

Variation of induced linear operators[☆]

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Abstract

Let V be an n -dimensional inner product space. Let λ be an irreducible character of the symmetric group S_m , and let V_λ be the symmetry class of tensors associated with it. Let A be a linear operator on V and let $K_\lambda(A)$ be the operator it induces on V_λ . We obtain an explicit expression for the norm of the derivative of the map $A \rightarrow K_\lambda(A)$ in terms of the singular values of A . Two special cases of this problem—antisymmetric and symmetric tensor products—have been studied earlier, and our results reduce to the earlier ones in these cases.

Keywords: Symmetry class of tensors; Induced linear operator; Derivative; Norm; Positive linear operator

1. Introduction

Let $\mathcal{L}(V)$ be the space of bounded linear operators on a Hilbert space V . The norm of an element A of $\mathcal{L}(V)$ is defined as

$$\|A\| = \sup \{ \|Av\| : v \in V, \|v\| = 1 \}.$$

In this paper V is finite-dimensional. Then $\|A\|$ is the largest singular value of A .

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Functions $f : \mathcal{L}(V) \rightarrow \mathcal{L}(W)$ are studied often in different contexts. Sometimes f is defined on an open subset of $\mathcal{L}(V)$ such as the set of invertible operators. In perturbation theory, numerical analysis, and physics, one often wants to know the effect of changes in A on $f(A)$. When the map f is differentiable, it is helpful to have estimates of the norm of its derivative. The derivative of f at A is a linear map $Df(A)$ from $\mathcal{L}(V)$ into $\mathcal{L}(W)$ and its norm is defined as

$$\|Df(A)\| = \sup \{ \|Df(A)(B)\| : B \in \mathcal{L}(V), \|B\| = 1 \}. \quad (1)$$

Estimates of this lead to first-order perturbation bounds for f . See the discussion in [1, Chapter X] and the papers [4,6,15,17] for different perspectives on this question.

Recall that

$$Df(A)(B) = \left. \frac{d}{dt} \right|_{t=0} f(A + tB). \quad (2)$$

Since A and B do not always commute several difficulties arise in estimating $\|Df(A)\|$. Finding *exact* values of $\|Df(A)\|$ is even more difficult, and very few such results are known. Some of them have led to intriguing questions [5,7].

In this paper we obtain exact formulas for $\|Df(A)\|$ when $f(A)$ is any of the operators induced by A on a symmetry class of tensors corresponding to the (full) symmetric group. Two special cases have been studied earlier [2,3]. To put our results in perspective we first recall these results. We need some basic facts, notations, and terminology of multilinear algebra. Further details may be found in [12] or [13].

Let $\dim V = n$, and for $A \in \mathcal{L}(V)$ let

$$\nu_1 \geq \nu_2 \geq \dots \geq \nu_n \geq 0$$

be the singular values of A . Let $\otimes^m V = V \otimes V \otimes \dots \otimes V$ be the m -fold tensor power of V and let $\otimes^m A$ be the corresponding tensor power of A . It is easy to see that [2]

$$\|D \otimes^m (A)\| = m \|A\|^{m-1} \quad (3)$$

Now let $1 \leq m \leq n$, let $\wedge^m V$ be the subspace of $\otimes^m V$ consisting of antisymmetric tensors, and let $\wedge^m A$ be the restriction of $\otimes^m A$ to this subspace. This is sometimes called the exterior power of A or the Grassmann power of A . In [2] it was shown that

$$\|D \wedge^m (A)\| = s_{m-1}(\nu_1, \nu_2, \dots, \nu_m), \quad (4)$$

where s_{m-1} is the $(m-1)$ th elementary symmetric polynomial in ν_1, \dots, ν_m ; i.e.,

$$s_{m-1}(\nu_1, \dots, \nu_m) = \sum_{j=1}^m \prod_{\substack{i=1 \\ i \neq j}}^m \nu_i. \quad (5)$$

The corresponding problem for the symmetric tensor power $\vee^m A$ (obtained by restricting $\otimes^m A$ to the space $\vee^m V$ of symmetric tensors) was studied in [3], where it was shown that

$$\|D \vee^m (A)\| = m \|A\|^{m-1} = m \nu_1^{m-1}, \quad (6)$$

and a speculation was made about a general result that would subsume (4) and (6). The precise formulation and proof of such a result is the principal outcome of this paper.

