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Uncertainty principles on certain Lie groups

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Abstract. There are several ways of formulating the uncertainty principle for the Fourier transform on $\mathbb{R}^n$. Roughly speaking, the uncertainty principle says that if a function f is 'concentrated' then its Fourier transform $\tilde{f}$ cannot be 'concentrated' unless f is identically zero. Of course, in the above, we should be precise about what we mean by 'concentration'. There are several ways of measuring 'concentration' and depending on the definition we get a host of uncertainty principles. As several authors have shown, some of these uncertainty principles seem to be a general feature of harmonic analysis on connected locally compact groups. In this paper, we show how various uncertainty principles take form in the case of some locally compact groups including $\mathbb{R}^n$, the Heisenberg group, the reduced Heisenberg group and the Euclidean motion group of the plane.

Keywords. Fourier transform; Heisenberg group; motion group; uncertainty principle.

1. Introduction

There are several ways of formulating the uncertainty principle for the Fourier transform on $\mathbb{R}^n$. Roughly speaking, the uncertainty principle says that if a function f is 'concentrated' then its Fourier transform $\tilde{f}$ cannot be concentrated unless f is identically zero. Of course, in the above, we should be precise about what we mean by 'concentration'. There are several ways of measuring 'concentration' and depending on the definition we get a host of uncertainty principles. As has been shown in [1], [2], [4], [9], [12], [13], [17] etc, some of these uncertainty principles seem to be a general feature of harmonic analysis on connected locally compact groups. We continue these investigations in this paper to see how various uncertainty principles take form in the case of some locally compact groups including $\mathbb{R}^n$, the Heisenberg group, the reduced Heisenberg group and the Euclidean motion group of the plane. In a forthcoming paper [14] we consider semi-simple Lie groups and also more general eigenfunction expansions on a manifold with respect to some elliptic operator.

One way of measuring concentration is by considering the decay of the function at infinity. In this context, a theorem of Hardy for the Fourier transform on $\mathbb{R}$ says the following:

Theorem 1. (*Hardy*) *Suppose f is a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant Ce^{-\alpha x^2}, \quad |\tilde{f}(\xi)| \leqslant Ce^{-\beta \xi^2}, \quad x, \xi \in \mathbb{R} \tag{1.1}$$

where α, β are positive constants. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying (1.1) *and if $\alpha\beta = \frac{1}{4}$ then $f(x) = Ce^{-\alpha x^2}$.*

For a proof of the above theorem see [3]. A more general theorem due to Beurling, from which Hardy's theorem can be deduced, can be found in [10]. In this paper we establish an analogue of the above theorem for the Heisenberg group $\mathscr{H}_n$ (see § 2 for the precise formulation). We also prove Hardy's theorem in the case of $\mathbb{R}^n$, $n \geq 2$ and show that though the exact analogue for the reduced Heisenberg group fails, a slightly modified version continues to hold. In the final section we prove an analogue of Hardy's theorem for the Euclidean motion group of the plane.

Another natural way of measuring 'concentration' is in terms of the supports of the function f and its Fourier transform $\tilde{f}$. If f is non-trivial and compactly supported then $\tilde{f}$ extends to an entire function, and so $\tilde{f}$ cannot have compact support. A non-trivial extension of this result due to Benedicks [1] says: If $f \in L^1(\mathbb{R}^n)$ is such that $m\{x : f(x) \neq 0\} < \infty$ and $m\{\xi : \tilde{f}(\xi) \neq 0\} < \infty$ then $f = 0$ a.e. Here m stands for the Lebesgue measure on $\mathbb{R}^n$. This result of Benedicks has been extended in [2], [12], [4] etc. to a wide variety of locally compact groups. In particular, one has the following result for the Heisenberg group:

Theorem 2. *(Price–Sitaram) Let $f \in L^1 \cap L^2(\mathscr{H}_n)$. Suppose that $m\{t \in \mathbb{R} : f(z, t) \neq 0\} < \infty$ for a.e. $z \in \mathbb{C}^n$ and $m\{\lambda \in \mathbb{R}^* : \hat{f}(\lambda) \neq 0\} < \infty$. Then $f = 0$ a.e.*

In the above $\hat{f}(\lambda)$ stands for the group Fourier transform on $\mathscr{H}_n$ and $\mathbb{R}^*$ means $\mathbb{R} \backslash \{0\}$. Roughly speaking, the above theorem says that if $f \in L^2(\mathscr{H}_n)$ is concentrated in the t direction then $\hat{f}(\lambda)$ cannot be concentrated. It is the concentration in the t direction, not that in the z direction, which forces the spreading out of the Fourier transform. In fact, as was shown by Thangavelu in [17], we can have L^2 functions with compact support in the z variable for which $\hat{f}$ is also compactly supported. The special role played by the t variable in the above theorem (as well as in our Hardy's theorem in § 2) should not come as a surprise. The Fourier transform on $\mathscr{H}_n$ is more or less the Euclidean Fourier transform as far as the t variable is concerned. If one goes through the proof of the above theorem, one observes that it is a consequence of the corresponding theorem for the Euclidean Fourier transform in the t variable.

In view of the preceding remarks one would like to have an analogue of the above theorem which respects the z variable. We formulate and prove such a theorem in § 3. We will show that when f has compact support in the z variable then $\hat{f}(\lambda)$ (as an operator) cannot have 'compact support'. We will give a precise meaning to this statement in § 3.

We now turn our attention towards quantitative versions of the uncertainty principle, namely uncertainty inequalities. The classical Heisenberg–Pauli–Weyl uncertainty inequality for the Fourier transform on $\mathbb{R}^n$ says that

$$\|f\|_2^4 \leq C_n\left(\int |x|^2 |f(x)|^2 \mathrm{d}x\right)\left(\int |\xi|^2 |\tilde{f}(\xi)|^2 \mathrm{d}\xi\right). \tag{1.2}$$

For a proof of (1.2) with the precise value of C_n we refer to [6]. A version of the above inequality for the Heisenberg group was established by Thangavelu in [17]. Here we are concerned with local versions of the above inequality for the Heisenberg group.

For the Fourier transform on $\mathbb{R}^n$ one has the following local uncertainty inequality: For any measurable $E \subset \mathbb{R}^n$, and $0 < \theta < \frac{1}{2}$,

$$\int_E |\tilde{f}(\xi)|^2 d\xi \leqslant C_\theta m(E)^{2\theta} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{2n\theta} dx. \tag{1.3}$$

An analogue of the above inequality is known on the Heisenberg group. The following result is proved in [13].

