Econometrica, Vol. 67, No. 3 (May, 1999), 605–628

SEPARABLE PREFERENCES, STRATEGYPROOFNESS, AND DECOMPOSABILITY

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We consider strategyproof social choice functions defined over product domains. If preferences are strict orderings and separable, then strategyproof social choice functions must be decomposable provided that the domain of preferences is rich. We provide several characterization results in the case where preferences are separable only with respect to the elements of some partition of the set of components and these partitions vary across individuals. We characterize the libertarian social choice function and show that no superset of the tops separable domain admits strategyproof nondictatorial social choice functions.

KEYWORDS: Strategyproof, social choice, product domains, separable preferences, decomposable.

1. INTRODUCTION

A *SOCIAL CHOICE FUNCTION* OR SCF is a mapping which associates a social alternative with every profile of individual preferences defined over a set of social alternatives. The value of a SCF at any profile is to be viewed as the "optimal" or "most desirable" outcome in that state of the world. A SCF is said to be *strategyproof* if no individual can ever profit by misrepresenting his true preferences. An issue of fundamental importance in incentive theory is the characterization of strategyproof SCFs.

The classic result of Gibbard (1973) and Satterthwaite (1975) asserts that if the domain of preferences is unrestricted, then the only strategyproof SCFs are the dictatorial ones. The principle of dictatorship is obviously an unsatisfactory method for the resolution of conflicts of interest; the Gibbard-Satterthwaite Theorem can therefore be interpreted as an expression of the enormous cost imposed on social decision problems by incentive considerations. A natural way to avoid this powerful negative result is to put more structure on the data of the problem, in particular on the set of social alternatives and the set of individual preferences. This direction has been pursued in several papers. For example, Groves and Clarke in their classic papers (Groves (1973), Clarke (1971)) consider allocation problems involving one private good and one public good with preferences assumed to be quasi-linear. Moulin (1980) considers an environment with one public good and single peaked preferences. In all these contexts,

[1]S. Subramanian played an important role in the initiation of this project. It is a pleasure for us to acknowledge his contribution. We also wish to thank Rabindran Abraham, Salvador Barberá, Bhaskar Dutta, three anonymous referees of the journal, and an editor, as well as seminar participants in Barcelona, Bellcore, Brown, Caen, Istanbul, LSE, Marseille, Oxford, Princeton, Rochester, Toulouse, and Yale for their comments.

an important result is that it is possible to avoid dictatorship without losing strategyproofness. Even though there is a price that has to be paid to get incentives "right," it is not maximal in terms of the allocation of decisive power.

In this paper, we pursue this line of research further. We assume that a social outcome comprises several components, say M, with each component representing an aspect of the outcome. An important assumption is that there is no interdependence between the choices over different components, i.e. the set of social alternatives is a product set. Two well-known models are covered by this assumption. The first is the "characteristics" or location model. Thus, in a political theory context, a political platform is a description of positions taken on a variety of separate issues. In a consumer theory setting, commodities are viewed as bundles of attributes and can be represented as points in a general attributes space. The second model that the product set assumption encompasses is the model of externalities first formulated in Sen (1970) in the context of the debate on liberalism (see Wriglesworth (1985)). Here, the number of components is equal to the number of individuals with the ith component being a description of individual i's personal issue.

Given this structure over the set of alternatives, we shall assume that individual preferences are "separable" over subsets of components in an appropriate sense. Each individual is assumed to have a partition of the set of components. By "separability" we mean that it is possible to unambiguously define preferences over components that together comprise an element of the partition. We refer to such preferences as preferences separable with respect to the given partition and to the preferences induced on subsets of components as marginal preferences. Examples of such preferences are those generated by additively separable utility functions. The structure of the partition is presumably determined by the degree of substitutability and complementarity between components. If social outcomes consist of three components, the levels of public expenditure on nuclear defense, conventional defense, and education, it would be appropriate to group the first two together as a single element of the partition. Separability would then mean that preferences over defense spending are independent of the level of spending on education and vice-versa. We note that the assumption of such preferences is common in applied consumption analysis. The reader is referred to Deaton and Muellbauer (1980, Chapter 5) for a more complete discussion of this issue.