Let S_m be the symmetric group of degree m . Each element σ of S_m gives rise to a linear operator $P(\sigma)$ on $\otimes^m V$. This is defined as

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \tag{7}$$

on decomposable tensors and then extended linearly to all of $\otimes^m V$.

The map $\sigma \rightarrow P(\sigma)$ is a unitary representation of S_m in $\otimes^m V$. In other words, $P(\sigma_1)P(\sigma_2) = P(\sigma_1\sigma_2)$ and $P(\sigma)^{-1} = P(\sigma^{-1}) = P(\sigma)^*$.

Let G be a subgroup of S_m , and let λ be an irreducible character of G . Let

$$T(G, \lambda) = \frac{\lambda(\text{id})}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma), \tag{8}$$

where id stands for the identity element and $|G|$ for the order of the group G . This linear operator on $\otimes^m V$ is an orthoprojector and is called a *symmetriser map*. Its range is called the *symmetry class of tensors* associated with λ and G .

We will study symmetry classes associated with the full symmetric group $G = S_m$. Then the alternating character $\lambda(\sigma) = \varepsilon_\sigma$ (the signature of the permutation σ) leads to the symmetry class $\wedge^m V$; whereas the principal character $\lambda(\sigma) \equiv 1$ leads to the symmetry class $\vee^m V$.

There is a standard canonical correspondence between irreducible characters of S_m and partitions of the integer m [10]. We use the same symbol λ to denote an irreducible character and the corresponding partition. Recall that a partition π of m is a k -tuple of positive integers $\pi = (\pi_1, \dots, \pi_k)$ such that $\pi_1 \geq \dots \geq \pi_k$ and $\pi_1 + \dots + \pi_k = m$. For convenience we think of a partition of m also as an m -tuple with nonnegative integer entries by putting some zeros at the end if necessary. We adopt a similar convention for decreasing sequences of nonnegative real numbers. If $\lambda = (1, \dots, 1)$, then $V_\lambda(S_m) = \wedge^m V$; and if $\lambda = (m, 0, \dots, 0)$, then $V_\lambda(S_m) = \vee^m V$.

Let $\ell(\lambda)$ be the length of the partition λ —this is the number of nonzero entries in λ . For each $1 \leq t \leq m$ we denote by $\lambda_{(t)}$ the m -tuple defined as

$$\lambda_{(t)} = \begin{cases} (\lambda_1, \dots, \lambda_{t-1}, \lambda_t - 1, \lambda_{t+1}, \dots, \lambda_m) & \text{if } t \leq \ell(\lambda), \\ (\lambda_1, \dots, \lambda_{t-1}, -\infty, \lambda_{t+1}, \dots, \lambda_m) & \text{if } \ell(\lambda) < t. \end{cases} \tag{9}$$

Given any n -tuple of nonnegative real numbers (v_1, v_2, \dots, v_n) , and a k -tuple $(\gamma_1, \dots, \gamma_k)$ whose entries are either nonnegative integers or $-\infty$, we define v^γ as

$$v^\gamma = v_1^{\gamma_1} v_2^{\gamma_2} \cdots v_n^{\gamma_n}$$

with the convention that $a^0 = 1$ and $a^{-\infty} = 0$ for every nonnegative a .

Now let λ be a partition of m and let $\ell(\lambda) \leq n$. Put

$$S_{\lambda, v} = \lambda_1 v^{\lambda_{(1)}} + \lambda_2 v^{\lambda_{(2)}} + \cdots + \lambda_m v^{\lambda_{(m)}}. \tag{10}$$

Note that if $\lambda = (1, 1, \dots, 1)$, then

$$S_{\lambda, \nu} = \sum_{j=1}^m \prod_{\substack{i=1 \\ i \neq j}}^m \nu_i = s_{m-1}(\nu_1, \dots, \nu_m).$$

If $\lambda = (m, 0, \dots, 0)$, then

$$S_{\lambda, \nu} = m\nu_1^{m-1}.$$

Now return to symmetry classes of tensors. It is well known that $V_\lambda(S_m) \neq \{0\}$ if and only if $\ell(\lambda) \leq n$; see [14]. Given any $A \in \mathcal{L}(V)$ we denote by $K_\lambda(A)$ the restriction of the operator $\otimes^m A$ to the subspace $V_\lambda(S_m)$. This is called the operator induced by A on the symmetry class $V_\lambda(S_m)$. Our principal result is the following theorem.

