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NOTE

INTERCHANGING THE ORDER OF STOCHASTIC INTEGRATION AND ORDINARY DIFFERENTIATION

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SUMMARY. We give conditions under which the order of differentiation w.r.t. a Euclidian parameter and stochastic integration w.r.t. a continuous semimartingale can be interchanged.

Suppose we have an observation $(x(t) : 0 \leqslant t \leqslant 1)$ on a process X known to satisfy

$$X(0) = 0$$
$$dX(t) = f(\theta_0, X(t))dt + d\beta(t)$$

for some $\theta_0 \in \mathcal{R}$, where β is a Brownian motion and $f(\theta, x)$ is a known function. In order to estimate the unknown parameter 'θ' using the well known 'Maximum Likelihood' method, we take $C[0, 1]$ as the sample space and μ_θ to be the 'distribution' of X_θ on $C[0, 1]$ (i.e. measure induced by X_θ) where X_θ is the solution of

$$X_\theta(0) = 0$$
$$dX_\theta(t) = f(\theta, X_\theta(t))dt + d\beta(t).$$

Under suitable conditions on f (e.g. $f(\theta, \cdot)$ is bounded for all θ), it is known that μ_θ has a density w.r.t. the Weiner measure P given by

$$\frac{d\mu_\theta}{dP}(w) = \exp\left[\int_0^1 f(\theta, w(t))dw(t) - \frac{1}{2}\int_0^1 f^2(\theta, w(t))dt\right]$$

where $w(t)$ is the co-ordinate process on $C[0, 1]$, which is a Brownian motion under P and $\int_0^1 f(\theta, w(t))dw(t)$ is interpreted as the Ito integral. So to estimate θ, we would like to know : under what conditions on f is $\int_0^1 f(\theta, w(t))dw(t)$ a differentiable function of θ and can we differentiate under the stochastic integral sign ?

In this note, we give simple conditions under which this can be done. The main tool is a strengthened form of Kolmogorov's theorem on existence of continuous modifications of stochastic processes. This has been used by Kunita (1981) to prove very strong results on solutions of SDE's. See also Stroock (1981) in this connection.

We first state and prove the result under 'simple' conditions for the Ito integral. Later we indicate how the techniques used earlier give the same result under more general conditions.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_t)_{t \geq 0}$ be an increasing family of sub σ fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$. All the processes we consider are $(\mathcal{F}_t)$ adapted. Let $(\beta_t)_{t \geq 0}$ be a $\mathcal{F}_t$- Brownian motion.

The first lemma that we state, without proof, is a generalisation of Kolmogorov's theorem. For a proof of this result, see Stroock (1981).

Lemma 1: *Let B be a separable Banach space. Let $[Z(\theta) : \theta \in \mathcal{R}^d]$ be a family of B-valued random variables such that for $C, \alpha, p : 0 < C < \infty$, $0 < \alpha \leq 1$, $p \geq d+\alpha$*

$$E\|Z(\theta_1)-Z(\theta_2)\|^p \leq C|\theta_1-\theta_2|^{d+\alpha};\ \theta_1, \theta_2 \in \mathcal{R}^d.$$

Then there exists a "continuous version" $Z_1(\theta)$ of $Z(\theta)$, i.e., $P(Z(\theta) = Z_1(\theta)) = 1$ for all θ, and for all $w \in \Omega$, $\theta \to Z(\theta, w)$ is a continuous map from $\mathcal{R}^d$ into B.

The next lemma is a form of Burkholder's inequality. See Stroock (1981) or Stroock and Varadhan (1979) for a proof of this.

Let $\mathcal{L}_2$ be the collection of all progressively measurable processes f on $[0, 1]$ such that $E \int_0^1 f^2(t, w)dt$ is finite.