Theorem 3. (*Price–Sitaram*) *Let* $\theta \in [0, \frac{1}{2})$. *Then, for each* $f \in L^1 \cap L^2(\mathscr{H}_n)$ *and measurable* $E \subset \mathbb{R}^*$, *one has*

$$\int_E \mathrm{tr}(\hat{f}(\lambda)^* \hat{f}(\lambda)) d\mu(\lambda) \leqslant C_\theta m(E)^{2\theta} \int_{\mathscr{H}_n} |f(z,t)|^2 |t|^{2\theta} dz\, dt. \tag{1.4}$$

(In the above tr stands for the canonical semifinite trace and $d\mu$ is the Plancherel measure on $\mathscr{H}_n$—see § 2.) Again we observe that the t variable plays a special role. As in the case of the Euclidean Fourier transform one would like to have an inequality which is more symmetric in all the variables. In § 4 we formulate and prove a local uncertainty inequality with the right hand side being

$$\int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\theta Q} dw \tag{1.5}$$

where $|w|^4 = |z|^4 + t^2$ and $Q = (2n+2)$ is the homogeneous dimension of $\mathscr{H}_n$. From the local uncertainty inequality we will also deduce a global inequality similar to the classical Heisenberg–Pauli–Weyl uncertainty inequality.

Finally, for various facts about the Heisenberg group we refer to the monographs of Folland [6] and Thangavelu [19]. We closely follow the notations of the latter which differ from the former by a factor of 2π.

2. Analogues of Hardy's theorem for $\mathbb{R}^n$ and $\mathscr{H}_n$

Before we prove Hardy's theorem for the Heisenberg group, consider the case of $\mathbb{R}^n$, $n \geqslant 2$. The proof of Hardy's theorem (for $n = 1$) depends heavily on complex analysis. As we have not found a reference in the literature for the higher dimensional case of Hardy's theorem we take this opportunity to present a proof which follows easily from the one-dimensional case via the Radon transform.

Theorem 4. *Let* f *be a measurable function on* $\mathbb{R}^n$ *and* α, β *two positive constants. Further assume that*

$$|f(x)| \leqslant C e^{-\alpha |x|^2}, |\tilde{f}(\xi)| \leqslant C e^{-\beta |\xi|^2}, \quad x, \xi \in \mathbb{R}^n. \tag{2.1}$$

If $\alpha\beta > \frac{1}{4}$, *then* $f = 0$ *a.e. If* $\alpha\beta < \frac{1}{4}$, *there are infinitely many linearly independent solutions for* (2.1) *and if* $\alpha\beta = \frac{1}{4}$, f *is a constant multiple of* $e^{-\alpha|x|^2}$.

Proof. As mentioned above, we will use theorem 1. So, assume that $n \geqslant 2$. We use the Radon transform to reduce the problem to the one-dimensional case. Recall that the

Radon transform Rg of an integrable function g on $\mathbb{R}^n$ is a function of two variables (ω, s) where $\omega \in S^{n-1}$ and $s \in \mathbb{R}$ and is given by

$$\mathrm{Rg}(\omega, s) = \int_{x.\omega = s} g(x)\mathrm{d}x. \tag{2.2}$$

where $\mathrm{d}x$ is the Euclidean measure on the hyperplane $x.\omega = s$. Actually, for each fixed ω, the above makes sense for almost all $s \in \mathbb{R}$ which may depend on ω. However for functions with sufficient rapid decay at infinity it makes sense for all s. For various properties of the Radon transform we refer to [5] and [8].

Our definition of the Fourier transform of a function f on $\mathbb{R}^n$ is

$$\tilde{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix.\xi} \mathrm{d}x. \tag{2.3}$$

Then it can be easily seen that

$$\tilde{f}(s\omega) = (Rf)^{\sim}(\omega, s) \tag{2.4}$$

where $s \in \mathbb{R}$, $\omega \in S^{n-1}$ and $(Rf)^{\sim}$ stands for the Fourier transform of Rf in the s-variable alone. From the definition of the Radon transform Rf and the relation (2.4), the conditions on f and $\tilde{f}$ translate into conditions on Rf and $(Rf)^{\sim}$. For each fixed ω, we therefore get

$$|Rf(\omega, r)| \leqslant C e^{-\alpha r^2}, \quad r \in \mathbb{R} \tag{2.5}$$

$$|(Rf)^{\sim}(\omega, s)| \leqslant C e^{-\beta s^2}, \quad s \in \mathbb{R}. \tag{2.6}$$

By appealing to Hardy's theorem for $\mathbb{R}$ we conclude that for $\alpha\beta > \frac{1}{4}$, $Rf(\omega, .) = 0$, for almost all ω. In view of the inversion theorem for the Radon transform this implies $f = 0$ a.e. When $\alpha\beta = \frac{1}{4}$, $(Rf)^{\sim}(\omega, s) = \tilde{f}(s\omega) = A(\omega)e^{-\alpha s^2}$ where A is a measurable function on the unit sphere S^{n-1}. Because $f \in L^1(\mathbb{R}^n)$, $\tilde{f}$ is continuous at zero and by taking $s \to 0$ we obtain $A(\omega) = \tilde{f}(0)$. Hence $\tilde{f}(\xi) = \tilde{f}(0)e^{-\beta|\xi|^2}$ so that $f(x) = Ce^{-\alpha|x|^2}$ for some constant C. If $\alpha\beta < \frac{1}{4}$, the n-dimensional suitably scaled Hermite functions Φ_μ satisfy (2.1).

We now consider the case of the Heisenberg group $\mathscr{H}_n = \mathbb{C}^n \times \mathbb{R}$. The multiplication law of the group $\mathscr{H}_n$ is given by

$$(z, t)(w, s) = (z + w, t + s + \tfrac{1}{2}\mathrm{Im}(z.\bar{w})), \tag{2.7}$$

where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$. Then $\mathscr{H}_n$ becomes a step-two nilpotent Lie group with Haar measure $\mathrm{d}z\mathrm{d}t$. In order to define the group Fourier transform we need to recall some facts about the representations of the Heisenberg group. For each $\lambda \in \mathbb{R}^*$, there is an irreducible unitary representation π_λ of $\mathscr{H}_n$ realised on $L^2(\mathbb{R}^n)$ and is given by

$$(\pi_\lambda(z, t)\phi)(\xi) = e^{i\lambda t} e^{i\lambda(x.\xi + \frac{1}{2}x.y)} \phi(\xi + y), \tag{2.8}$$

where $z = x + iy$ and $\phi \in L^2(\mathbb{R}^n)$. A theorem of Stone–von Neumann says that all the infinite dimensional irreducible unitary representations of $\mathscr{H}_n$ are given by π_λ, $\lambda \in \mathbb{R}^*$, (up to unitary equivalence). The Plancherel measure $\mathrm{d}\mu = |\lambda|^n \mathrm{d}\lambda$ is supported on $\mathbb{R}^*$. (There is another family of one-dimensional representations of $\mathscr{H}_n$ which do not play a role in the Plancherel theorem.)