Theorem 1. *Let V be an n -dimensional Hilbert space. Let m be a positive integer. Let λ be a partition of m such that $\ell(\lambda) \leq n$. Let $A \rightarrow K_\lambda(A)$ be the map that associates to each element A of $\mathcal{L}(V)$ the induced operator $K_\lambda(A)$ on the symmetry class $V_\lambda(S_m)$. Then the norm of the derivative of this map at A is given by the formula*

$$\|DK_\lambda(A)\| = S_{\lambda, \nu}, \quad (11)$$

where $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$ are the singular values of A , and $S_{\lambda, \nu}$ is the polynomial defined by (10).

Note that Theorem 1 includes as very special cases the results (4) and (6) obtained in [2,3].

To guide the reader through the proof we highlight its salient features. Let A have the singular value decomposition $A = U_1 P U_2$. Using the unitary invariance of the norm and of the singular values one sees that $\|DK_\lambda(A)\| = \|DK_\lambda(P)\|$. So, one may replace A by the positive diagonal matrix P . Then one observes that $DK_\lambda(P)$ is a positive linear map between two matrix algebras. By a general theorem of Russo and Dye, such a map between any two unital C^* -algebras attains its norm at the identity I . This simplifies our calculations immensely because we do not have to consider arbitrary A and B in expression (2) for derivatives. Even after this simplification some difficulties remain. While in the special examples $\wedge^m V$ and $\vee^m V$ good orthonormal bases corresponding to the standard basis in V can be found immediately, this is not the case in other symmetry classes. We explain how a suitable basis may be chosen for our purposes. This choice leads to a partition of m ; and finally we have to study the relation between this partition and λ , and the corresponding functions $S_{\lambda, \nu}$. Here we prove a majorisation theorem that is of interest in its own right.

The idea of replacing A by P in calculating $\|D \wedge^k(A)\|$ occurs in [2]. It is also shown there that $\|D \wedge^k(P)\| = \|D \wedge^k(P)(I)\|$. The idea of proving the same result using completely positive maps is due to Sunder [16].

2. Preliminaries

Given a symmetriser map $T(G, \lambda)$ let

$$v_1 * v_2 * \cdots * v_m = T(G, \lambda)(v_1 \otimes v_2 \otimes \cdots \otimes v_m).$$

These vectors belong to $V_\lambda(G)$ and are called decomposable symmetrised tensors.

Let $\Gamma_{m,n}$ be the set of all maps from the set $\{1, \dots, m\}$ into the set $\{1, \dots, n\}$. This set can be identified with the collection of all multiindices $\{(i_1, \dots, i_m) : 1 \leq i, j \leq n\}$. If $\alpha \in \Gamma_{m,n}$, this correspondence associates the index $(\alpha(1), \dots, \alpha(m))$ with it. We order $\Gamma_{m,n}$ by the lexicographic order.

Every subgroup G of S_m acts on $\Gamma_{m,n}$ by the action $(\sigma, \alpha) \rightarrow \alpha\sigma^{-1}$, $\sigma \in G$, $\alpha \in \Gamma_{m,n}$. The subgroup G_α of G defined as

$$G_\alpha = \{\sigma \in G : \alpha\sigma = \alpha\}$$

is called the *stabiliser* of α .

Let $\{e_1, \dots, e_n\}$ be a basis of V . Then $\{e_\alpha^\otimes := e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$ is a basis for $\otimes^m V$. Hence the set

$$\{e_\alpha^* := T(\lambda, G)e_\alpha^\otimes : \alpha \in \Gamma_{m,n}\}$$

spans the space $V_\lambda(G)$. However, the elements of this set need not be linearly independent. Some of them may even be zero. Let

$$\Omega = \Omega_\lambda = \left\{ \alpha \in \Gamma_{m,n} : \sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0 \right\} \tag{12}$$

It is easy to see that

$$\|e_\alpha\|^2 = \frac{\lambda(\text{id})}{|G|} \sum_{\sigma \in G_\alpha} \lambda(\sigma).$$

So the set $\{e_\alpha^* : \alpha \in \Omega\}$ consists of the nonzero elements of $\{e_\alpha^* : \alpha \in \Gamma_{m,n}\}$.

