

Space-Time Symmetries in Noncommutative Gauge Theory:  
A Hamiltonian Analysis

Subir Ghosh<sup>1</sup>

Physics and Applied Mathematics Unit,  
Indian Statistical Institute,  
203 B. T. Road, Kolkata 700108,  
India.

**Abstract:**

We study space-time symmetries in Non-Commutative (NC) gauge theory in the (constrained) Hamiltonian framework. The specific example of NC  $CP(1)$  model, posited in [9], has been considered. Subtle features of Lorentz invariance violation in NC field theory were pointed out in [13]. Out of the two - Observer and Particle - distinct types of Lorentz transformations, symmetry under the former, (due to the translation invariance), is reflected in the conservation of energy and momentum in NC theory. The constant tensor  $\theta_{\mu\nu}$  (the noncommutativity parameter) destroys invariance under the latter.

In this paper we have constructed the Hamiltonian and momentum operators which are the generators of time and space translations respectively. This is related to the Observer Lorentz invariance. We have also shown that the Schwinger condition and subsequently the Poincare algebra is not obeyed and that one can not derive a Lorentz covariant dynamical field equation. These features signal a loss of the Particle Lorentz symmetry. The basic observations in the present work will be relevant in the Hamiltonian study of a generic noncommutative field theory.

Keywords: Noncommutative field theory, Seiberg-Witten map, Violation of Lorentz invariance,

## Introduction

In recent years, Non-Commutative (NC) field theories have become the focus of intense research activity after its connection to low energy string physics was elucidated by Seiberg and Witten [1, 2]. Specifically, the open string boundaries, attached to D-branes [3], in the presence of a two-form background field, turn into NC spacetime coordinates [1]. (This phenomenon has been recovered from various computational schemes [4].) The noncommutativity induces an NC D-brane world volume and hence field theories on the brane become NC field theories.

NC field theories have revealed unexpected textures in the conventional field theory framework - UV-IR mixing [5], soliton solutions in higher dimensional scalar theories [6], dipole-like elementary charged excitations [7], etc. are some of them. The inherent non-locality, (or equivalently the introduction of a length scale by  $\theta^{\rho\sigma}$  - the noncommutativity parameter), of the NC field theory is manifested through these peculiar properties that are absent in the corresponding ordinary spacetime theories.

Solitons in NC  $CP(1)$  model have been found [8], very much in analogy to their counterpart in ordinary spacetime. In this framework, NC theories are treated as systems of operator valued fields and one works directly with operators on the quantum phase space, characterized by the noncommutativity condition

$$[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}. \quad (1)$$

With only spatial noncommutativity on the NC plane, the above simplifies to a Heisenberg algebra,  $[x^1, x^2]_* = x^1 * x^2 - x^2 * x^1 = i\theta^{12} = i\epsilon^{12}\theta = i\theta$  which in the complex coordinates reduces to the creation annihilation operator algebra for the simple Harmonic Oscillator. Thus to a function in the NC spacetime, through Weyl transform, one associates an operator acting on the Hilbert space, in a basis of a simple Harmonic Oscillator eigenstates. Explicit details of the computations in this particular model are found in [8].

In the present work, we will concentrate on a specific NC gauge theory, that was recently proposed [9] as an alternative formulation of the NC extension of the  $CP(1)$  model. Due to the presence of the  $U(1)$  invariance, (induced by the  $CP$  variables), the Seiberg-Witten map [1] plays a pivotal role in our scheme. It is used to convert the NC to a theory expressed in terms of ordinary fields, with noncommutative effects appearing as  $\theta$ -dependent interaction terms. We found in [9] that our model allows solutions obeying a (Bogolmolny) lower bound in energy, (protected by topological considerations), which are the solitons of the NC  $CP(1)$  model. We also noted in [9] that (unlike the commutative case) additional restrictions on the  $CP(1)$  variables appear when the BPS equations are considered as a subset of the variational equation of motion. The reason might be the perturbative (in  $\theta$ ) nature of the formalism. In fact it is well known that there are complications in the definition of the Energy-Momentum (EM) tensor in NC field theory [10].

The EM tensor in a generic NC field theory has been discussed in [10, 11] in the Lagrangian framework. Hamiltonian analysis of NC theories have been performed in [12]. The novel feature of the present work is the study of the Poincare algebra, leading to the Lorentz invariance violation.