Given a function f, say in $L^1(\mathscr{H}_n)$, its group Fourier transform $\hat{f}$ is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{\mathscr{H}_n} f(z,t)\pi_\lambda(z,t)\mathrm{d}z\,\mathrm{d}t. \tag{2.9}$$

(The above integral being interpreted suitably). For each $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$. A simple calculation shows that $\hat{f}(\lambda)$ is an integral operator with kernel $K_f^\lambda(\xi,\eta)$ given by

$$K_f^\lambda(\xi,\eta) = \mathscr{F}_{13} f\left(\frac{\lambda(\xi+\eta)}{2},\ \xi-\eta,\ \lambda\right), \tag{2.10}$$

where we have written $f(z,t) = f(x,y,t)$ and $\mathscr{F}_{13}f$ stands for the Fourier transform of f in the first and the third set of variables. For f in $L^1 \cap L^2(\mathscr{H}_n)$ a simple calculation shows that

$$\|\hat{f}(\lambda)\|_{HS}^2 = C|\lambda|^{-n}\int_{\mathbb{C}^n} |\mathscr{F}_3 f(z,\lambda)|^2\mathrm{d}z, \tag{2.11}$$

(for a suitable constant C) where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm. From this and the Euclidean–Plancherel theorem, the Plancherel theorem for the Heisenberg group follows:

$$\|f\|_2^2 = C_n \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda), \tag{2.12}$$

where $\mathrm{d}\mu(\lambda) = |\lambda|^n \mathrm{d}\lambda$ and C_n is a constant depending only on the dimension.

We now state and prove the following analogue of Hardy's theorem for $\mathscr{H}_n$.

Theorem 5. *Suppose f is a measurable function on $\mathscr{H}_n$ satisfying the estimates*

$$|f(z,t)| \leqslant g(z)e^{-\alpha t^2}, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}, \tag{2.13}$$

$$\|\hat{f}(\lambda)\|_{HS} \leqslant Ce^{-\beta|\lambda|^2}, \quad \lambda \in \mathbb{R}^*, \tag{2.14}$$

where $g \in L^1 \cap L^2(\mathbb{C}^n)$ and α, β are positive constants. Then, if $\alpha\beta > \frac{1}{4}$, $f = 0$ a.e.; if $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying the above estimates.

Proof. For a function f on $\mathscr{H}_n$ define f^* to be the function $f^*(z,t) = \overline{f(z,-t)}$ and let $f *_3 f^*$ stand for the convolution of f and f^* in the t-variable. Then, a simple calculation shows that

$$\begin{aligned}\int_{\mathscr{H}_n} (f *_3 f^*)(z,t)e^{i\lambda t}\mathrm{d}z\,\mathrm{d}t &= \int_{\mathbb{C}^n} \mathscr{F}_3(f *_3 f^*)(z,\lambda)\mathrm{d}z \\ &= \int_{\mathbb{C}^n} |\mathscr{F}_3 f(z,\lambda)|^2\mathrm{d}z \end{aligned}\tag{2.15}$$

which, in view of (2.11), equals $C^{-1}|\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2$. Define a function h on $\mathbb{R}$ by

$$h(t) = \int_{\mathbb{C}^n} (f *_3 f^*)(z, t)\,\mathrm{d}z. \tag{2.16}$$

Then one has

$$\tilde{h}(\lambda) = C^{-1}|\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2. \tag{2.17}$$

Now the conditions (2.13) and (2.14) on f and $\hat{f}$ translate into the conditions

$$|h(t)| \leqslant Ce^{-(\alpha/2)t^2}, \quad |\tilde{h}(\lambda)| \leqslant Ce^{-2\beta'|\lambda|^2}, \quad t\in\mathbb{R}, \quad \lambda\in\mathbb{R}^*, \tag{2.18}$$

where β' can be chosen so that $\alpha\beta' > \frac{1}{4}$ or $< \frac{1}{4}$ according as $\alpha\beta > \frac{1}{4}$ or $< \frac{1}{4}$. If $\alpha\beta > \frac{1}{4}$, then $\alpha\beta' > \frac{1}{4}$, so that Hardy's theorem for $\mathbb{R}$ implies that $h = 0$ a.e. This means $\|\hat{f}(\lambda)\|_{HS}^2 = 0$ for all $\lambda\in\mathbb{R}^*$ and consequently $f = 0$ a.e. by the Plancherel theorem for $\mathscr{H}_n$. If $\alpha\beta < \frac{1}{4}$, then any function of the form $g(z)h_k(t)$ where h_k is a suitably scaled Hermite function satisfies the hypothesis of the theorem.

The following is the exact analogue of Hardy's theorem for $\mathscr{H}_n$.

COROLLARY 6

Suppose f is a measurable L^1-function on $\mathscr{H}_n$ and

$$|f(z, t)| \leqslant Ce^{-\alpha(|z|^2 + |t|^2)}, \quad z\in\mathbb{C}^n, \quad t\in\mathbb{R} \tag{2.19}$$

$$\|\hat{f}(\lambda)\|_{HS} \leqslant Ce^{-\beta|\lambda|^2}, \quad \lambda\in\mathbb{R}^* \tag{2.20}$$

for some positive constants α and β. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$, then there are infinitely many such linearly independent functions.

We shall now consider the case of the reduced Heisenberg group $\mathscr{H}_n^{\mathrm{red}} = \mathbb{C}^n \times S^1$. The multiplication law is as in (2.7) except for the understanding that t is a real number modulo 1. The reduced Heisenberg group $\mathscr{H}_n^{\mathrm{red}}$ is also a step two nilpotent Lie group with Haar measure $\mathrm{d}z\mathrm{d}t$ where $\mathrm{d}t$ denotes the normalized Lebesgue measure on S^1. For each $m\in Z^* = Z\backslash\{0\}$, there is an irreducible unitary representation π_m of $\mathscr{H}_n^{\mathrm{red}}$ realized on $L^2(\mathbb{R}^n)$ and is defined exactly as in (2.8). As in the case of $\mathscr{H}_n$, we get (up to unitary equivalence) that all the infinite dimensional irreducible unitary representations of $\mathscr{H}_n^{\mathrm{red}}$ are given by π_m, $m\in Z^*$. Apart from this there is a class of one dimensional representations, $\pi_{a,b}$, $a, b\in\mathbb{R}^n$ given by

$$\pi_{a,b}(z, t) = e^{2\pi i(ax + by)} \quad \text{for} \quad (z, t)\in\mathscr{H}_n^{\mathrm{red}}. \tag{2.21}$$

The dual $\hat{\mathscr{H}}_n^{\mathrm{red}}$ can be thought of as the disjoint union of Z^* and $\mathbb{R}^{2n}$. The Plancherel measure is the counting measure on Z^* with a weight function $C|m|^n$ (for a suitable constant C) and the Lebesgue measure on $\mathbb{R}^{2n}$. (This is in sharp contrast to the case of Heisenberg group.)