Let Δ be the system of distinct representatives for the set $\Gamma_{m,n}/G$, constructed by choosing the smallest element (in the lexicographic order) from each orbit. Let

$$\bar{\Delta} = \bar{\Delta}_\lambda = \Delta \cap \Omega_\lambda.$$

It can be proved that $\{e_\alpha^* : \alpha \in \bar{\Delta}\}$ is a linearly independent set. Since the set $\{e_\alpha^* : \alpha \in \Omega\}$ spans $V_\lambda(G)$ there exists a set $\hat{\Delta}$ such that $\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega$ and

$$\{e_\alpha^* : \alpha \in \hat{\Delta}\} \tag{13}$$

is a basis for $V_\lambda(G)$, not necessarily orthonormal. See [13] for details.

Each element α of $\Gamma_{m,n}$ gives rise to a partition of m in the following way. Let $\text{range } \alpha = \{i_1, \dots, i_\ell\}$, where i_1, \dots, i_ℓ are labelled in such a way that

$$|\alpha^{-1}(i_1)| \geq |\alpha^{-1}(i_2)| \geq \cdots \geq |\alpha^{-1}(i_\ell)|.$$

Then

$$\mu^{(\alpha)} := (|\alpha^{-1}(i_1)|, |\alpha^{-1}(i_2)|, \dots, |\alpha^{-1}(i_\ell)|) \tag{14}$$

is a partition of m of length ℓ .

On the set of partitions of m , we define a partial order $<$ as follows: we say that $\mu < \lambda$ if for all $1 \leq k \leq m$

$$\sum_{j=1}^k \mu_j \leq \sum_{j=1}^k \lambda_j.$$

(This is the usual *majorisation* order between m -tuples [1] when we identify partitions with m -tuples.) We will need the following theorem of Merris [14].

Theorem 2 (Merris). *Let λ be a partition of m and α an element of $\Gamma_{m,n}$. Let Ω_λ and $\mu^{(\alpha)}$ be as defined in (12) and (14). Then $\alpha \in \Omega_\lambda$ if and only if $\mu^{(\alpha)} < \lambda$.*

Let λ, μ be two partitions of m . We say that $\mu \triangleleft \lambda$, if there exist indices $i, j \in \{1, \dots, m\}$ such that

- (i) $i < j$;
- (ii) $\mu_i = \lambda_i - 1, \mu_j = \lambda_j + 1$, and $\lambda_k = \mu_k$ for $k \neq i, j$;
- (iii) either $i = j - 1$ or $\mu_i = \mu_j$.

We will need the following result [10, p. 24].

Proposition 3. *If $\mu < \lambda$, then there exists a sequence of partitions $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}$ such that*

$$\mu = \lambda^{(1)} \triangleleft \lambda^{(2)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda.$$

For brevity we say that $A \in \mathcal{L}(V)$ is *positive* if it is positive semidefinite. A linear map $\Phi : \mathcal{L}(V) \rightarrow \mathcal{L}(W)$ is called *positive* if it maps positive elements of $\mathcal{L}(V)$ into positive elements of $\mathcal{L}(W)$. We say that Φ is unital if $\Phi(I) = I$.

Positive linear maps Φ enjoy a very special property: $\|\Phi\| = \|\Phi(I)\|$. This is a consequence of the well-known Russo–Dye Theorem [11] valid in C^* -algebras.

3. Proofs

Let $V_\lambda = V_\lambda(S_m)$ be the symmetry class of tensors associated with λ and let $K_\lambda : \mathcal{L}(V) \rightarrow \mathcal{L}(V_\lambda)$ be the induced map. For brevity let $D_\lambda(A, B) = DK_\lambda(A)(B)$, the image of B under the derivative $DK_\lambda(A)$. Then $D_\lambda(A, B)$ is the restriction to V_λ of the operator on $\otimes^m V$ defined as

$$D(A, B) := B \otimes A \otimes A \otimes \dots \otimes A + A \otimes B \otimes A \otimes \dots \otimes A + \dots + A \otimes \dots \otimes A \otimes B.$$

Note that if A and B are positive, then so is $D(A, B)$.