In this framework, we establish three different types of results. First, we prove a very general decomposability result in the case where all individuals have preferences that are separable with respect to a common partition of the set of components.[2] We introduce the key notion of a rich domain within the set of separable orderings. We show that a SCF defined over this domain is strate-

[2] For notational simplicity, we shall present the result in the case where individual preferences are separable over all components.

gyproof if and only if it can be decomposed into "marginal" SCFs defined over appropriate marginal preference profiles. Richness of the domain is therefore a sufficient condition for the decomposability of a strategyproof SCF. An attractive feature of the definition of richness is that it is extremely general; in particular it assumes virtually nothing about marginal orderings. A special case of a rich domain is the domain consisting of all separable orderings, in which case the marginal domains are all unrestricted. Using characterization results for the marginal SCFs we derive several general characterization results, including the main results of Barbera, Sonnenschein, and Zhou (1991) (henceforth BSZ) and Barbera, Gul, and Stachetti (1993) (henceforth BGS). We also indicate how many new results can be obtained along similar lines.

We next consider the case where individuals have different partitions of the set of components. The domain of a particular individual's preference orderings is the set of all orderings separable with respect to his partition. We prove that a SCF is strategyproof over this domain if and only if it is a SCF where the "right to choose" every component is assigned to some individual. Furthermore, this rights assignment must be consistent with the vector of individual partitions in a well-defined sense. An extreme case occurs when an individual is assigned the right to all components. This is dictatorship and is, of course, strategyproof. However, depending on the partitions, there are other nondictatorial strategyproof SCFs. In fact, we use the general result to obtain a characterization of the libertarian SCF where each individual is assigned the right to his personal issue. We show however, that if SCFs are required to be efficient as well as strategyproof, then we are left with nothing other than dictatorial SCFs.

Our third set of results is concerned with the largest preference domain that admits nondictatorial strategyproof SCFs. We show that the characterization on separable domains can be extended to a larger domain that we call the tops separable domain. A tops separable preference ordering with respect to a given partition allows the identification of a maximal element of the subset of components that comprise an element of the partition. However, we prove that if the domain is extended beyond the set of separable orderings, then strategyproofness implies dictatorship. This result suggests that a minimal degree of separability, in particular the property of being able to unambiguously identify a maximal element for some subset of components, is a necessary condition for the existence of nontrivial strategyproof SCFs.

The results described so far depend critically on the assumption that admissible preference orderings are antisymmetric. In a companion paper, Le Breton and Sen (1995), we explore the consequences of relaxing this assumption.

The paper is organized as follows. Sections 2 and 3 introduce the notation and the central idea of a rich domain respectively. Sections 4 and 5 are concerned with the main decomposability result and various applications. Section 6 presents characterization results when individual partitions differ. Section 7 considers the issue of the maximality of the tops separable domain. All the proofs are gathered in Section 8.

2. PRELIMINARIES

The set $I = \{1, \dots, N\}$ is the set of individuals. The set of social states (outcomes) is the set $A = A_1 \times \cdots \times A_M$. We assume that A is finite.[3] Elements of A will be denoted by $a, b, c, d, x \dots$, where it is understood that a is the M-tuple $(a_1, \dots, a_M)$ with $a_j \in A_j$ for all $j = 1, \dots, M$. For any $k \subset \{1, \dots, M\}$, $A_{M-k} = \prod_{j \notin k} A_j$ and elements of A_{M-k} will be denoted by $a_{M-k}, b_{M-k}, \dots$ etc.

Let $P(A)$ denote the set of all strict orderings (no indifference) of the elements of A. Each individual is assumed to have a preference ordering over A represented by an element of $P(A)$. However, all elements of $P(A)$ need not be admissible. For all $i \in I$, we denote by $\mathscr{D}^i (\mathscr{D}^i \subseteq P(A))$ the set of admissible preference orderings of individual i.

An N-tuple of preference orderings $P \equiv (P^1, P^2, \dots, P^N)$ where $P^i \in \mathscr{D}^i$ for all $i \in I$ will be referred to as an (admissible) preference profile.

A *Social Choice Function* (SCF) f is a mapping $f \colon \prod_{i \in I} \mathscr{D}^i \to A$.

A SCF f is *nonimposed* if for all $a \in A$, there exists a preference profile $P \in \prod_{i \in I} \mathscr{D}^i$ such that $f(P) = a$.

Throughout the paper, attention is restricted to nonimposed SCFs.