Given f in $L^1(\mathscr{H}_n^{\mathrm{red}})$, we can write

$$f(z, t) = \sum_{k=-\infty}^{\infty} \Psi_k(z)e^{ikt} \tag{2.22}$$

as a Fourier series in the central variable t. (Here f can be thought of as the L^1-limit of the Cesàro means of the right hand side of (2.22).) Hence, as in the case of $\mathscr{H}_n$, if we compute the group Fourier transform $\hat{f}(m)$, $m \in Z^*$ we see that it is an integral operator with kernel $K_f^m(\xi, \eta)$ given by

$$K_f^m(\xi, \eta) = \mathscr{F}_1 \Psi_{-m}\left(\frac{m(\xi+\eta)}{2}, \xi - \eta\right) \tag{2.23}$$

where $\mathscr{F}_1 \Psi_{-m}$ stands for the Fourier transform of Ψ_{-m} in the first set of variables. Therefore, for $f \in L^1 \cap L^2(\mathscr{H}_n^{\text{red}})$, a simple calculation shows that

$$\|\hat{f}(m)\|_{HS}^2 = |m|^{-n} \|\mathscr{F}_1 \Psi_{-m}\|_{L^2(\mathbb{C}^n)}^2, \quad m \in Z^*. \tag{2.24}$$

Remark 7. We will now show by an example that the exact analogue of Hardy's theorem on $\mathscr{H}_n^{\text{red}}$ is not valid. Since t varies over a compact set in this case, one might be tempted to consider the following analogue of Hardy's theorem:

Suppose f is a measurable L^1-function on $\mathscr{H}_n^{\text{red}}$ and f satisfies the following estimates:

$$|f(z,t)| \leqslant Ce^{-\alpha|z|^2}, \quad \|\hat{f}(m)\|_{HS} \leqslant Ce^{-\beta|m|^2}, \quad z \in \mathbb{C}^n, m \in Z^*, \tag{2.25}$$

for positive constants α, β. Then if $\alpha\beta > \frac{1}{4}$, $f = 0$ a.e.

However, the following demonstrates that this is not the case.

Observe that as f satisfies (2.25), f belongs to $L^1 \cap L^2(\mathscr{H}_n^{\text{red}})$ and the series in (2.22) converges to f in L^2-sense. Now take $f(z,t) = e^{-\alpha|z|^2} e^{ikt}$, for some $k \in Z^*$. Using (2.24) one can see that f is a non-trivial function satisfying the conditions (2.25).

However the following, which can be viewed as a "sort of" uncertainty principle still holds:

Suppose f is a measurable L^1-function on $\mathscr{H}_n^{\text{red}}$ satisfying

$$|f(z,t)| \leqslant \alpha(z)\beta(t), \quad z \in \mathbb{C}^n, t \in S^1 \tag{2.26}$$

$$\|\hat{f}(m)\|_{HS} \leqslant Ce^{-\gamma|m|}, \quad m \in Z^*, \tag{2.27}$$

where α is any function with reasonably rapid decay at infinity, β is any function that vanishes to infinite order at some point $t_0 \in S^1$ and γ is a positive constant. Then $f = 0$ a.e.

Remark 8. Since S^1 is compact the point t_0 can be "viewed" as the point at infinity and therefore condition (2.26) can be thought of as the analogue of the decay of the function at infinity.

3. An uncertainty principle for the Heisenberg group

In this section we formulate and prove an uncertainty principle for the Fourier transform on the Heisenberg group. In the uncertainty principle stated in theorem 2 as well as in the analogue of Hardy's theorem the Fourier transform has been considered as a function of the continuous parameter λ. The properties of the given function f as a function of the t variable are reflected in $\hat{f}(\lambda)$ as a function of λ. But if we want to investigate how the properties of f as a function of z are affecting $\hat{f}(\lambda)$ one has to view the Fourier transform as a function of two parameters, one continuous and the other discrete.

To justify the above claim let us write down the formula for $\hat{f}(\lambda)$ when f is a radial function. In what follows, by a radial function we mean a function which is radial in the z variable. In order to state the formula we need to introduce some more notation. For each multi index $\alpha\in\mathbb{N}^n$ let $\Phi_\alpha(x)$ stand for the normalized Hermite functions on $\mathbb{R}^n$. For $\lambda\in\mathbb{R}^*$ we let $\Phi_\alpha^\lambda(x)=|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$ and define $P_k(\lambda)$ to be the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda:|\alpha|=k\}$. By $\varphi_k^\lambda(r)$ we denote the scaled Laguerre function

$$\varphi_k^\lambda(r)=L_k^{n-1}(\tfrac{1}{2}|\lambda|r^2)e^{-(1/4)|\lambda|r^2}, \tag{3.1}$$

$L_k^{n-1}(t)$ being the kth Laguerre polynomial of type $(n-1)$.

Now let $f(z,t)$ be a radial function and write $f(r,t)$ in place of $f(z,t)$ when $|z|=r$. Then we have the following formula for the Fourier transform of f:

$$\hat{f}(\lambda)=\sum_{k=0}^{\infty}R_k(\lambda,f)P_k(\lambda) \tag{3.2}$$

where the coefficients $R_k(\lambda,f)$ are given by

$$R_k(\lambda,f)=C_n\frac{k!}{(k+n-1)!}\int_0^\infty \tilde{f}(r,\lambda)\varphi_k^\lambda(r)r^{2n-1}\,\mathrm{d}r. \tag{3.3}$$

In the above $\tilde{f}(r,\lambda)$ stands for the Fourier transform of $f(r,t)$ in the t-variable and C_n is a constant. From the above formula it follows that we can identify $\hat{f}(\lambda)$ with the sequence of functions $\{R_k(\lambda,f)\}$. The support properties of f as a function of t are reflected on the properties of $R_k(\lambda,f)$ as a function of λ. Likewise, one expects that the z support of f will influence the properties of $R_k(\lambda,f)$ as a function of k. We will show that this is indeed the case.