Let $A = U_1 P U_2$ be the singular value decomposition of A . Using unitary invariance of the norm and the fact that $K_\lambda(U)$ is unitary if U is unitary, we see that

$$\|DK_\lambda(A)\| = \|DK_\lambda(P)\|.$$

From the description above it is clear that $DK_\lambda(P)$ is a positive linear map. Hence by the Russo–Dye Theorem

$$\|DK_\lambda(A)\| = \|D_\lambda(P, I)\|. \tag{15}$$

So we have to calculate the maximum eigenvalue of $D_\lambda(P, I)$. We will do this by finding a basis for V_λ in which $D_\lambda(P, I)$ is diagonal. Then the diagonal entries of this matrix are the eigenvalues of $D_\lambda(P, I)$; our basis need not be orthonormal for this.

Let $\alpha \in \Gamma_{m,n}$ and let $\mu^{(\alpha)}$ be the partition of length ℓ associated with α as in (14). Let

$$v_\alpha = (v_{i_1}, \dots, v_{i_\ell}) \tag{16}$$

be the largest (in the lexicographic order) sequence such that (i_1, \dots, i_ℓ) satisfies (14). (For example, if $\ell = 4$ and $|\alpha^{-1}(6)| = |\alpha^{-1}(7)| > |\alpha^{-1}(4)| = |\alpha^{-1}(3)|$, then $v_\alpha = (v_6, v_7, v_3, v_4)$.)

Given any partition λ , let ω_λ be the element of $\Gamma_{m,n}$ defined as

$$\omega_\lambda = \left(\underbrace{1, \dots, 1}_{\lambda_1 \text{ times}}, \underbrace{2, \dots, 2}_{\lambda_2 \text{ times}}, \dots, \underbrace{\ell(\lambda), \dots, \ell(\lambda)}_{\lambda_{\ell(\lambda)} \text{ times}} \right). \tag{17}$$

Then clearly

$$\mu^{(\omega_\lambda)} = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) = \lambda, \quad v_{\omega_\lambda} = (v_1, \dots, v_{\ell(\lambda)}). \tag{18}$$

Proposition 4. *Let P be a positive linear operator on V , and suppose $E = \{e_1, \dots, e_n\}$ is an orthonormal basis for V in which the matrix of P is diagonal with diagonal entries $v_1 \geq \dots \geq v_n$. Let $\{e_\alpha^* : \alpha \in \widehat{\Delta}\}$ be a basis for V_λ as in (13). Then in this basis $D_\lambda(P, I)$ is diagonal and its (α, α) entry is*

$$D_\lambda(P, I)_{\alpha, \alpha} = \sum_{j=1}^m \prod_{\substack{i=1 \\ i \neq j}}^m v_{\alpha(i)} = S_{\mu^{(\alpha)}, v_\alpha}, \quad \alpha \in \widehat{\Delta}. \tag{19}$$

Proof. Recall that for any $\alpha \in \Gamma_{m,n}$

$$e_\alpha^* = \frac{\lambda(\text{id})}{m!} \sum_{\sigma \in S_m} \lambda(\sigma) e_{\alpha\sigma}^\otimes.$$

Note that

$$D_\lambda(P, I) \left(\sum_{\sigma} \lambda(\sigma) e_{\alpha\sigma}^\otimes \right)$$

$$\begin{aligned}
 &= D(P, I) \left(\sum_{\sigma} \lambda(\sigma) e_{\alpha\sigma}^{\otimes} \right) \\
 &= (I \otimes P \otimes P \otimes \dots \otimes P) \left(\sum_{\sigma} \lambda(\sigma) e_{\alpha\sigma}^{\otimes} \right) \\
 &\quad + \dots + (P \otimes P \otimes \dots \otimes I) \left(\sum_{\sigma} \lambda(\sigma) e_{\alpha\sigma}^{\otimes} \right) \\
 &= \left(\sum_{\sigma} \lambda(\sigma) \prod_{i=2}^m v_{\alpha\sigma(i)} e_{\alpha\sigma}^{\otimes} \right) + \dots + \left(\sum_{\sigma} \lambda(\sigma) \prod_{i=1}^{m-1} v_{\alpha\sigma(i)} e_{\alpha\sigma}^{\otimes} \right).
 \end{aligned}$$