Of special interest will be an explicit demonstration of the validity of the ideas introduced in [13], in the context of Lorentz symmetry violation in NC field theories. The issue is subtle since there exists [14] *two* distinct types of Lorentz transformations: Observer and Particle Lorentz transformations. The NC action, (as well as the fields and the constant tensor  $\theta_{\mu\nu}$

comprising it), transform covariantly under the former, thereby yielding conservation of energy and momentum, at least when only spatial noncommutativity is present. This is expected due to the translation invariance of the theory. On the other hand, the essential ingredients of a relativistic theory - the Schwinger condition [15] and subsequent Poincare algebra - are not respected. This indicates a loss of the Particle Lorentz symmetry. These generic features will emerge naturally in our Hamiltonian formulation. We have restricted the discussion to spatial noncommutativity only so that a conventional Hamiltonian analysis can be carried through.

Let us put our work in its proper perspective. This paper is a sequel to [9] where we provide a field theoretic analysis of the  $CP(1)$  model in a Hamiltonian framework, keeping in mind the future possibility of quantization of the model. In particular, we study in detail the nature (of the violation) of the Poincare algebra. This is probably the first example of an in depth study of a specific NC field theory model in the Hamiltonian scheme, enunciated by Dirac [16]. Our analysis reveals both expected and unexpected features of the model. Some of the results are generic to any NC field theory and some are specific to the (spacetime) dimensionality of the problem. With its non-trivial but simpler structure, 2+1-dimensional NC field theories can become successful laboratories for higher dimensional studies, as the present work indicates.

Furthermore, our analysis goes on to show that the alternative definition of the NC  $CP(1)$  model that we have posited in [9], leads to a consistent and well defined NC gauge theory, which conforms to the expected features of such a system.

The paper is organized as follows: Section II introduces the noncommutative spacetime and provides a brief digression of the NC  $CP(1)$  model [9]. The canonical EM tensor is studied in Section III. Section IV discusses the Hamiltonian formulation of the model with the associated constraint analysis. It also exhibits the transformation properties of the fields under gauge and spacetime transformations and the Hamiltonian equations of motion. Section V is devoted to the study of the Schwinger condition and Poincare algebra. The major contributions of the present work are in Sections IV and V. Section VI provides a summary and conclusions.

## Section II: Noncommutative $CP(1)$ model - a brief digression

The  $CP(1)$  model in ordinary spacetime is described by the gauge invariant action <sup>2</sup>,

$$S = \int d^3x [(D^\mu \phi)^* D_\mu \phi + \Lambda(\phi^* \phi - 1)], \quad (2)$$

where  $D_\mu \phi_a = (\partial_\mu - iA_\mu)\phi_a$  defines the covariant derivative and the multiplier  $\Lambda$  enforces the  $CP(1)$  constraint. The equation of motion for  $A_\mu$  leads to the identification,

$$A_\mu = -i\phi^* \partial_\mu \phi. \quad (3)$$

The "gauge field"  $A_\mu$  - being a dependent variable - can be removed from the action classically using (3). The infinitesimal gauge transformation of the variables are,

$$\delta\phi_a^* = -i\lambda\phi_a^*; \quad \delta\phi_a = i\lambda\phi_a; \quad \delta A_\mu = \partial_\mu \lambda. \quad (4)$$

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<sup>2</sup>Since the scenario is classical, hermitian conjugate operator  $\phi^\dagger$  is replaced by complex conjugate  $\phi^*$  and operator ordering ambiguities are not taken in to account anywhere. Adjacent  $\phi$ -terms without any Roman index are assumed to be summed.

Let us now enter the noncommutative spacetime. In constructing the NC  $CP(1)$  (or any generic) model the following steps are taken [9]:

(i) The appropriate NC field theory is constructed in terms of NC analogue fields ( $\hat{\psi}$ ) of the fields ( $\psi$ ) with the replacement of ordinary products of fields ( $\psi\varphi$ ), by the Moyal-Weyl  $*$ -product ( $\hat{\psi} * \hat{\varphi}$ ),

$$\hat{\psi}(x) * \hat{\varphi}(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{\sigma\mu}\partial_{\xi\nu}} \hat{\psi}(x+\sigma)\hat{\varphi}(x+\xi) |_{\sigma=\xi=0} = \hat{\psi}(x)\hat{\varphi}(x) + \frac{i}{2}\theta^{\rho\sigma}\partial_{\rho}\hat{\psi}(x)\partial_{\sigma}\hat{\varphi}(x) + O(\theta^2). \quad (5)$$

The *hatted* variables are NC degrees of freedom. We take  $\theta^{\rho\sigma}$  to be a real constant antisymmetric tensor, as is customary [1], (but this need not always be the case [17]). The NC spacetime (1) follows from the above definition.