More generally we consider the Fourier transform $\hat{f}(\lambda)$ as a family of linear functionals $F(\lambda,\alpha)$ on $L^2(\mathbb{R}^n)$ indexed by $(\lambda,\alpha)\in\mathbb{R}^*\times\mathbb{N}^n$. For each (λ,α) the linear functional $F(\lambda,\alpha)$ is given by

$$F(\lambda,\alpha)\varphi=(\varphi,\hat{f}(\lambda)\Phi_\alpha^\lambda),\quad \varphi\in L^2(\mathbb{R}^n). \tag{3.4}$$

With the above notations the uncertainty principle stated in theorem 2 can be restated as follows. If $m\{t:f(z,t)\neq 0\}<\infty$ for a.e. z and $m\{\lambda:F(\lambda,\alpha)\neq 0\}<\infty$ then $f=0$. Now to state our uncertainty principle let

$$A(\lambda)=\{z:\tilde{f}(z,\lambda)\neq 0\} \tag{3.5}$$

and

$$B(\lambda)=\{\alpha:F(\lambda,\alpha)\neq 0\}. \tag{3.6}$$

Then we have the following result.

Theorem 9. *Suppose $f\in L^1\cap L^2(\mathscr{H}_n)$ is such that $m(A(\lambda))<\infty$ and $B(\lambda)$ is finite for a.e. $\lambda\in\mathbb{R}^*$. Then $f=0$.*

Before going into the proof of the theorem we make the following remarks concerning the statement of the theorem. If there exists a compact set $K \subset \mathbb{C}^n$ such that $f(z,t)=0$ whenever $z \notin K$ and $t \in \mathbb{R}$ then it follows that $A(\lambda)$ is compact for each λ and hence $m(A(\lambda))<\infty$ is satisfied. The condition $B(\lambda)$ is finite simply means that $\hat{f}(\lambda)\Phi_\alpha^\lambda \neq 0$ only for finitely many α and consequently there is a $k=k(\lambda)$ such that $\hat{f}(\lambda)P_j(\lambda)=0$ for all $j>k$. Let S_k^λ be the span of $\{\Phi_\alpha^\lambda : |\alpha|=k\}$. Then it has been observed by Geller in [7] that S_k^λ are the analogues of the spheres $|x|=r$ in $\mathbb{R}^n$. In other words we can think of S_k^λ as a sphere in $L^2(\mathbb{R}^n)$ of radius $(2k+n)|\lambda|$. This view has turned out to be fruitful in other problems also as can be seen from [18].

Thus we can let B_k^λ to be the span of $\{\Phi_\alpha^\lambda : |\alpha| \leqslant k\}$ which is the analogue of a ball in $\mathbb{R}^n$ and the condition $\hat{f}(\lambda)P_j(\lambda)=0$ for $j>k$ simply means that $\hat{f}(\lambda)=0$ in the orthogonal complement of B_k^λ in $L^2(\mathbb{R}^n)$. Let us say that $\hat{f}(\lambda)$ has compact support in B_k^λ when the above holds. With this definition we can restate the above theorem in the following form.

Theorem 10. *Let $f \in L^1 \cap L^2(\mathscr{H}_n)$. Suppose for each λ the Fourier transform $\hat{f}(\lambda)$ is compactly supported. Then $\tilde{f}(.,\lambda)$ cannot have compact support for each λ unless $f=0$.*

We now come to the proof of theorem 9. We need to use some facts about the special Hermite expansions for which we refer the reader to [19]. If $f \in L^2(\mathbb{C}^n)$ then we have the expansion

$$f=(2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k. \tag{3.7}$$

In the above $\varphi_k(z)=L_k^{n-1}(\frac{1}{2}|z|^2)e^{-(1/4)|z|^2}$ and $f \times \varphi_k$ stands for the twisted convolution

$$(f \times \varphi_k)(z)=\int_{\mathbb{C}^n} f(z-w)\varphi_k(w)e^{(i/2)\mathrm{im}(z.\bar{w})}\mathrm{d}w. \tag{3.8}$$

The functions φ_k are eigenfunctions of the operator

$$L=-\Delta+\frac{1}{4}|z|^2-i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\right) \tag{3.9}$$

with eigenvalues $(2k+n)$ and $f \to f \times \varphi_k$ is the projection of $L^2(\mathbb{C}^n)$ onto the k-th eigenspace of the operator L. We also have for any m

$$L^m(f \times \varphi_k)=(2k+n)^m f \times \varphi_k \tag{3.10}$$

and in view of the orthogonality the relation

$$\|L^m f\|_2^2=(2\pi)^{-2n}\sum_{k=0}^{\infty}(2k+n)^{2m}\|f \times \varphi_k\|_2^2. \tag{3.11}$$

We need the following proposition in order to prove theorems 9 and 10.

PROPOSITION 11

Suppose $f \in L^2(\mathbb{C}^n)$ is such that $\|f \times \varphi_k\|_2 \leqslant Ce^{-\alpha(2k+n)}$ for some $\alpha > 0$. Then f is real analytic.

Proof. By the Sobolev's embedding theorem it is easy to see that f is in $C^\infty(\mathbb{C}^n)$. We want to apply an elliptic regularity theorem of Kotakè–Narasimhan to prove the proposition (see [11], theorem 3.8.9). In view of their theorem it suffices to show that for any positive integer m

$$\|L^m f\|_2 \leqslant M^{m+1}(2m)! \tag{3.12}$$

holds with some constant M. Under the assumption on f, the relation (3.11) gives

$$\|L^m f\|_2^2 \leqslant (2\pi)^{-2n} \sum_{k=0}^{\infty} (2k+n)^{2m} e^{-2\alpha(2k+n)}. \tag{3.13}$$

The series can be estimated by

$$\int_0^\infty t^{2m} e^{-2\alpha t} \mathrm{d}t \tag{3.14}$$

which gives the estimate

$$\|L^m f\|_2^2 \leqslant C^{2m+1}(2m)! \tag{3.15}$$

which is more than what we need.

Now we can give proofs of theorems 9 and 10. Define a radial function $G_j(z,t)$ by

$$G_j(z,t) = \int_{\mathbb{R}} e^{-i\lambda t} e^{-(1/2)\lambda^2} \varphi_k^\lambda(z) |\lambda|^n \mathrm{d}\lambda. \tag{3.16}$$

:n it follows from (3.2) that

$$\hat{G}_j(\lambda) = C_n e^{-(1/2)\lambda^2} P_j(\lambda), \tag{3.17}$$

where C_n is some constant which we do not bother to calculate. Setting $g_j = f * G_j$ and taking the (group) Fourier transform we get

$$\hat{g}_j(\lambda) = \hat{f}(\lambda)\hat{G}_j(\lambda) = C_n e^{-(1/2)\lambda^2} \hat{f}(\lambda) P_j(\lambda). \tag{3.18}$$

Now fix λ. Then under the hypothesis of the theorem we have $\hat{g}_j(\lambda) = 0$ for $j > k$ which in view of (2.11) means that for a.e. z in $\mathbb{C}^n g_j^\lambda(z) = 0$ for $j > k$ where we have set $g_j^\lambda(z)$ to stand for $\tilde{g}_j(z,\lambda)$ the Fourier transform of g_j in the t-variable.