For any preference profile P and any preference ordering $\bar{P}^i \in \mathscr{D}^i$, $P|\bar{P}^i$ denotes the preference profile $(P^1, \dots, P^{i-1}, \bar{P}^i, P^{i+1}, \dots, P^N)$.

The SCF f is *manipulable* at $P \in \prod_{i \in I} \mathscr{D}^i$ if there exists $i \in I$ and $\bar{P}^i \in \mathscr{D}^i$ such that $f(P|\bar{P}^i) P^i f(P)$. It is *strategyproof* if it is not manipulable at any profile.

Let $P^i \in \mathscr{D}^i$. Then $\tau(P^i, A)$ is the maximal element in A according to P^i. Clearly, such an element exists and is unique.

The SCF f is *dictatorial* if there exists an individual $l \in I$ such that for all $P \in \prod_{i \in I} \mathscr{D}^i$, $f(P) = \tau(P^l, A)$.

The class of strategyproof SCFs clearly depends on the set of admissible profiles. The classic result of Gibbard (1973) and Satterthwaite (1975) provides a full characterization of this class where the domain is unrestricted.

THEOREM 2.1 (Gibbard (1973), Satterthwaite (1975)): *Assume* $|A| \geq 3$. *Then a nonimposed SCF* $f \colon [P(A)]^N \to A$ *is strategyproof if and only if it is dictatorial.*

We now make precise the domain restrictions considered in this paper.

DEFINITION 2.1: The ordering P^i is *separable* if for all $j \in \{1, \dots, M\}$, for all $a_j, b_j \in A_j$ and for all $x_{M-j}, y_{M-j} \in A_{M-j}$, $(a_j, x_{M-j}) P^i (b_j, x_{M-j})$ implies $(a_j, y_{M-j}) P^i (b_j, y_{M-j})$.

[3] This is a simplifying assumption in the sense that the results of this section remain valid if it is assumed that a maximal element of A exists for all preference orderings under consideration. However, the requirement that orderings over A are strict sits uncomfortably with the notion that A is nonfinite. We therefore directly assume the finiteness of A.

DEFINITION 2.2: The ordering P^i is *additively representable* if for all $j = 1, \dots, M$ there exist functions W_j^i: $A_j \to \mathbb{R}$ such that for all $a, b \in A(a \neq b)$, aP^ib if and only if $\sum_{j=1}^{M} W_j^i(a_j) > \sum_{j=1}^{M} W_j^i(b_j)$.

We denote by $\mathscr{D}$ and $\mathscr{D}^A$ the set of separable and additively representable preference orderings respectively.

REMARK 2.1: It is clear that $\mathscr{D}^A \subset \mathscr{D}$. A separable ordering is not necessarily additively representable (see Fishburn (1970) for a counterexample).

REMARK 2.2: Every $P^i \in \mathscr{D}$ induces an ordering P_j^i over A_j in a natural way: for all $a_j, b_j \in A_j, a_j P_j^i b_j$ if $(a_j, x_{M-j}) P^i (b_j, x_{M-j})$ for all x_{M-j}. The ordering P_j^i will be referred to as the *marginal ordering* over A_j induced by P^i.

Given a marginal ordering P_j^i over A_j, we denote by $\tau(P_j^i, A_j)$ the maximal element in A_j according to P_j^i.

We note that every separable ordering induces a unique marginal ordering over every component. However, the converse is not true as the following example shows.

EXAMPLE 2.1: $M = 2$, $A_1 = \{a_1, b_1\}$, $A_2 = \{a_2, b_2\}$. Consider the pair of marginal orderings $a_1 P_1^i b_1$ and $a_2 P_2^i b_2$. Any separable ordering which induces these marginal orderings must rank $(a_1 a_2)$ first and $(b_1 b_2)$ last. However, no restrictions are imposed on the relative ranking of $(a_1 b_2)$ and $(b_1 a_2)$. Thus, both P^i and $\bar{P}^i$ are consistent with P_1^i and P_2^i where $(a_1 a_2) P^i (a_1 b_2) P^i (b_1 a_2) P^i (b_1 b_2)$ and $(a_1 a_2) \bar{P}^i (b_1 a_2) \bar{P}^i (a_1, b_2) \bar{P}^i (b_1 b_2)$.

3. RICH DOMAINS

In this subsection, we introduce and discuss rich domains, a concept that is central to our paper. We say that a domain is rich if it satisfies two properties called A and B which we describe below.

