xi_i/\mathcal{L}_{i-1})^2) \leq \sum_{i=1, k+1} E\xi_i^2 \\ &\leq (1 + 2\sum_{i=1, u} \alpha_1^{1/2}) N B^{1/2} \leq (1 + 2\sum_{i=1, u} \alpha_1^{1/2})(D/Z)D. \end{aligned} \quad \dots (2.12)$$

Now, combining the estimates obtained through (2.9)–(2.11), and using Chebychev inequality, we see that if $c_{11} > 1 + 2\sum_{t=1, \infty} \alpha_t^{1/s}$, l.h.s. of (2.8) can not exceed

$$P(|E\sum_{t=1, \infty} E(\xi_t/\sigma_{t-1}) + (3/4)ZD^{-1}\sum_{t=1, \infty} (E(\eta_t^2/\sigma_{t-1}) - E\eta_t^2)| \\ > (c_{11} - 1 - 2\sum_{t=1, \infty} \alpha_t^{1/s})D) \leq c_{12}N^2\alpha_p D^{-2}.$$

The proof of Proposition. 2.2 is omitted but we offer an explanatory remark about it.

Remark 2.1 : The restriction $r < 1/2 - 4/(s+2)$ in Proposition 2.2 emerges out of the requirement that $n^2 D^{-2} \alpha_{D/4Z} \ll n^{-2-s}$ while dealing with I_{2n} , since in that particular application of Lemma 2.2, we have got to take $Z = 4I_n$ and $D = (nI_n)^{1/2}r$. The other requirements are automatically met if $r < 1/2 - 4/(s+2)$.

Next, we give bounds for probabilities of deviations of E_r which are direct extensions of Lemma 3.1 of S1 to mixing r.v.s.

Proposition 2.3 : If $\{X_n\}$ is ϕ -mixing with $\sum \phi_t^{1/s} < \infty$ for some $s > 1$, then for any positive numbers a, b and r satisfying

$$0 \leq r < \min((2+2a)^{-1}(1-(s+1)^{-1}), b^{-1}) \quad \dots (2.13)$$

there exists a positive constant $c_\phi = c_\phi(a, b, r)$ such that if $y^2 > c_\phi I_n$

$$P(E_r > n^{-1/2}y) \leq n^{-a}y^{-b}. \quad \dots (2.14)$$

Proof : Admittedly, the proof is pretty much similar to that of Lemma 3.1 of S1, once we have the exponential bound of Lemma 2.1. A sketch of the proof is given below.

Let us write $(0, 1) = U_{t=1, t} J_{1n}$, where

$$J_{1n} = (0, n^{-1-a}y^{-b}), \quad J_{2n} = (n^{-1-a}y^{-b}, 1/2), \\ J_{3n} = (1/2, 1 - n^{-1-a}y^{-b}) \text{ and } J_{4n} = [1 - n^{-1-a}y^{-b}, 1).$$

As in S1, for all $r \in (0, 1/2)$,

$$P(\sup\{|V_n(t, r)| : t \in J_{1n}\} \geq n^{-1/2}) \leq n^{-a}y^{-b}.$$

We divide the interval J_{3n} into subintervals of length $n^{-1-a}y^{-b} = \gamma_n$ and see that

$$\sup\{|V_n(t, r)| : t \in J_{3n}\} \leq 2 \sup\{|V_n(t) - t|t^{-r} : t \in J_{3n}\} \\ < 4 \max\{|V_n(t) - t|t^{-r} : t = j\gamma_n, j = 1, 2, \dots, [\gamma_n^{-1/2}] + 1\} + n^{-1/2}. \quad \dots (2.15)$$