Recalling the definition of the convolution $g_j = f * G_j$ on $\mathscr{H}_n$ and taking the Fourier transform in the t-variable we get with the same notation as above

$$g_j^\lambda(z) = f^\lambda *_\lambda G_j^\lambda(z), \tag{3.19}$$

where the λ-twisted convolution is given by

$$f^\lambda *_\lambda G_j^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-w) G_j^\lambda(w) e^{i(\lambda/2)\mathrm{im}(z.\bar{w})} \mathrm{d}w. \tag{3.20}$$

Let $f^{\lambda}_{\lambda}(z) = f^{\lambda}(2^{-1}|\lambda|^{-(1/2)}z)$. Then it follows from the definition of G_j that

$$(f^{\lambda} *_{\lambda} G^{\lambda}_j)(2^{-1}|\lambda|^{-(1/2)}z) = C_n e^{-(1/2)\lambda^2}(f^{\lambda}_{\lambda} \times \varphi_j)(z). \tag{3.21}$$

Under the hypothesis of either of the theorems we have $(f^{\lambda}_{\lambda} \times \varphi_j)(z) = 0$ for $j > k$. This means that f^{λ}_{λ} satisfies the conditions of proposition 11 and consequently $\tilde{f}(z, \lambda)$ is real analytic for a.e. λ as a function of (x, y). But then the set $\{z: \tilde{f}(z, \lambda) \neq 0\}$ cannot have finite measure unless $\tilde{f}(z, \lambda) = 0$ for a.e. z. This implies $f = 0$ and hence theorem 9 follows. It is clear that the hypothesis of theorem 10 implies that of theorem 9. Hence both theorems are proved.

4. Some uncertainty inequalities for the Heisenberg group

In this section we establish a local uncertainty inequality for the Fourier transform on $\mathscr{H}_n$ and deduce a global inequality too. As we have remarked in the previoussection we consider the Fourier transform $\hat{f}(\lambda)$ as a family of linear functionals $F(\lambda, \alpha)$ indexed by $(\lambda, \alpha) \in \mathbb{R}^* \times \mathbb{N}^n$. From the definition of $F(\lambda, \alpha)$ it follows that

$$\operatorname{tr}(\hat{f}(\lambda)^* \hat{f}(\lambda)) = \sum_{\alpha} \|\hat{f}(\lambda)\Phi^{\lambda}_{\alpha}\|_2^2 = \sum_{\alpha} \|F(\lambda, \alpha)\|^2, \tag{4.1}$$

where $\|F(\lambda, \alpha)\|$ is the norm of the linear functional $F(\lambda, \alpha)$. In this notation the uncertainty inequality of theorem 3 can be written as

$$\sum_{\alpha} \int_A \|F(\lambda, \alpha)\|^2 \mathrm{d}\mu(\lambda) \leqslant C_{\theta} m(A)^{2\theta} \int_{\mathscr{H}_n} |f(z,t)|^2 |t|^{2\theta} \mathrm{d}z\,\mathrm{d}t. \tag{4.2}$$

In the next theorem we will prove an inequality which is more symmetric in both variables.

Let ν be the counting measure on $\mathbb{N}^n$ and let $\sigma = \mu \times \nu$ on $\mathbb{R}^* \times \mathbb{N}^n$. We now prove the following inequality. We let $Q = (2n + 2)$ and $|w|^4 = |z|^4 + t^2$ for $w = (z, t) \in \mathscr{H}_n$.

Theorem 12. *Given $\theta \in [0, \frac{1}{2})$, for each $f \in L^1 \cap L^2(\mathscr{H}_n)$ and $E \subset \mathbb{R}^* \times \mathbb{N}^n$ with $\sigma(E) < \infty$ one has*

$$\int_E \|F(\lambda, \alpha)\|^2 \mathrm{d}\sigma \leqslant C^2_{\theta} \sigma(E)^{2\theta} \int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\theta Q} \mathrm{d}w, \tag{4.3}$$

where C_{θ} depends only on θ and Q.

Proof. Let $r > 0$ be a positive number to be chosen later. We write $f = g + h$ where $g(w) = f(w)$ when $|w| \leqslant r$ and $g(w) = 0$ otherwise. We then have

$$\int_E \|F(\lambda, \alpha)\|^2 \mathrm{d}\sigma \leqslant 2\left\{\int_E \|\hat{g}(\lambda)\Phi^{\lambda}_{\alpha}\|_2^2 \mathrm{d}\sigma + \int_E \|\hat{h}(\lambda)\Phi^{\lambda}_{\alpha}\|_2^2 \mathrm{d}\sigma\right\}. \tag{4.4}$$

Since

$$\|\hat{g}(\lambda)\Phi^{\lambda}_{\alpha}\|_2 \leqslant \|\hat{g}(\lambda)\| \, \|\Phi^{\lambda}_{\alpha}\|_2 = \|\hat{g}(\lambda)\|, \tag{4.5}$$

where $\|\hat{g}(\lambda)\|$ is the operator norm of $\hat{g}(\lambda)$ on $L^2(\mathbb{R}^n)$ and as $\|\hat{g}(\lambda)\| \leqslant \|g\|_1$ we obtain

$$\int_E \|\hat{g}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \leqslant \sigma(E)\left(\int_{\mathscr{H}^n} |g(w)|\mathrm{d}w\right)^2 \tag{4.6}$$

$$\leqslant \sigma(E)\left(\int_{\mathscr{H}^n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right)\left(\int_{|w|\leqslant r} |w|^{-2\theta Q}\mathrm{d}w\right)$$

$$\leqslant C\sigma(E)r^{-(2\theta-1)Q}\left(\int_{\mathscr{H}^n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right)$$

where we have applied Cauchy–Schwarz to get the second inequality.