Note that the effects of spacetime noncommutativity has been accounted for by the introduction of the  $*$ -product. For gauge theories the Seiberg-Witten Map [1] plays a crucial role in connecting  $\hat{\phi}(x)$  to  $\phi(x)$ . This formalism allows us to study the effects of noncommutativity as  $\theta^{\rho\sigma}$  dependent interaction terms in an ordinary spacetime field theory format. This is the prescription we will follow.

The first task is to generalize the scalar gauge theory (2) to its NC version, keeping in mind that the latter must be  $*$ -gauge invariant. The NC action (without the  $CP(1)$  constraint) is,

$$\hat{S} = \int d^3x (\hat{D}^{\mu}\hat{\phi})^* * \hat{D}_{\mu}\hat{\phi} = \int d^3x (\hat{D}^{\mu}\hat{\phi})^* \hat{D}_{\mu}\hat{\phi}, \quad (6)$$

where the NC covariant derivative is defined as

$$\hat{D}_{\mu}\hat{\phi}_a = \partial_{\mu}\hat{\phi}_a - i\hat{A}_{\mu} * \hat{\phi}_a.$$

The NC action (6) is invariant under the  $*$ -gauge transformations,

$$\delta\hat{\phi}_a^* = -i\hat{\lambda} * \hat{\phi}_a^* ; \quad \delta\hat{\phi}_a = i\hat{\lambda} * \hat{\phi}_a ; \quad \delta\hat{A}_{\mu} = \partial_{\mu}\hat{\lambda} + i[\hat{\lambda}, \hat{A}_{\mu}]_*. \quad (7)$$

We now exploit the Seiberg-Witten Map [1, 18] to revert back to the ordinary spacetime degrees of freedom. The explicit identifications between NC and ordinary spacetime counterparts of the fields, to the lowest non-trivial order in  $\theta$  are,

$$\begin{aligned} \hat{A}_{\mu} &= A_{\mu} + \theta^{\sigma\rho} A_{\rho}(\partial_{\sigma}A_{\mu} - \frac{1}{2}\partial_{\mu}A_{\sigma}) \\ \hat{\phi} &= \phi - \frac{1}{2}\theta^{\rho\sigma} A_{\rho}\partial_{\sigma}\phi ; \quad \hat{\lambda} = \lambda - \frac{1}{2}\theta^{\rho\sigma} A_{\rho}\partial_{\sigma}\lambda. \end{aligned} \quad (8)$$

As stated before, the "hatted" variables on the left are NC degrees of freedom and gauge transformation parameter. The higher order terms in  $\theta$  are kept out of contention as there are certain non-uniqueness involved in the  $O(\theta^2)$  mapping. The significance of the Seiberg-Witten map is that under an NC or  $*$ -gauge transformation of  $\hat{A}_{\mu}$  by,

$$\delta\hat{A}_{\mu} = \partial_{\mu}\hat{\lambda} + i[\hat{\lambda}, \hat{A}_{\mu}]_*,$$

$A_{\mu}$  will undergo the transformation

$$\delta A_{\mu} = \partial_{\mu}\lambda.$$

Subsequently, under this mapping, a gauge invariant object in conventional spacetime will be mapped to its NC counterpart, which will be  $*$ -gauge invariant. This is crucial as it ensures that the ordinary spacetime action that we recover from the NC action (6) by applying the Seiberg-Witten Map will be gauge invariant. Thus the NC action (6) in ordinary spacetime variables reads,

$$\hat{S} = \int d^3x [(D^\mu \phi)^* D_\mu \phi + \frac{1}{2} \theta^{\alpha\beta} \{F_{\alpha\mu} ((D_\beta \phi)^* D^\mu \phi + (D^\mu \phi)^* D_\beta \phi) - \frac{1}{2} F_{\alpha\beta} (D^\mu \phi)^* D_\mu \phi\}] \quad (9)$$

The above action is manifestly gauge invariant. Remember that so far we have not introduced the  $CP^1$  target space constraint in the NC spacetime setup. *Let us assume the constraint to be identical to the ordinary spacetime one* [9], i.e.,

$$\phi^* \phi = 1. \quad (10)$$

The reasoning is as follows [9]. Basically, after utilizing the Seiberg-Witten Map, we have returned to the ordinary spacetime and its associated dynamical variables and the effects of noncommutativity appears only as additional interaction terms in the action. Hence it is natural to keep the  $CP^1$  constraint unchanged. (For more details, see [9].)