Let $\mathscr{D}^i \subset \mathscr{D}$. Let $\mathscr{D}_j^i = \{P_j^i | \exists \bar{P}^i \in \mathscr{D}^i$ such that $\bar{P}_j^i = P_j^i\}$, i.e., it is the set of marginal orderings over component j induced by the orderings in $\mathscr{D}^i$.

Property A is a weak restriction on admissible marginal orderings. It requires that every element in every marginal component set be maximal according to some admissible marginal ordering.

DEFINITION 3.1: The domain $\mathscr{D}^i \subset \mathscr{D}$ satisfies *Property A*, if (i) for all $a_j \in A_j$, there exists $P_j^i \in \mathscr{D}_j^i$ such that $\tau(P_j^i, A_j) = a_j$.

Property B is the heart of the richness condition. It says the following. Consider an arbitrary collection of admissible marginal orderings, one for each component. Pick a component, say j. Then there exists an admissible ordering (of the elements of A), such that the induced marginal orderings over every component coincides with the ones in the arbitrary collection. Moreover, component j lexicographically dominates all other components in this preference ordering. In other words, the preference relation between any two elements in the set A that differ in their jth components, is determined completely by the ranking of these distinct elements in the jth component set, by the induced marginal ordering on the jth component. This is the content of part (i) of Property B. Part (ii) of the Property requires the existence of another ordering whose induced component orderings agree with the ones in the arbitrary collection we started with, but with the property that component j is lexicographically dominated by all other components. Thus, in the ranking of two elements in the set A, the values of component j "matter" only in the event that the values of all other components are the same.

DEFINITION 3.2: The domain $\mathscr{D}^i \subset \mathscr{D}$ satisfies *Property B* if, for all $(P_1^i, \ldots, P_M^i) \in \prod_{j=1}^{M} \mathscr{D}_j^i$, for all $j = 1, \ldots, M$, there exists $\bar{P}^i, \hat{P}^i \in \mathscr{D}^i$ with $\bar{P}_k^i = \hat{P}_k^i = P_k^i$ for all $k = 1, \ldots, M$, such that:

(i) For all $x, y \in A$, $[x_j \neq y_j] \rightarrow [x\bar{P}^i y \leftrightarrow x_j \bar{P}_j^i y_j]$.

(ii) For all $x, y \in A$, $x \neq y$, $[x\hat{P}^i y$ and $x_j \hat{P}_j y_j] \rightarrow$ [either $x_k \hat{P}_k^i y_k$ with $x_k \neq y_k$ for some $k \neq j$ or $x_{M-j} = y_{M-j}$].

DEFINITION 3.3: A domain $\mathscr{D}^i \subset \mathscr{D}$ is *rich* if it satisfies Properties A and B.

REMARK 3.1: Richness imposes no restrictions on admissible marginal domains other than the extremely weak Property A. Richness requires instead that available marginal orderings be combined in specific lexicographic ways. We shall see in the next subsection how this lack of association between richness and the marginal domains increases the applicability of our main result.

We now give some examples of rich domains.

EXAMPLE 3.1: Let $\mathscr{D}_j^i$, $j = 1, \ldots, M$, be an arbitrary set of marginal orderings satisfying Property A. Let $\mathscr{D}^{i,W} = \{P \in \mathscr{D} | P_j^i \in \mathscr{D}_j^i,\ j = 1, \ldots, M\}$. In other words, $\mathscr{D}^{i,W}$ is the largest set of conditional orderings that induces marginal orderings in the given sets $\mathscr{D}_j^i$. We shall refer to the set $\mathscr{D}^{i,W}$ as the *closure* of the sets of marginal orderings $\mathscr{D}_j^i$, $j = 1, \ldots, M$. It is perhaps the most natural example of a rich domain and it highlights a feature of rich domains that we have noted before, viz. rich domains can be constructed from arbitrary marginal orderings.

EXAMPLE 3.2: The domains $\mathscr{D}$ and $\mathscr{D}^A$ are rich. In each case the associated set of marginal orderings over component j consists of all orderings over A_j.