For some t as in the r.h.s. of (2.15), and r satisfying (2.13), we apply Lemma (2.1) with $N = n$, $D = t^r n^{1/2} y / 10c_1$, where c_1 is the constant appearing in Lemma 2.1, $Z = (2a+1)l_n + 2bl_y$ (where $l \equiv \log y$) and $B = t$ to see that if $y^2 > (10c_1)^2((2a+1)l_n + 2bl_y)$,

$$P(|V_n(t) - t| t^{-r} > n^{1/2} y / 5) \ll c_{13} n^{-2a-1} y^{-2b}. \quad \dots (2.16)$$

The condition (2.13) is actually needed to ensure that for c sufficiently large

$$n y_n^{-1} (D/Z)^{-2c} \ll n n^{(c-1)/(1+\delta)} [(n-1-a_j b)^r n^{1/2} y]^{-2c} \ll n^{-2a-1} y^{-2b}.$$

The requirement $y^2 > (10c_1)^2((2a+1)l_n + 2bl_y)$ is met if $y^2/2 > (10c_1)^2(2a+1)l_n$ and $y^2/2 > (10c_1)^2 2bl_y$, but the second condition is contained in the first one for all n large enough. Now (2.15) and (2.16) together yield

$$P(\sup\{|V_n(t, r)| : t \in J_{2n}\} > n^{-1/2} y) \ll n^{-a} y^{-b}$$

provided y^2/l_n is large enough. Similar bounds on J_{3n} and J_{4n} complete the proof.

Proposition 2.4 : If $\{X_n\}$ is strong mixing with $\Sigma \alpha_i^{1/\delta} < \infty$ for some $\delta > 1$, then for any given positive numbers a, b and r satisfying

$$0 \leq r < \min\{(2a+2)^{-1} - (2a+3)(\delta+2)^{-1}(1+a)^{-1}, \delta^{-1} - 2(\delta+2)^{-1}\} \quad \dots (2.17)$$

there exists a positive constant $c_a = c_a(a, b, r)$ such that if $y^2 > c_a l_n$,

$$P(K_r > n^{-1/2} y) \ll n^{-a} y^{-b}.$$

Proof: The proof is same as that of Proposition (2.3) except that (2.16, is to be derived using Lemma 2.2. To do so, we make the same choices of Z, N, B and choose D to be $t^r n^{1/2} y / 10c_a$, where c_a is the constant appearing in Lemma 2.2. We have got to check that

$$n^2 D^{-3} \alpha_{(D/Z)} \ll n^{-2a-1} y^{-2b} \quad \dots (2.18)$$

and

$$2r \leq 1 - \delta^{-1}. \quad \dots (2.19)$$

The second requirement comes in verifying that $ZNB^{1/\delta} \ll D^3$ where $\delta^{-1} + \delta^{-1} = 1$.

Since $t > n^{-1-a} y^{-b}$, (2.18) holds provided

$$(1/2 - (1+a)r)(\delta+2) > (2a+3) \quad \dots (2.20)$$

and

$$(1-br)(\delta+2) > 2b. \quad \dots (2.21)$$

(2.17) contains (2.19), (2.20) and (2.21).

Finally, in this section, we study the asymptotic nature of

$$E_{**} = E_{**}(n) = \sup\{|V_n(V_n^{-1}(t)) - t| : t \in (0, 1)\}.$$

The statement $E_{**} \leq n^{-1}$ u.s. just amounts to telling that the random variables $\{U_i\}$ take distinct values u.s. and this is obviously so if U_i 's are independent and have $U[0, 1]$ distribution. This idea can be stretched easily to see that $E_{**} \leq m/n$ a.s. for m -dependent r.v.s. But, the method works no more once we move to mixing processes. We have weaker statements in these cases, which we present now.