This allows us to write,

$$A_\mu = -i\phi^* \partial_\mu \phi + a_\mu(\theta) \quad (11)$$

with  $a_\mu$  denoting the  $O(\theta)$  correction, obtained from (9,10). For  $\theta = 0$ .  $A_\mu$  reduces to its original form. Note that  $a_\mu$  is *gauge invariant*. Thus the  $U(1)$  gauge transformation of  $A_\mu$  remains intact, at least to  $O(\theta)$ . Keeping in mind the constraint  $\phi^* \phi = 1$ , let us now substitute (11) in the NC action (9). Since we are concerned only with the  $O(\theta)$  correction, in the  $\theta$ -term of the action, we can use  $A_\mu = -i\phi^* \partial_\mu \phi$ . However, in the first term in the action, we must incorporate the full expression for  $A_\mu$  given in (11). Remarkably, the constraint condition conspire to cancel the effect of the  $O(\theta)$  correction term  $a_\mu$ . Finally it boils down to the following: the action for the NC  $CP^1$  model to  $O(\theta)$  is given by (9) with the *identifications*  $A_\mu = -i\phi^* \partial_\mu \phi$ ,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and the *constraint*  $\phi^* \phi = 1$  [9].

### Section III - Energy-momentum tensor for the NC $CP(1)$ model

As is well-known, in general, it is not possible to obtain a symmetric, gauge invariant and conserved EM tensor in an NC field theory, with noncommutativity of the form of (1). There are two forms of EM tensor in vogue [10]: a manifestly symmetric form, obtained from the variation of the action with respect to the metric, and the canonical form, following the Noether prescription. The former is covariantly conserved whereas the latter is conserved. Since we are interested in the space-time invariance properties and Poincare generators, we will concentrate on the canonical (Noether) form, which is given by,

$$T_{\mu\nu} = \frac{\delta \hat{L}}{\delta (\partial^\mu \phi^*)} \partial_\nu \phi^* + \frac{\delta \hat{L}}{\delta (\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \hat{L}. \quad (12)$$

In the present case, for the action (9),

$$\hat{S} = \int d^3x [(D^\mu \phi)^* D_\mu \phi + \frac{1}{2} \theta^{\alpha\beta} \{F_{\alpha\mu} ((D_\beta \phi)^* D^\mu \phi + (D^\mu \phi)^* D_\beta \phi) - \frac{1}{2} F_{\alpha\beta} (D^\mu \phi)^* D_\mu \phi\}]$$

$$+ \Lambda(\phi^* \phi - 1)] \quad (13)$$

the EM tensor  $T_{\mu\nu}$  is,

$$\begin{aligned} T_{\mu\nu} = & [D_\mu \phi^* D_\nu \phi + (\mu \Leftrightarrow \nu)] \left(1 - \frac{1}{4} \theta^{\alpha\beta} F_{\alpha\beta}\right) + \frac{1}{2} \theta^{\alpha\beta} [F_{\alpha\nu} (D_\beta \phi^* D_\mu \phi + D_\mu \phi^* D_\beta \phi) + (\mu \Leftrightarrow \nu)] \\ & - \frac{1}{2} \theta_{\alpha\mu} F_{\nu}^\alpha |D\phi|^2 + \frac{1}{2} \theta_{\mu\beta} [F_{\nu\alpha} (D^\beta \phi^* D^\alpha \phi + D^\alpha \phi^* D^\beta \phi) + F^{\alpha\beta} (D_\alpha \phi^* D_\nu \phi + D_\nu \phi^* D_\alpha \phi)]. \end{aligned} \quad (14)$$

In order to obtain a symmetric  $T_{\mu\nu}$ , at least for  $\theta = 0$ , we have rewritten the covariant derivatives in the following equivalent form,

$$\begin{aligned} (D^\mu \phi)_a &= \partial^\mu \phi_a - (\phi^* \partial^\mu \phi) \phi_a = \partial^\mu \phi_a - \frac{1}{2} (\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi) \phi_a, \\ (D^\mu \phi)_a^* &= \partial^\mu \phi_a^* + (\phi^* \partial^\mu \phi) \phi_a = \partial^\mu \phi_a^* + \frac{1}{2} (\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi) \phi_a^*. \end{aligned} \quad (15)$$

Note that  $\mu \Leftrightarrow \nu$  symmetry of  $T_{\mu\nu}$  is destroyed by some of the  $\theta$ -terms. However,  $T_{\mu\nu}$  is manifestly gauge invariant and conserved (as it is derived from the canonical definition (12)).