On the other hand by the Plancherel theorem

$$\int_E \|\hat{h}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \leqslant \int_{\mathbb{R}^*\times\mathbb{N}^n} \|\hat{h}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \tag{4.7}$$

$$= \int_{\mathbb{R}^*} \|\hat{h}(\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda)$$

$$= C_n\int_{\mathscr{H}^n} |h(w)|^2 \mathrm{d}w$$

$$= C_n\int_{\mathscr{H}^n} |h(w)|^2|w|^{-2\theta Q}|w|^{2\theta Q}\mathrm{d}w$$

$$\leqslant C_n r^{-2\theta Q}\left(\int_{\mathscr{H}^n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right).$$

Therefore, we have proved the inequality

$$\int_E \|F(\lambda,\alpha)\|^2 \mathrm{d}\sigma \leqslant (2C\sigma(E)r^{(1-2\theta)Q} + 2C_n r^{-2\theta Q})\left(\int_{\mathscr{H}^n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right). \tag{4.8}$$

Minimizing the right hand side by a judicious choice of r we get the inequality

$$\int_E \|F(\lambda,\alpha)\|^2 \mathrm{d}\sigma \leqslant C_\theta\sigma(E)^{2\theta}\left(\int_{\mathscr{H}^n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right). \tag{4.9}$$

This completes the proof of the theorem.

As in the case of $\mathbb{R}^n$ we can now deduce a global uncertainty inequality from the above local inequality. To state the inequality we need some more notation. Let $\mathscr{L}$ be the sublaplacian on the Heisenberg group and let $H(\lambda)$ be the Hermite operator whose spectral decomposition is given by

$$H(\lambda) = \sum_{k=0}^{\infty} (2k+n)|\lambda|P_k(\lambda). \tag{4.10}$$

For the definition of $\mathscr{L}$ we refer to [16] and we remark that when $\lambda = 1$, $H(\lambda) = -\Delta + |x|^2$ on $\mathbb{R}^n$. The relation between $\mathscr{L}$ and $H(\lambda)$ is given by

$$(\mathscr{L}f)\hat{\ }(\lambda) = \hat{f}(\lambda)H(\lambda), \tag{4.11}$$

for any reasonable function f on $\mathscr{H}_n$. We can define any fractional power $\mathscr{L}^\gamma$ by the equation

$$(\mathscr{L}^\gamma f)\hat{\ }(\lambda) = \hat{f}(\lambda)(H(\lambda))^\gamma, \tag{4.12}$$

where $(H(\lambda))^\gamma$ is given by the decomposition

$$(H(\lambda))^\gamma = \sum_{k=0}^{\infty} ((2k+n)|\lambda|)^\gamma P_k(\lambda). \tag{4.13}$$

We can now prove the following global uncertainty inequality for $\mathscr{H}_n$.

Theorem 13. *For f in $L^2(\mathscr{H}_n)$, $0 \leqslant \gamma < Q/2$ one has*

$$\|f\|_2^4 \leqslant K\left(\int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\gamma} \mathrm{d}w\right)\left(\int_{\mathscr{H}_n} |\mathscr{L}^{\gamma/2} f(w)|^2 \mathrm{d}w\right) \tag{4.14}$$

where K is a constant.

Before going into the proof of the above inequality the following remarks are in order. When $\gamma = 1$ the above inequality reduces to

$$\|f\|_2^4 \leqslant K\left(\int_{\mathscr{H}_n} |f(w)|^2 |w|^2 \mathrm{d}w\right)\left(\int_{\mathscr{H}_n} |\mathscr{L}^{1/2} f(w)|^2 \mathrm{d}w\right) \tag{4.15}$$

and this is the analogue of the classical uncertainty inequality for the Fourier transform on $\mathbb{R}^n$. The analogy can be seen clearly if we write the inequality (1.2) in the form

$$\|f\|_2^4 \leqslant K\left(\int |f(x)|^2 |x|^2 \mathrm{d}x\right)\left(\int |(-\Delta)^{1/2} f(x)|^2 \mathrm{d}x\right). \tag{4.16}$$

The inequality (4.15) is valid even if we replace $|w|$ by $|z|$ as was shown in [17] and then a precise value for K can also be obtained.

Now we prove theorem 13. As in the case of the previous theorem the proof is modelled after the proof in the Euclidean case. Let E_r denote the set

$$E_r = \{(\lambda, \alpha) : (2|\alpha| + n)|\lambda| \leqslant r^2\}. \tag{4.17}$$

We claim that $\sigma(E_r) \leqslant Cr^Q$. To see this we first note that

$$E_r = \bigcup_{k=0}^{\infty} \bigcup_{|\alpha|=k} \{\lambda : (2|\alpha| + n)|\lambda| \leqslant r^2\} \times \{\alpha\} \tag{4.18}$$

and therefore

$$\sigma(E_r) \leqslant \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \mu\{\lambda : (2k+n)|\lambda| \leqslant r^2\}. \tag{4.19}$$

Since $\mu\{\lambda:(2k+n)|\lambda|\leqslant r^2\}\leqslant Cr^Q(2k+n)^{-n-1}$ and $\Sigma_{|\alpha|=k}1\leqslant C(2k+n)^{n-1}$ we get

$$\sigma(E_r)\leqslant Cr^Q\sum_{k=0}^{\infty}(2k+n)^{-2}\leqslant Cr^Q \tag{4.20}$$

and this proves the claim.

Let E_r' stand for the complement of E_r and write

$$\begin{aligned}\|f\|_2^2&=C_n\int_{\mathbb{R}}\|\hat{f}(\lambda)\|_{HS}^2\,\mathrm{d}\mu(\lambda)\\&=C_n\int\|F(\lambda,\alpha)\|^2\,\mathrm{d}\sigma\\&=C_n\left(\int_{E_r}\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma+\int_{E_r'}\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma\right).\end{aligned} \tag{4.21}$$

Applying the local uncertainty inequality to the first integral with $\theta=\gamma/Q<\frac{1}{2}$ and making use of the claim we obtain

$$\int_{E_r}\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma\leqslant Cr^{2\gamma}\int_{\mathscr{H}n}|f(w)|^2|w|^{2\gamma}\,\mathrm{d}w. \tag{4.22}$$

For the second integral one has the following chain of inequalities:

$$\begin{aligned}\int_{E_r'}\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma&\leqslant r^{-2\gamma}\int_{E_r'}((2|\alpha|+n)|\lambda|)^\gamma\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma\\&=r^{-2\gamma}\int_{E_r'}\|\hat{f}(\lambda)(H(\lambda))^{\gamma/2}\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma\\&\leqslant r^{-2\gamma}\int\|(\mathscr{L}^{\gamma/2}f)^\wedge(\lambda)\Phi_\alpha^\lambda\|_2^2\,\mathrm{d}\sigma\\&=Cr^{-2\gamma}\int_{\mathscr{H}n}|\mathscr{L}^{\gamma/2}f(w)|^2\,\mathrm{d}w.\end{aligned} \tag{4.23}$$

Thus we have obtained the inequality

$$\|f\|_2^2\leqslant C\left\{r^{2\gamma}\int|f(w)|^2|w|^{2\gamma}\,\mathrm{d}w+r^{-2\gamma}\int|\mathscr{L}^{\gamma/2}f(w)|^2\,\mathrm{d}w\right\}. \tag{4.24}$$

Minimizing the right hand side we obtain

$$\|f\|_2^4\leqslant K\left(\int|f|^2|w|^{2\gamma}\,\mathrm{d}w\right)\left(\int|\mathscr{L}^{\gamma/2}f(w)|^2\,\mathrm{d}w\right). \tag{4.25}$$

which proves the theorem.