EXAMPLE 3.3: Once again, let $\mathscr{D}_j^i$, $j = 1, \ldots, M$ be an arbitrary set of marginal orderings satisfying Property A. Let $\mathscr{D}^{i,L}$ denote the set of all lexicographic orderings, the set of whose induced marginal orderings over component j is $\mathscr{D}_j^i$,

$j = 1, \ldots, M$. In other words, let $P_j^i \in \mathscr{D}_j^i$ for all j and pick a strict ordering of the set $\{1, \ldots, M\}$, say $1 > 2 > \cdots > M-1 > M$. There exists $\bar{P}^i \in \mathscr{D}^{i,L}$ such that $\bar{P}_j^i = P_j^i$, $j = 1, \ldots, M$, and component k dominates component $k+1$ lexicographically, $k = 1, \ldots, M-1$. Thus, $a\bar{P}^i b$ $(a \neq b)$ if and only if $a_j \bar{P}_j^i b_j$ where j is the lowest index such that $a_j \neq b_j$. It follows immediately that $\mathscr{D}^{i,L}$ is rich. Moreover, $\mathscr{D}^{i,L}$ is a subset of the closure of the marginal orderings $\mathscr{D}_j^i$, $j = 1, \ldots, M$. In the special case where $M = 2$, $\mathscr{D}^{i,L}$ is the set of all separable orderings that induce $\mathscr{D}_j^i$, $j = 1, 2$.

EXAMPLE 3.4: Let A_j be strictly ordered by $>_j$, $j = 1, \ldots, M$. For all $a_j, b_j \in A_j$, let $B_j(a_j, b_j) = \{c_j \in A_j | b_j >_j c_j >_j a_j\}$. The ordering P_j^i is *single-peaked* (on component j) if for all $a_j \in A_j$, $b_j \in B_j(\tau(P_j^i, A_j), a_j) \rightarrow b_j P_j^i a_j$. Let $\mathscr{D}_j^{SP}$, $j = 1, \ldots, M$, be the set of all single peaked orderings over A_j and let $\mathscr{D}^{SP}$ denote the closure of these marginal orderings. Clearly, $\mathscr{D}^{SP}$ is rich.

We note an important feature of the domain $\mathscr{D}^{SP}$ that is relevant to subsequent discussion.

DEFINITION 3.4 (BGS): The ordering P^i is *multi-dimensional single-peaked*, if for all $a, b \in A$, $[b_j \in B_j(\tau(P_j^i, A_j), a_j)$ for all $j = 1, \ldots, M] \rightarrow bP^i a$.

Let $\tilde{\mathscr{D}}^{SP}$ denote the set of all multi-dimensional single-peaked orderings. We claim that $\mathscr{D}^{SP} \subset \tilde{\mathscr{D}}^{SP}$ and the inclusion is strict. Let $P^i \in \mathscr{D}^{SP}$. Since P^i is separable, it follows immediately that P^i is multi-dimensional single-peaked. However, the converse is not true. To make matters simple, assume $M = 2$. Let P^i be an ordering such that the ordering induced on component j depends on the value of the other component, i. Assume, however that all these orderings are single-peaked and have a common peak. It is easy to check that $P^i \in \tilde{\mathscr{D}}^{SP}$, but $P \notin D^{SP}$ since it is not separable.

4. DECOMPOSABILITY RESULTS

Our objective in this section is to establish a fundamental connection between strategyproofness and decomposability, a property we introduce below.

DEFINITION 4.1: For all $i \in I$, let $\mathscr{D}^i \subset \mathscr{D}$. The SCF f: $\prod_{i \in I} \mathscr{D}^i \rightarrow A$ is *decomposable* if, for all $j = 1, \ldots, M$, there exists a strategyproof SCF f_j: $\prod_{i \in I} \mathscr{D}_j^i \rightarrow A_j$ with the following property: for all $P \in \prod_{i \in I} \mathscr{D}^i$ and for all $a \in A$, $f(P) = a$ implies that $a_j = f_j(P_j)$.

If f is decomposable then the SCF f_j: $\prod_{i \in I} \mathscr{D}_j^i \rightarrow A_j$ specified in the definition of decomposability will be referred to as a *marginal* SCF.

Let P be an admissible preference profile such that each P^i is separable. We know that P induces marginal preference profiles P_j for all $j \in \{1, \ldots, M\}$. If f

is decomposable, then the jth component of $f(P)$ is the value of a strategyproof SCF f_j at the marginal preference profile P_j. Thus each component of $f(P)$ is determined by a strategyproof marginal SCF.