#### Section IV - Hamiltonian formulation and constraint analysis

Let us now perform a Hamiltonian constraint analysis, in the Dirac [16] scheme, which entails in obtaining the full set of constraints in a given theory. Furthermore, the constraints are classified in to the largest set of commuting constraints (the First Class Constraints (FCC)) and the remaining non-commuting constraints (Second Class Constraints (SCC)). The presence of FCCs indicate local gauge invariances. For a consistent quantization programme, the SCCs are taken into account by replacing the Poisson brackets by Dirac brackets [16], that is defined below for two generic variables  $A$  and  $B$ ,

$$\{A, B\}_{DB} = \{A, B\} - \{A, \chi_i\} \{\chi_i, \chi_j\}^{-1} \{\chi_j, B\}, \quad (16)$$

In (16), Poisson brackets are used in the right hand side and  $\{\chi_i, \chi_j\}^{-1}$  denotes the inverse of the constraint Poisson bracket matrix  $\{\chi_i, \chi_j\}$ , the latter being invertible for SCC  $\chi_i$ . Notice that using Dirac brackets allows us to put the SCCs strongly equal to zero since they commute with everything in the Dirac bracket sense,

$$\{A, \chi_i\}_{DB} = \{\chi_j, B\}_{DB} = 0.$$

We closely follow the earlier work [19] on  $CP(1)$  model in ordinary spacetime.<sup>3</sup> Only spatial noncommutativity is being considered here, that is  $\theta^{01} = \theta^{02} = 0$ ,  $\theta^{12} = \theta \epsilon^{12}$ . The canonically conjugate momenta, as obtained from the action (13), are

$$\begin{aligned} \pi_a &= (1 + C) D^0 \phi_a^* - i\theta \epsilon^{ij} (D^0 \phi^* D^j \phi) D^i \phi_a^*, \\ \pi_a^* &= (1 + C) D^0 \phi_a + i\theta \epsilon^{ij} (D^0 \phi D^j \phi^*) D^i \phi_a, \end{aligned} \quad (17)$$

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<sup>3</sup>In an alternative extended space quantization scheme [20], the  $CP(1)$  model has been discussed in [21].

where  $C \equiv -\frac{1}{4}\theta\epsilon^{ij}F^{ij}$ . We immediately find the following two primary constraints,  $\psi_2$ ,  $\psi_3$ , along with the  $CP(1)$  primary constraint  $\psi_1$ ,

$$\psi_1 \equiv \phi^*\phi - 1 \approx 0 ; \psi_2 \equiv \phi\pi \approx 0 ; \psi_3 \equiv \phi^*\pi^* \approx 0. \quad (18)$$

Using the basic canonical Poisson brackets,

$$\{\phi_a(x), \pi_b(y)\} = \delta_{ab}\delta(x-y) ; \{\phi_a^*(x), \pi_b^*(y)\} = \delta_{ab}\delta(x-y) \quad (19)$$

it is revealed that the linear combination

$$\xi \equiv \phi\pi - \phi^*\pi^* \approx 0 \quad (20)$$

commutes with the other two constraints,

$$\chi_1 \equiv \phi^*\phi - 1 = 0 ; \chi_2 \equiv \phi\pi + \phi^*\pi^* = 0. \quad (21)$$

However, the non-vanishing bracket,

$$\{\chi_1(x), \chi_2(y)\} = 2(\phi^*(x)\phi(x))\delta(x-y) \approx 2\delta(x-y), \quad (22)$$

indicates that the above pair of constraints are Second Class Constraints (SCC) [16]. In the present case the basic Dirac brackets are computed below,

$$\begin{aligned} \{\phi_a(x), \phi_b(y)\} &= \{\phi_a(x), \phi_b^*(y)\} = 0, \\ \{\phi_a(x), \pi_b(y)\} &= (\delta_{ab} - \frac{1}{2}\phi_a\phi_b^*)\delta(x-y) ; \{\phi_a(x), \pi_b^*(y)\} = -\frac{1}{2}\phi_a\phi_b\delta(x-y), \\ \{\pi_a(x), \pi_b(y)\} &= \frac{1}{2}(\pi_a\phi_b^* - \pi_b\phi_a^*)\delta(x-y) ; \{\pi_a(x), \pi_b^*(y)\} = \frac{1}{2}(\pi_a\phi_b - \pi_b^*\phi_a^*)\delta(x-y). \end{aligned} \quad (23)$$