5. The Euclidean motion group

In this section we shall state and prove an analogue of Hardy's theorem for the Euclidean motion group, $M(2)$. The group $G=M(2)$ is the semidirect product of

$SO(2)(\simeq S^1)$ and $\mathbb{R}^2(\simeq \mathbb{C})$. A typical element of G is denoted by (z,α) and this element acts on $\mathbb{R}^2$ as $t(z)r(\alpha)$ where $t(z)$ is the translation by $z\in\mathbb{C}(\simeq\mathbb{R}^2)$ and $r(\alpha)$ is the rotation by an angle α, $0\leqslant\alpha\leqslant 2\pi$. The multiplication law is given by the composition of such maps. Haar measure on G is $\mathrm{d}z\,\mathrm{d}\alpha$ where $\mathrm{d}z$ is Lebesgue measure on $\mathbb{C}(\simeq\mathbb{R}^2)$ and $\mathrm{d}\alpha$ is the normalized Haar measure on $SO(2)(\simeq S^1)$. For any unexplained terminology and notation in this section see [15].

For $a\in\mathbb{R}^+=(0,\infty)$, we have the unitary irreducible representation U^a of G as operators in $\mathcal{U}(L^2(S^1))$ defined by

$$(U^a(z,\alpha)\phi)(\theta)=e^{i\langle z,r(\theta)a\rangle}\phi(\theta-\alpha), \tag{5.1}$$

where $\phi\in L^2(S^1)$, $0\leqslant\theta\leqslant 2\pi$ and $\langle .,.\rangle$ is the inner product on $\mathbb{R}^2$. Here one is identifying $a\in\mathbb{R}^+$ with $(0,a)\in\mathbb{C}$. The Plancherel measure μ on $\hat{G}$ is supported on this family of representations parametrized by $\mathbb{R}^+$, and is given by $a\,\mathrm{d}a$, where $\mathrm{d}a$ is Lebesgue measure on $\mathbb{R}^+$.

The Fourier transform $\hat{f}$ of $f\in L^1(G)$ is a function on $\mathbb{R}^+$ taking values in $\mathfrak{B}(L^2(S^1))$, and is defined by

$$\hat{f}(a)=U^a(f)=\int_{M(2)}U^a(z,\alpha)f(z,\alpha)\mathrm{d}z\,\mathrm{d}\alpha \tag{5.2}$$

(the integral interpreted suitably) and therefore we have

$$(\hat{f}(a)\phi)(\theta)=\int_{\mathbb{C}}\int_{SO(2)}f(z,\alpha)e^{i\langle z,r(\theta)a\rangle}\phi(\theta-\alpha)\mathrm{d}z\,\mathrm{d}\alpha \tag{5.3}$$

for $\phi\in L^2(S^1)$ and $\theta\in[0,2\pi)$.

The following is an analogue of Hardy's theorem for the Euclidean motion group$M(2)$:

Theorem 14. *Suppose f is a measurable function on G satisfying the following conditions for some positive constants α, β and C:*

$$|f(z,\theta)|\leqslant Ce^{-\alpha|z|^2},\quad (z,\theta)\in G, \tag{5.4}$$

$$\|\hat{f}(a)\|_{HS}\leqslant Ce^{-\beta|a|^2},\quad a\in\mathbb{R}^+. \tag{5.5}$$

If $\alpha\beta>\frac{1}{4}$, then $f=0$ a.e.

Remark 15. Since functions on $\mathbb{R}^2$ can be thought of as functions on G invariant under right action by $SO(2)$, Hardy's theorem for $\mathbb{R}^2$ shows that $\frac{1}{4}$ is the best possible constant.

Proof. For $n\in Z$, define χ_n on $SO(2)$ as $\chi_n(\theta)=e^{in\theta}$. It is enough to show that if $\alpha\beta>\frac{1}{4}$, $\chi_n*f*\chi_m=0$ for all n,m. This is because if f is a L^1-function (or more generally a distribution) and $\chi_n*f*\chi_m$ is zero for all n, $m\in Z$, then f is itself zero. A simple calculation shows that if f satisfies (5.4) and (5.5) then for all n, m, $\chi_n*f*\chi_m$ also satisfy (5.4) and (5.5). For $n,m\in Z$, define

$$L^1_{n,m}(G)=\{g\in L^1(G){:}g(r(\theta)xr(\gamma))=\chi_n(\theta)g(x)\chi_m.(\gamma)$$
$$\text{a.e. } x\in G,\quad \text{a.e. } r(\theta),r(\gamma)\in SO(2)\}.$$

Observe that if $h = \chi_n * f * \chi_m$ then h belongs to $L^1_{n,m}(G)$. Therefore it is enough to prove the theorem for a function h in $L^1_{n,m}(G)$. It is easy to check that if $h \in L^1_{n,m}(G)$ then $\hat{h}(a)$ maps $\chi_m \in L^2(S^1)$ to a multiple of χ_n and is zero on the orthogonal complement of χ_m. In fact,

$$\hat{h}(a)\chi_m = \langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)} \chi_n,$$
$$\hat{h}(a)\chi_l = 0 \quad \text{for } l \neq m.$$

Therefore

$$\| \hat{h}(a) \|_{HS} = | \langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)} |.$$

Using the transformation property of h, it can be shown that

$$| \langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)} | = | \mathscr{F}_1 h(r(\theta)a, \gamma) | \tag{5.6}$$

for a.e. θ and γ in $[0, 2\pi)$ where $\mathscr{F}_1 h$ denotes the Euclidean Fourier transform of h in the $\mathbb{C}(\simeq \mathbb{R}^2)$-variable z. Thus from (5.5) and (5.6) it will follow that:

$$| \mathscr{F}_1 h(\xi, \gamma) | \leqslant C e^{-\beta|\xi|^2} \tag{5.7}$$

for $\xi \in \mathbb{C}(\simeq \mathbb{R}^2)$ and a.e. γ in $[0, 2\pi)$. But h also satisfies (5.4). Using the analogue of Hardy's theorem for $\mathbb{R}^2(\simeq \mathbb{C})$ we conclude that $h(., \gamma) = 0$ for a.e. γ in $[0, 2\pi)$. This implies that $h = 0$ a.e.

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