Lemma 2: *Let $p \geq 2$. There exists a universal constant C_p such that for all $f \in \mathcal{L}_2$,*

$$E \sup_{0 \leq t \leq 1} \left| \int_0^t f(u, \cdot)d\beta(u) \right|^p \leq C_p E \int_0^1 |f(u, \cdot)|^p du.$$

The next theorem, which is well known, is on existence of continuous versions of stochastic integral depending on a parameter.

Theorem 3: *Let $[g(\theta, \cdot, \cdot) : \theta \in \mathcal{R}^d] \subseteq \mathcal{L}_2$ be such that (C1) holds:*

(C1) *There exist $C, \beta, 0 < C < \infty, 0 < \beta \leq 1$ such that*
$|g(\theta_1, t, w)-g(\theta_2, t, w)| \leq C|\theta_1-\theta_2|^\beta$, for all t, w, θ_1, θ_2.

Then there exists a 'continuous versions' $X(\theta, t, \cdot)$ of $\int_0^t g(\theta, u, \cdot)d\beta(u)$, i.e., for all $w \in \Omega$, $\theta \to X(\theta, \cdot, w)$ is a continuous map from $\mathcal{R}^d$ into $C[0, 1]$ (equipped with the sup norm).

Proof: Choose $p \geqslant 2$ such that $p\beta > d$.

Let $Z(\theta, t) = \int_0^t g(\theta, u) d\beta(u)$. Then by Lemma 2

$$E \sup_{0 \leqslant t \leqslant 1} |Z(\theta, t) - Z(\theta_2, t)|^p \leqslant C_p E \int_0^1 |g(\theta_1, u) - g(\theta_2, u)|^p \, du$$

$$\leqslant C_p . C |\theta_1 - \theta_2|^{p\beta}.$$

Now, the assertion follows from Lemma 1.

Theorem 4: *Let $[f(\theta, \cdot, \cdot) : \theta \in \mathcal{R}] \subseteq \mathcal{L}_2$ be such that for all t, w, $\frac{d}{d\theta} f(\theta, t, w) = f'(\theta, t, w)$ exists and $[f'(\theta, \cdot, \cdot) : \theta \in \mathcal{R}] \subseteq \mathcal{L}_2$. Further, assume that* (C2) *holds*:

(C2) *There exist constants C, β_1, β_2; $0 < C < \infty$, $0 < \beta_1 \leqslant 1$, $0 < \beta_2 \leqslant 1$ such that*

$$|f(\theta_1, t, w) - f(\theta_2, t, w)| \leqslant C |\theta_1 - \theta_2|^{\beta_1}$$

$$|f'(\theta_1, t, w) - f'(\theta_2, t, w)| \leqslant C |\theta_1 - \theta_2|^{\beta_2}$$

Then, there exists a version $X(\theta, t, \cdot)$ of $\int_0^t f(\theta, u) d\beta(u)$ such that for all w, t; $\theta \to X(\theta, t, w)$ is differentiable in θ and

$$\frac{d}{d\theta} X(\theta, t, .) = \int_0^t f'(\theta, u, \cdot) d\beta(u).$$

(In other words, we can differentiate under the stochastic integral sign.)

Proof: First choose a continuous version $X_1(\theta, t)$ of $\int_0^t f(\theta, u) d\beta(u)$. This can be done as (C2) implies (C1).

Let

$$Y(\theta_1, \theta_2, t, w) = \frac{X_1(\theta_1, t, w) - X_1(\theta_2, t, w)}{\theta_1 - \theta_2} \quad \text{if } \theta_1 \neq \theta_2$$

$$= \int_0^t f'(\theta_1, u) d\beta(u) \qquad \text{if } \theta_1 = \theta_2.$$

We will show that Y has a 'continuous version'. As Y is continuous on $[(\theta_1, \theta_2) : \theta_1 \neq \theta_2]$ and any two continuous versions agree outside a null set, this will imply that outside a null set N, $X_1(\theta, t, w)$ is a differentiable function of θ for all t. The result follows from this by putting $X(\theta, t, w) = 0$ on this null set N and equal to $X_1(\theta, t, w)$ elsewhere