Since in the subsequent analysis only Dirac brackets are used, we have avoided the notation  $\{, \}_{DB}$ . Complex conjugation reproduces rest of the Dirac brackets. From now on we will be utilising Dirac brackets and exploit the SCC relations strongly, that is  $\chi_1 = \chi_2 = 0$ .

In order to find the full set of constraints, we now compute the canonical Hamiltonian. This means that the time derivatives are to be expressed in terms of the phase space variables. This is carried out to  $O(\theta)$ . To that end, we first note from (17) that

$$D^0\phi_a^* = \pi_a + O(\theta) , \quad D^0\phi_a = \pi_a^* + O(\theta). \quad (24)$$

This yields the following relations,

$$\begin{aligned} \pi_a &= (1 + C)D^0\phi_a^* - i\theta\epsilon^{ij}(\pi D^j\phi)D^i\phi_a^* + O(\theta^2), \\ \pi_a^* &= (1 + C)D^0\phi_a + i\theta\epsilon^{ij}(\pi^* D^j\phi^*)D^i\phi_a + O(\theta^2). \end{aligned} \quad (25)$$

The above equations allow us to rewrite the time derivatives as,

$$\begin{aligned} D^0\phi_a &= (1 - C)\pi_a^* - i\theta\epsilon^{ij}(\pi^* D^j\phi^*)D^i\phi_a + O(\theta^2), \\ D^0\phi_a^* &= (1 - C)\pi_a + i\theta\epsilon^{ij}(\pi D^j\phi)D^i\phi_a^* + O(\theta^2). \end{aligned} \quad (26)$$

This leads us to the canonical Hamiltonian, (with noncommutative effects up to  $O(\theta)$ ),

$$\begin{aligned} T_{00} &= 2(1+C)D_0\phi^*D_0\phi + \theta\epsilon_{ij}F_{i0}(D_0\phi^*D_j\phi + D_j\phi^*D_0\phi) - \mathcal{L} \\ &= (\pi^*\pi + D^k\phi^*D^k\phi)(1-C) + i\theta\epsilon^{ij}(\pi^*D^i\phi^*)(\pi D^j\phi). \end{aligned} \quad (27)$$

The total Hamiltonian, in the terminology of Dirac [16] is

$$\mathcal{H} = T_{00} + \Lambda(x)\xi(x). \quad (28)$$

It is now straightforward (though tedious) to check explicitly that

$$\{\xi(x), H\} = \{\xi(x), \int d^2y T_{00}(y)\} = 0. \quad (29)$$

This demonstration is very significant as it shows that there are no further constraints and that  $\xi \equiv \phi\pi - \phi^*\pi^* \approx 0$  is the only First Class Constraint (FCC) [16]. The presence of the FCC signals a gauge invariance -  $U(1)$  in the present case. This fact was apparent in the explicit form of the action (13) as well. It is trivial to ensure that the FCC  $\xi(x)$  functions properly as the generator  $Q$  of  $U(1)$  gauge transformation ( $Q \equiv \int d^2x \alpha(x)\xi(x)$ ) by evaluating the brackets,

$$\begin{aligned} \delta_\alpha\phi_a(x) &\equiv \{\phi_a(x), Q\} = \alpha(x)\phi_a(x) ; \delta_\alpha\phi_a^*(x) \equiv \{\phi_a^*(x), Q\} = -\alpha(x)\phi_a^*(x), \\ \delta_\alpha\pi_a(x) &\equiv \{\pi_a(x), Q\} = -\alpha(x)\pi_a(x) ; \delta_\alpha\pi_a^*(x) \equiv \{\pi_a^*(x), Q\} = \alpha(x)\pi_a^*(x), \end{aligned} \quad (30)$$

where  $\alpha(x)$  denotes the infinitesimal gauge transformation parameter.

## Section V: Schwinger condition, Poincare algebra and the equation of motion

Finally we are ready to tackle the question of the exact nature of Lorentz symmetry violation induced by noncommutativity. We will closely follow the conventional field theoretic approach in ordinary spacetime [19].