Proposition 2.5: If $\{X_t\}$ is a ϕ -mixing process with $\Sigma\phi_t^h < \infty$ for some $s > 1$ and γ is a positive numbers s.t. $\gamma < 1 - (2s+2)^{-1}$, then

$$P\{E_{**} > (2c_1+2)n^{-\gamma}\} \ll n^{-a}$$

for any positive number a .

$$\begin{aligned} \text{Proof: } E_{**} &= \sup\{|V_n(V_n^{-1}(t)) - t| : t \in [0, 1]\} \\ &\leq \sup\{|V_n(V_n^{-1}(t)) - V_n(V_n^{-1}(t) - 0)| : t \in [0, 1]\} \\ &\leq \sup\{|V_n(V_n^{-1}(t)) - V_n(V_n^{-1}(t) - n^{-\gamma})| : t \in [0, 1]\} \\ &\leq \max\{|V_n(t) - V_n(t + 2n^{-\gamma})| : t = jn^{-\gamma}, j = 0, 1, 2, \dots, [n^\gamma]\} \\ &\leq \max\{|V_n(t) - V_n(t + 2n^{-\gamma}) + 2n^{-\gamma}| : t = jn^{-\gamma}, \\ &\quad j = 0, 1, \dots, [n^\gamma]\} + 2n^{-\gamma}. \end{aligned} \quad \dots (2.22)$$

Now as usual, we apply Lemma 2.1 appropriately to derive the proposition.

Proposition 2.6: If $\{X_t\}$ is a strong mixing process with $\Sigma\alpha_t^h < \infty$ for some $s > 1$ and γ is s.t. $0 < \gamma < 1 - 3/(s+3)$, then

$$P\{E_{**} > c_s n^{-\gamma}\} \ll n^{-s+(s+3)/s}$$

and if $\alpha_t \ll e^{-t}$ then, for all $a > 0$, there exists a constant c_a s.t.

$$P\{E_{**} > c_a n^{-1/2}\} \ll n^{-a}.$$

Proof: For the first part, we use the bound (2.22) and apply Lemma 2.2. For the second part, we mimic the steps of (2.22) to see that

$$E_{**} \leq \max\{|V_n(t) - V_n(t + 2n^{-\gamma}) + 2n^{-\gamma}| : t = jn^{-\gamma}, j = 0, 1, \dots, [n^\gamma]\} + 2n^{-\gamma}$$

and then apply Lemma 2.2 with $D = (c_s - 2)^2 n^\gamma$, $B = 2n^{-\gamma}$, $Z = (a+2)n^\gamma$. The constant c_a is adjusted to be large enough.

Corollary (I): In the ϕ -mixing case with $\Sigma \phi_t^{1/s} < \infty$, for some $s < 1$, $E_{s,s} \ll n^{-\gamma}$ a.s. for all $\gamma < 1 - (2s+2)^{-1}$.

(II): In the strong mixing case with $\Sigma \alpha_t^{1/s} < \infty$, for some $s < 1$, $E_{s,s} \ll n^{-\gamma}$ a.s. for all $\gamma < 1 - 3(s+3)^{-1}$.

(III): In the strong mixing case with $\alpha_t \ll e^{-t}$, $E_{s,s} \ll n^{-1/s}$ a.s.

The first and the third part of the corollary are direct application of Borel-Cantelli lemma. The arguments for the second part go as follows:

If $\gamma < 1 - 3(s+3)^{-1}$, then $P(E_{s,s} > c_{1s} n^{-\gamma})$ is summable along the subsequence $\{2^r\}$. Thus, $E_{s,s}(2^r) \ll c_{1s}(2^r)^{-\gamma}$ for all γ sufficiently large a.s. Since $E_{s,s}(n)$ is nothing but the amount of the biggest jump of the function $V_n(t)$, $t \in [0, 1]$; hence $nE_{s,s}(n) =$ cardinality of the biggest subset of $\{U_1, U_2, \dots, U_n\}$ with equal values. Therefore, $nE_{s,s}(n)$ is a non-decreasing function of n . In view of this, for some n a.s. $2^r \leq n \leq 2^{r+1}$

$$nE_{s,s}(n) \leq 2^{r+1} E_{s,s}(2^{r+1}) \leq 2^{r+1} c_{1s}(2^{r+1})^{-\gamma} \leq 2c_{1s}n^{1-\gamma}$$

for all r large enough, a.s. Thus we have the conclusion.