Now, observe that for all θ_1, θ_2,

$$Y(\theta_1, \theta_2, t, .) = \int_0^t \left[\int_0^1 f'(\lambda\theta_1 + (1-\lambda)\theta_2, u, w) d\lambda \right] d\beta(u)$$

Let

$$g(\theta_1, \theta_2, u, w) = \int_0^1 f'(\lambda\theta_1 + (1-\lambda)\theta_2, u, w) d\lambda.$$

Then

$$|g(\theta_1, \theta_2, u, w) - g(\theta_3, \theta_4, u, w)|$$

$$\leqslant \int_0^1 |f'(\lambda\theta_1 + (1-\lambda)\theta_2, u, w) - f'(\lambda\theta_3 + (1-\lambda)\theta_4, u, w)| d\lambda$$

$$\leqslant C \int_0^1 |\lambda\theta_1 + (1-\lambda)\theta_2 - \lambda\theta_3 - (1-\lambda)\theta_4|^\beta d\lambda$$

$$\leqslant C_1(|\theta_1 - \theta_3|^\beta + |\theta_2 - \theta_4|^\beta)$$

$$\leqslant C_2(|\theta_1 - \theta_3|^2 + |\theta_2 - \theta_4|^2)^{\beta/2},$$

where C_1, C_2 are some constants.

This in view of Theorem 3 implies that Y has a continuous modification, which completes the proof as remarked earlier.

Remarks : Theorems 3 and 4 were presented here in a very simple case. We had considered processes on $[0, 1]$, but the results easily extend to cover processes on $[0, \infty)$. The constant 'C' appearing in conditions (C1) and (C2) can be replaced by a locally bounded process by using a 'localisation' procedure. The integrator 'Brownian motion' can be replaced by a 'continuous semi-martingale' by using L^p estimates on stochastic integrals. We state the results in general form without proof as the main idea of the proof is contained in the simple case presented earlier.

Say that $[g(\theta, \cdot, \cdot) : \theta \in \mathcal{R}^d]$ satisfies (C3) if :

(C3) For all $n \geq 1$, there exist constants C_n, α_n; $0 < C_n < \infty$, $0 < \alpha_n \leq 1$, stop times T_n, such that
(i) T_n increases to ∞.
(ii) $|g(\theta, t, w)|_{[t<T_n(w)]} \leq C_n$
(iii) $|g(\theta_1), t, w)-g(\theta_2, t, w)|\,|_{[t<T_n(w)]} \leq C_n|\theta_1-\theta_2|^{\alpha_n}$, if $|\theta_1| \leq n$, $|\theta_2| \leq n$.

Theorem 5 : *Let S be a continuous semimartingale and* $[f(\theta, \cdot, \cdot) : \theta \in \mathcal{R}^d]$ *be a family of progressively measurable processes.*

(1.) *If* $[f(\theta, ., .) : \theta \in \mathcal{R}^d]$ *satisfies* (C3), *then there exists a continuous version* $X(\theta, t, .)$ *of* $\int_0^t f(\theta, u, .)dS(u)$.

(2.) *If* $g(\theta, t, w) = \frac{d}{d\lambda} f(\theta+\lambda e, t, w)|_{\lambda=0}$ *exists for all* θ, *where* $e \in \mathcal{R}^d$, *and* $[g(\theta, ., .) : \theta \in \mathcal{R}^d]$ *satisfies* (C3), *then there exists a version* $X(\theta, t, .)$ *of* $\int_0^t f(\theta, u)dS(u)$ *such that*

$$Y(\theta, t, w) = \frac{d}{d\lambda} X(\theta+\lambda e, t, w)|_{\lambda=0}$$

exists, and

$$Y(\theta, t, w) = \int_0^t g(\theta, u, .)dS(u).$$

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