First of all, let us now compute the spatial momenta

$$P_i = \int d^2x T_{0i}(x), \quad (31)$$

where  $T_{\mu\nu}$  in (14) reproduces

$$\begin{aligned} T_{0i} &= (D_0\phi^*D_i\phi + D_i\phi^*D_0\phi)(1+C) \\ &+ \frac{1}{2}\theta\epsilon_{jk}[F_{ji}(D_0\phi^*D_k\phi + D_k\phi^*D_0\phi) + F_{j0}(D_k\phi^*D_i\phi + D_i\phi^*D_k\phi)] \\ &= \pi D_i\phi + \pi^*D_i\phi^* = \pi\partial_i\phi + \pi^*\partial_i\phi^* + \xi(x)\phi^*\partial_i\phi \approx \pi\partial_i\phi + \pi^*\partial_i\phi^*. \end{aligned} \quad (32)$$

We immediately find that  $P_i$  generates correct transformations among the degrees of freedom:

$$\{\phi(x), P_i\} = \{\phi(x), \int d^2y T_{0i}(y)\} = \partial_i\phi(x) ; \{\pi(x), P_i\} = \{\pi(x), \int d^2y T_{0i}(y)\} = \partial_i\pi(x), \quad (33)$$

where the Dirac brackets in (23) are used. Next we show that the Hamiltonian (27) represents the generator of time translation,

$$\begin{aligned} \{\phi_a(x), H\} &\equiv \{\phi_a(x), \int d^2x T_{00}(x)\} \\ &= \pi_a^* + i\theta\epsilon^{ij}[(\pi^* D^i \phi^*) D^j \phi_a - \frac{1}{2}(D^i \phi^* D^j \phi) \pi_a^*] = D^0 \phi_a, \end{aligned} \quad (34)$$

where we have used (25) to replace the momenta. Now notice an interesting fact that

$$\begin{aligned} \phi^* \partial_0 \phi &= \frac{1}{2}(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*) \\ &= -\frac{1}{2}\xi(1 - i\theta\epsilon^{ij} D^i \phi^* D^j \phi). \end{aligned} \quad (35)$$

Putting (35) back in (34) this means that modulo the FCC  $\xi(x)$ ,

$$\{\phi_a(x), H\} = \partial_0 \phi_a, \quad (36)$$

so that the correct Hamiltonian equation of motion is reproduced. The above results confirm the conservation of the energy and momenta, thereby ensuring Observer Lorentz invariance [13]. Obviously this is expected since the action is manifestly translation invariant, but the cancellation of the  $\theta$ -term is indeed non-trivial. However, we emphasize that this is the first explicit demonstration of the conservation principle in a particular noncommutative field theory model, in the Hamiltonian framework. This one of our main results.

We now consider how far it is possible to construct a Lorentz covariant equation of motion for  $\phi_a$ . Let us start with the ordinary spacetime  $CP(1)$  model ( $\theta = 0$ ). The Hamiltonian equations of motion yields,

$$\partial_0 \phi_a = \pi_a^* ; \quad \partial_0 \pi_a^* = -(\pi^* \pi) \phi_a + D^k (D^k \phi_a). \quad (37)$$

Once again exploiting the previous argument leading to (35), we can express (37) in a manifestly covariant form, (modulo the FCC  $\xi$ ),

$$D^\mu (D_\mu \phi_a) = -(\pi^* \pi) \phi_a. \quad (38)$$

The  $O(\theta)$  correction to the above equation is straightforward to compute but the the explicit form is quite involved and not particularly illuminating. Writing the  $O(\theta)$  equation of motion schematically,

$$D^\mu (D_\mu \phi_a) = -(\pi^* \pi) \phi_a + \epsilon^{\mu\nu\lambda} \theta_\mu A_{\nu\lambda}, \quad (39)$$

we note that  $A_{\nu\lambda}$  contains manifestly Lorentz covariant and non-covariant terms, comprising of field variables. Hence one is not able to recover a fully Lorentz covariant dynamical equation of motion in NC field theory.