3. ALMOST SURE REPRESENTATION OF L -STATISTICS

For some bounded function w on $[0, 1]$, consider the following linear combination of order statistics

$$L_n = \int_{-\infty}^{\infty} xw(F(x))dF_n(x)$$

and the corresponding parametric value

$$L = \int_{-\infty}^{\infty} xw(F(x))dF(x).$$

If G denotes any inverse of F , then we can also write

$$L_n = \int_0^1 G(u)w(V_n(u))dV_n(u) \quad \text{and} \quad L = \int_0^1 G(u)w(u)du.$$

Further, let

$$Y_t = Y(U_t) = \int_0^1 (u - I(U_t \leq u))w(u)G(u),$$

$$Y = Y(U) \quad \text{and} \quad \bar{Y}_n = n^{-1} \sum_{t=1, n} Y_t.$$

Let us say that a point $x \in (0, 1)$ is a jump point of the function w if x is a discontinuity point of w but w is either left continuous or right continuous at x . For convenience, let us say that the pair (F, w) is of type A if w has bounded second derivative on $(0, 1)$ and F is continuous on \mathcal{X} , and (F, w) is of type B if w has bounded second derivative throughout $(0, 1)$ except at finitely many jump points, F is continuous on \mathcal{X} and in a neighbourhood of each of the jump points, G has bounded derivative. Findings of the proof of Theorem 1 of $S1$ can be summarized as

Proposition 3.1: Let $\{X, X_1, X_2, \dots\}$ be a stationary sequence of r.v.s. with $E|X| < \infty$. If $E^0 \ll n^{-1/2}l_n^{1/2}$ n.s., then

$$|L_n - L - \bar{Y}_n| \ll E^0 \int_0^1 (u(1-u))^{2r} dG(u) + E(A, B) + n^{-1}l_n$$

for all $r > 0$ where $E(A, B) = 0$ if (F, w) is of type A , $= E_{**}$ if (F, w) is of type B .

Now we have all the tools needed for asymptotic representation of L -statistics in case of mixing r.v.s. We simply have to combine Propositions 2.1-2.6 and 3.1 to write down the theorems stated below.

Theorem 1: If $\{X, X_1, X_2, \dots\}$ is a stationary ϕ -mixing sequence with $\Sigma \phi_i^{1/s} < \infty$ for some $s > 1$ and $E|X|^{1+s-1+\delta} < \infty$ for some $\delta > 0$, then

$$\begin{aligned} |L_n - L - \bar{Y}_n| &\ll n^{-1}l_n \text{ a.s. if } (F, w) \text{ is of type } A \\ &= n^{-1+(s+\delta)^{-1+\delta_0}} \text{ if } (F, w) \text{ is of type } B \text{ a.s.} \end{aligned}$$

for all $\delta_0 > 0$.

Theorem 2: If $\{X, X_1, X_2, \dots\}$ is a stationary strong mixing sequence with $\Sigma \alpha_i^{1/s} < \infty$ for some $s > 0$ and $E|X|^{1+s(s-\delta)^{-1+\delta} < \infty$ for some $\delta > 0$, then

$$\begin{aligned} |L_n - L - \bar{Y}_n| &\ll n^{-1}l_n \text{ a.s. if } (F, w) \text{ is of type } A \\ &\ll n^{-1+(s+\delta)^{-1+\delta_0}} \end{aligned}$$

a.s. for all $\delta_0 > 0$ if (F, w) is of type B . In case $\alpha_t \ll e^{-t}$, $E|X|^{1+\delta} < \infty$ and (F, w) is of type B , $|L_n - L - \bar{Y}_n| \ll n^{-1}l_n$ a.s.

4. CONVERGENCE RATES TO NORMALITY

For the sake of convenient reference, we state below a few results of Statulevicius briefly which are used to establish some uniform convergence rates to normality for L_n .