For each $1 \leq k \leq m$

$$\prod_{\substack{i=1 \\ i \neq j}}^m v_{\alpha\sigma(i)} = \prod_{\substack{i=1 \\ i \neq \sigma(j)}}^m v_{\alpha(i)}.$$

This shows that

$$D_{\lambda}(P, I) e_{\alpha}^* = \left(\sum_{j=1}^m \prod_{\substack{i=1 \\ i \neq j}}^m v_{\alpha(i)} \right) e_{\alpha}^*.$$

Thus the matrix of $D_{\lambda}(P, I)$ in the basis $\{e_{\alpha}^* : \alpha \in \widehat{\Delta}\}$ is diagonal with entries given in (19).

By definitions (14) and (19)

$$\begin{aligned}
 \sum_{\substack{j=1 \\ i \neq j}}^m \prod_{i=1}^m v_{\alpha(i)} &= \mu_1^{(\alpha)} v_{i_1}^{\mu_1^{(\alpha)}-1} v_{i_2}^{\mu_2^{(\alpha)}} \dots v_{i_{\ell}}^{\mu_{\ell}^{(\alpha)}} + \mu_2^{(\alpha)} v_{i_1}^{\mu_1^{(\alpha)}} v_{i_2}^{\mu_2^{(\alpha)}-1} \dots v_{i_{\ell}}^{\mu_{\ell}^{(\alpha)}} \\
 &\quad + \dots + \mu_{\ell}^{(\alpha)} v_{i_1}^{\mu_1^{(\alpha)}} v_{i_2}^{\mu_2^{(\alpha)}} \dots v_{i_{\ell}}^{\mu_{\ell}^{(\alpha)}-1} \\
 &= S_{\mu^{(\alpha)}} v_{\alpha}. \quad \square
 \end{aligned}$$

Proposition 5. *Let λ and μ be partitions of m , and let $v_1 \geq \dots \geq v_m \geq 0$ be any decreasing sequence of nonnegative numbers. If $\mu \triangleleft \lambda$, then $S_{\mu, v} \leq S_{\lambda, v}$.*

Proof. By Proposition 3, it is enough to prove this when $\mu \triangleleft \lambda$. Assume $v_m > 0$; the general case follows from this by continuity.

Use the notations as in definition of $\mu \triangleleft \lambda$, before Proposition 3. Then for $k \neq i, j$

$$\lambda_k v^{\lambda(k)} = \lambda_k v_k^{\lambda_k-1} \prod_{r \neq k} v_r^{\lambda_r}$$

$$\begin{aligned}
 &= \mu_k v_k^{\mu_k - 1} \prod_{r \neq k} v_r^{\lambda_r} \\
 &\geq \mu_k v_k^{\mu_k - 1} \left[\prod_{r \neq i, j, k} v_r^{\lambda_r} \right] v_i^{\lambda_i - 1} v_j^{\lambda_j + 1} \\
 &= \mu_k v_k^{\mu_k - 1} \prod_{r \neq k} v_r^{\mu_r} \\
 &= \mu_k v^{\mu^{(k)}}.
 \end{aligned}$$

Next note that

$$\begin{aligned}
 &[\lambda_i v^{\lambda^{(i)}} + \lambda_j v^{\lambda^{(j)}}] - [\mu_i v^{\mu^{(i)}} + \mu_j v^{\mu^{(j)}}] \\
 &= \frac{v^\lambda}{v_i^2 v_j} [\lambda_i v_i v_j + \lambda_j v_i^2 - (\lambda_i - 1)v_j^2 - (\lambda_j + 1)v_i v_j] \\
 &= \frac{v^\lambda}{v_i^2 v_j} [(\lambda_i - 1)v_i v_j - (\lambda_i - 1)v_j^2 + \lambda_j(v_i^2 - v_i v_j)] \\
 &\geq \frac{v^\lambda}{v_i^2 v_j} [(\lambda_i - 1)(v_i v_j - v_j^2)] \quad (\text{since } v_i^2 \geq v_i v_j) \\
 &\geq 0 \quad (\text{since } \lambda_i = \mu_i + 1 \geq 1).
 \end{aligned}$$