Our next objective is the study of the Schwinger condition [15], the validity of which is necessary and sufficient to guarantee Poincare invariance. This requires verification of the all important bracket [15],

$$\{T_{00}(x), T_{0i}(y)\} = (T^{0i}(x) + T^{0i}(y)) \partial_{(x)}^i \delta(x - y), \quad (40)$$

known as the Schwinger condition. In the present case, after a long calculation, we find,

$$\{T_{00}(x), T_{0i}(y)\} = (T^{0i}(x) + T^{0i}(y))\partial_{(x)}^i \delta(x - y) + (\tau^i(x) + \tau^i(y))\partial_{(x)}^i \delta(x - y), \quad (41)$$

where

$$\begin{aligned} \tau^i = i\theta\epsilon^{ij} & \left[ \frac{1}{2}(\pi^* \pi + D^k \phi^* D^k \phi)(\pi D^j \phi - \pi^* D^j \phi^*) \right. \\ & \left. + (\pi^* D^k \phi^*)(D^j \phi^* D^k \phi) - (\pi D^k \phi)(D^k \phi^* D^j \phi) \right]. \end{aligned} \quad (42)$$

Clearly in the present instance, the Schwinger condition is not satisfied because  $\tau_i \neq 0$  in general, indicating a violation of Particle Lorentz symmetry [13]. The reason is clearly the introduction of a constant tensor, in the form of the noncommutativity parameter  $\theta_{\mu\nu}$ , which singles out a fixed direction in spacetime. This is the other major result that we had advertised.

It is interesting to note that, in 2+1-dimensions, even for  $\theta \neq 0$ , Schwinger pcondition and subsequently Poincare invariance can be maintained in this model provided the fields are such that  $\tau_i$  vanishes identically. This type of scenario has not been reported before.

If we define the Poincare generators in the conventional way,

$$P_\mu \equiv \int d^2x T_{0\mu} ; J_{\mu\nu} \equiv \int d^2x (x_\mu T_{0\nu} - x_\nu T_{0\mu}), \quad (43)$$

the sector of the Poincare algebra involving the Lorentz boost generators  $J_{0i}$  will be violated. It is not surprising that, although we have introduced spatial noncommutativity, still the angular momentum algebra remains intact, indicating rotational invariance. This is because the asymmetry (via noncommutativity) is actually introduced in the time direction, which is easily seen if we look at  $\theta^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda}\theta_{\nu\lambda}$ , the dual of  $\theta_{\mu\nu}$ . In the present example, a non-zero  $\theta_{12}$  yield a non-zero  $\theta^0$ .

## Section VI: Conclusions and Future Prospects

Let us summarize our work. In [9] we had provided an alternative formulation of the noncommutative extension of the  $CP(1)$  model, that was distinct from the existing ones [8]. The present work deals with the Hamiltonian formulation of the model of [9]. We emphasize that probably this is the first instance where a noncommutative field theory has been studied in the Hamiltonian framework and most of the basic observations will be relevant for the study of a generic noncommutative field theory in Hamiltonian framework.

The aim is to study in detail the characteristic features of the violation of Lorentz invariance in a noncommutative field theory. This subtle issue first appeared in [13], where it was pointed out that two distinct types of Lorentz transformations are to considered: the Observer and the Particle Lorentz transformations. Symmetry under the former is maintained (in noncommutative theories), due to the translation invariance of the theory, thereby leading to the conservation of energy and momentum. The latter is destroyed owing to the presence of the constant tensor  $\theta_{\mu\nu}$ .

We have successfully addressed the above issues in the present work. We have constructed the Hamiltonian and momentum operators and have shown that they act properly as the generators time and space translations. This is related to the Observer Lorentz invariance. On

the other hand, we have shown that Schwinger condition and subsequently the Poincare algebra is not respected and that one can not derive a Lorentz covariant dynamical field equation. These features signal a loss of the Particle Lorentz symmetry.

The age old Hamiltonian constraint analysis, as formulated by Dirac [16], has been instrumental in our analysis. The model contains both First Class and Second Class constraints, conforming to the classification scheme of Dirac [16]. The Dirac brackets [16] have been computed and they are exploited throughout in arriving at the above mentioned results.

An intriguing open problem is the introduction of the Hopf term in the  $CP(1)$  model and its subsequent noncommutative extension. The classic work of [22] was the demonstration that the Hopf term is able to impart fractional spin and statistics to the solitons of the Non-linear  $\sigma$ -model. This was further corroborated in [19] in a Hamiltonian formulation. The advantage in an alternative  $CP(1)$  representation of the Non-linear  $\sigma$ -model is that the Hopf term is reduced to a local expression [23] in terms of  $CP(1)$  degrees of freedom. One of our future projects is to study the effects of noncommutativity on the anyons induced by the Hopf term.

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