Theorem (S) I: If $\{Y, Y_1, \dots\}$ is a stationary ϕ -mixing process with $EY = 0$, $E|Y|^3 < \infty$, $V(Y_1) + 2\sum_{t=1, \dots, \infty} \text{cov}(Y_1, Y_{1+t}) > 0$ and $\sum \phi_t^{1/2} < \infty$ then

$$\sup |P(n\bar{Y}_n \leq (V(n\bar{Y}_n))^{1/2}x) - \Phi(x)| \ll n^{-1/2}l_n.$$

II: If $\{Y, Y_1, \dots\}$ is a strong mixing process with $E|Y|^3 < \infty$ for some integer $k > 3$, $\alpha_l \ll i^{-3^{k-1}(k-3)^{-1}}$ and $V(n\bar{Y}_n) \rightarrow \infty$ then

$$\sup |P(n\bar{Y}_n \leq (V(n\bar{Y}_n))^{1/2}x) - \Phi(x)| \ll n^{-1/2}l_n.$$

If $x = 3$, one needs $\alpha_l \ll e^{-l}$.

Now, we state the convergence rates derived for L_n using the above Theorem (S) and the tools developed in Section 2.

Let us assume that $0 < \sigma^2 = V(Y_1) + 2\sum_{t=1, \dots, \infty} \text{cov}(Y_1, Y_{1+t})$. We define

$$T_n(x) = |P(L_n - L \leq (V(\bar{Y}_n))^{1/2}x) - \Phi(x)|.$$

Theorem 3 (I): If $\{X_t\}$ is a ϕ -mixing process with $\sum \phi_t^{1/2} < \infty$ for some $s > 2$, $E|Y|^3 < \infty$ and $E|X|^{s/2+4} < \infty$, then $\|T_n(x)\|_\infty \ll n^{-c(s)l_n}$ when $c(s) = 1/2$ if (F, w) is of type A and $c(s)$ is any number $< 1/2 - 1/2(s+1)$ if (F, w) is of type B.

(II): If $\{X_t\}$ is strong mixing, $E|Y|^4 < \infty$, $\alpha_l \ll i^{-s}$ and $E|X|^4 < \infty$, then $\|T_n\|_\infty \ll n^{-1/2}l_n$ if (F, w) is of type A.

(III): If $\{X_t\}$ is strong mixing, $E|Y|^3 < \infty$, $E|X|^{2s+4} < \infty$ and $\alpha_l \ll e^{-l}$ then $\|T_n(x)\|_\infty \ll n^{-1/2}l_n$ if (F, w) is of type A and $\|T_n(x)\|_\infty \ll n^{-1/2}l_n^2$ if (F, w) is of type B.

A sketch of the proof is as follows. We fix a set $\Omega_n = \{E_0 \leq c_{17}n^{-1/2}l_n^{1/2}\}$ using Proposition 2.3 or 2.4, depending upon the mixing case, s.t. $P(\Omega_n^c) \ll n^{-1/2}$. Then, proceeding along the lines of the proof of Theorem 1 of S1, we see that, on Ω_n ,

$$\begin{aligned} |L_n - L - \bar{Y}_n| &\ll (E_0^2 + n^{-1}) \int G(u) dV_n(u) \\ &+ E_0^2 + E_0 n^{-1/2}l_n^{1/2} + E_7^2 \int_{|u|(1-u)}^{2r} dG_1(u) + E(A, B). \end{aligned}$$

The required convergence rates for \bar{Y}_n is supplied by Theorem (S). The convergence rates for $(L_n - L)$ are obtained using the same for \bar{Y}_n , Lemma 3.3(a) of S1 and Propositions 2.3-2.6 of the previous section.

With the same methods and tools, we can also extend the non-uniform convergence rates of Babu, Ghosh and Singh (1977), and the moderate deviation bound of Babu and Singh (1977) to L -statistics.

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