Taken together, the two inequalities we have obtained, prove the proposition. \square

Let $\lambda = (\lambda_1, \dots, \lambda_\ell, 0, \dots, 0)$ be a partition of m . Then we denote by λ^* the partition of $m - \ell$ given as

$$\lambda^* = (\lambda_1^*, \dots, \lambda_{m-\ell}^*),$$

where $\lambda_i^* = \lambda_i - 1$ if $i \leq \ell$ and $\lambda_i^* = 0$ if $i > \ell$.

Given an m -tuple $(\theta_1, \dots, \theta_m)$ of real numbers we denote by θ^\downarrow its decreasing rearrangement; i.e., $\theta^\downarrow = (\theta_1^\downarrow, \dots, \theta_m^\downarrow)$, where $\theta_1^\downarrow \geq \dots \geq \theta_m^\downarrow$ are the numbers $\theta_1, \dots, \theta_m$ rearranged. We use the notation $v \geq \theta$ to mean $v_j \geq \theta_j$ for all j .

Proposition 6. *Let v, θ be m -tuples of nonnegative real numbers such that v is decreasing and $v \geq \theta^\downarrow$. Then for every partition λ of m we have $S_{\lambda, v} \geq S_{\lambda, \theta}$.*

Proof. Note first that

$$v^\lambda \geq (\theta^\downarrow)^\lambda \geq \theta^\lambda. \tag{20}$$

For any m -tuple $\rho = (\rho_1, \dots, \rho_m)$ of nonnegative real numbers let

$$T_{\lambda, \rho} = \sum_{i=1}^m \rho^{\lambda^{(i)}}.$$

Then, bearing in mind that $\lambda_{(i)}(i) = -\infty$ if $i > \ell$ we have

$$\begin{aligned} T_{\lambda,\rho} &= \sum_{i=1}^{\ell} \rho^{\lambda_{(i)}} \\ &= \rho_1^{\lambda_1-1} \rho_2^{\lambda_2-1} \cdots \rho_{\ell}^{\lambda_{\ell}-1} (\rho_2 \cdots \rho_{\ell} + \rho_1 \rho_3 \cdots \rho_{\ell} + \cdots + \rho_1 \cdots \rho_{\ell-1}) \\ &= \rho^{\lambda^*} s_{\ell-1}(\rho_1, \dots, \rho_{\ell}), \end{aligned}$$

where $s_{\ell-1}$ is the $(\ell - 1)$ th elementary symmetric polynomial in ℓ variables. So from (20) and using the symmetry of $s_{\ell-1}$ we have

$$T_{\lambda,\nu} \geq T_{\lambda,\theta}. \tag{21}$$

Next note that

$$\begin{aligned} S_{\lambda,\rho} &= T_{\lambda,\rho} + (\lambda_1 - 1)\rho^{\lambda_{(1)}} + (\lambda_2 - 1)\rho^{\lambda_{(2)}} + \cdots + (\lambda_{\ell} - 1)\rho^{\lambda_{(\ell)}} \\ &= T_{\lambda,\rho} + \rho_1 \cdots \rho_{\ell} \left(\lambda_1^* \rho^{\lambda_{(1)}} + \cdots + \lambda_{m-\ell}^* \rho^{\lambda_{(m-\ell)}} \right) \\ &= T_{\lambda,\rho} + \rho_1 \cdots \rho_{\ell} S_{\lambda^*,\rho}. \end{aligned} \tag{22}$$

We prove the assertion $S_{\lambda,\nu} \geq S_{\lambda,\theta}$ by induction on the integer λ_1 . If $\lambda_1 = 1$, then

$$S_{\lambda,\nu} = s_{m-1}(\nu_1, \dots, \nu_m) \geq s_{m-1}(\theta_1, \dots, \theta_m) = S_{\lambda,\theta}.$$

If $\lambda_1 > 1$, use (22) to write

$$S_{\lambda,\nu} = T_{\lambda,\nu} + \nu_1 \cdots \nu_{\ell} S_{\lambda^*,\nu}.$$

Then use (21), the inequalities $\nu \geq \theta^{\downarrow}$, and the induction hypothesis to conclude that $S_{\lambda,\nu} \geq S_{\lambda,\theta}$. \square

Combining Propositions 5 and 6 we have:

Proposition 7. *Let ν, θ be m -tuples of nonnegative real numbers such that ν is decreasing and $\theta^{\downarrow} \leq \nu$. Let λ, μ be partitions of m such that $\mu \prec \lambda$. Then*

$$S_{\mu,\theta} \leq S_{\lambda,\nu}.$$

Proof of Theorem 1. By Proposition 4, the matrix of $D_{\lambda}(P, I)$ is diagonal in the basis $\{e_{\alpha}^* : \alpha \in \widehat{\Lambda}\}$, and the diagonal elements are given by $S_{\mu^{(\alpha)}, \nu_{\alpha}}$.

Let ω_{λ} be the element of $\Gamma_{m,n}$ associated with λ by (17). Then $\omega_{\lambda} \in \overline{\Delta} \subseteq \widehat{\Lambda}$. By the proof of Proposition 4, we also have $D_{\lambda}(P, I)_{\omega_{\lambda}, \omega_{\lambda}} = S_{\lambda,\nu}$ (see the relations (18)). So $\|D_{\lambda}(P, I)\| \geq S_{\lambda,\nu}$.

By Theorem 2, $\mu^{(\alpha)} \prec \lambda$. It is obvious that $\nu_{\alpha}^{\downarrow} \leq \nu$. Hence $S_{\mu^{(\alpha)}, \nu_{\alpha}} \leq S_{\lambda,\nu}$ by Proposition 7.

Since $S_{\mu^{(\alpha)}, \nu_{\alpha}}$, $\alpha \in \widehat{\Delta}$, is an enumeration of all the eigenvalues of the positive operator $D_{\lambda}(P, I)$, this implies $\|D_{\lambda}(P, I)\| \leq S_{\lambda,\nu}$. Thus $\|D_{\lambda}(P, I)\| = S_{\lambda,\nu}$. Use (15) to complete the proof. \square

4. Remarks

- Using standard results of Calculus [1, Chapter X] we can obtain from Theorem 1 perturbation bounds for K_λ . Thus we have for B close to A the first-order perturbation bound

$$\|K_\lambda(A) - K_\lambda(B)\| \leq S_{\lambda, \nu} \|A - B\| + O(\|A - B\|^2). \tag{23}$$

- Given an irreducible character λ of S_m , let

$$d_\lambda(A) = \frac{1}{\lambda(\text{id})} \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{j=1}^m a_{j\sigma(j)}.$$

This is called an *immanant* of A . These functions are important in representation theory and combinatorics.

Let $m = n$. When $\lambda(\sigma) = \varepsilon(\sigma)$ the function d_λ is the determinant, and when $\lambda(\sigma) \equiv 1$, it is the permanent. It is well known that we can choose an orthonormal basis for $V_\lambda(S_n)$ such that $d_\lambda(A)$ is one of the diagonal entries of $K_\lambda(A)$ in this basis. So from (23) we obtain

$$|d_\lambda(A) - d_\lambda(B)| \leq S_{\lambda, \nu} \|A - B\| + O(\|A - B\|^2). \tag{24}$$

- For simplicity we have restricted our discussion to symmetry classes associated with the full symmetric group. Similar results can be obtained for general symmetry classes. Let G be a subgroup of S_m and let λ be a complex irreducible character of G . Denote by π_λ the multilinearity partition of λ [8]. Using arguments similar to those that have been used to prove Theorem 1 and the results in [9], we can see that

$$\|DK_\lambda(A)\| \leq S_{\pi_\lambda, \nu}.$$

Furthermore if the inner product $(\pi_\lambda, \lambda)_G$ is different from zero, then it can be proved that

$$\|DK_\lambda(A)\| = S_{\pi_\lambda, \nu}.$$

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