THEOREM 4.1: *Let $\mathscr{D}^i \subset \mathscr{D}$ be rich for all $i \in I$. A SCF defined over the domain $\prod_{i \in I} \mathscr{D}^i$ is strategyproof if and only if it is decomposable.*

The Theorem says that strategyproofness over a rich domain of separable preferences is sufficient to "separate" the choice over different components.

REMARK 4.1: Papers such as (BSZ) and (BSG) establish separability results for particular domains where strategyproofness also implies a "tops only" property. In the domains that they consider, if a SCF is strategyproof, then it depends only on the maximal element of each individual's preference ordering. Theorem 4.1 and our formulation of the richness condition (in particular, the fact that marginal domains are virtually unrestricted) makes it clear that the decomposability issue and the "tops only" issue are logically independent. We can always find domains where strategyproof SCFs are decomposable but the marginal SCFs (and hence the SCF itself) are (is) not "tops only."

REMARK 4.2: The notion of rich domain that we have is somewhat stronger than what we require for Theorem 4.1. Our proof of the result uses only the property described in Lemma 1 together with part (i) of Property B. It is possible to combine these two requirements and obtain a strictly weaker notion of richness that also yields decomposability. We believe, however, that this notion is less transparent than the one we have; in our presentation, we choose therefore to sacrifice a little generality in the interests of clarity.

REMARK 4.3: We have assumed at the outset in this paper that preferences are separable with respect to every component. We could be more general in this regard and assume only that preferences are separable with respect to some partition of the set $\{1, \dots, M\}$. Thus, marginal preference orderings can only be defined for various elements of the partition rather than for each component separately. For example, components i and j may be "close substitutes" and it may only be possible to define marginal orderings only over i and j lumped together. Redefining richness appropriately, Theorem 4.1 can then be restated in terms of decomposability across elements of the partition.

REMARK 4.4: One may wish to impose ethical requirements on the SCF over and above strategyproofness. Suppose we require, for example, the SCF to be anonymous and strategyproof. We can show quite easily (using Theorem 4.1) that the SCF must be decomposable and that each marginal SCF must be anonymous. Similarly, if we introduce a neutrality over components requirement, then the marginal SCFs must all be identical.

5. APPLICATIONS

In this subsection, we provide several applications of Theorem 4.1. In each case we impose some structure on admissible marginal domains and invoke decomposability.

Consider the case where $\mathscr{D}^i = \mathscr{D}$ for all $i \in I$ and $|A_j| \geq 3$, $j = 1, \ldots, M$. We know from Theorem 4.1 that the jth component of f is determined by a strategyproof SCF f_j: $\prod_{i \in I} \mathscr{D}_j^i \to A_j$. The domain $\mathscr{D}_j^i$ consists of all strict orderings over A_j. Since $|A_j| \geq 3$, the Gibbard-Satterthwaite Theorem applies and we have a dictator for every component. We therefore have the following result.

THEOREM 5.1: *Assume that* $|A_j| \geq 3$ *for all* $j = 1, \ldots, M$ *and that* $\mathscr{D}^i = \mathscr{D}$ *for all* $i \in I$. *The SCF* f: $[\mathscr{D}]^N \to A$ *is strategyproof if and only if for all* $j \in \{1, \ldots, M\}$ *there exists* $\sigma(j) \in I$ *such that* $f(P) = (\tau(P_1^{\sigma(1)}, A_1), \ldots, \tau(P_M^{\sigma(M)}, A_M))$ *for all* $P \in [\mathscr{D}]^N$.

Now consider the case where $A_j = \{a_j, b_j\}$ for all $j = 1, \ldots, M$ and $\mathscr{D}^i = \mathscr{D}$ for all $i \in I$ for all $i \in I$. Applying Theorem 4.1 and a well-known result in social choice theory, we deduce that each marginal SCF must be a voting by committee rule.

DEFINITION 5.1: A *committee* is a pair $C = (I, W)$ where W is a nonempty collection of nonempty coalitions of I satisfying the following property: $U \in W$ and $U \subseteq U'$ implies $U' \in W$.

The following result is the main result of (BSZ).

THEOREM 5.2 (BSZ): *Assume that* $A_j = \{a_j, b_j\}$ *for all* $j = 1, \ldots, M$ *and that* $\mathscr{D}^i = \mathscr{D}$ *for all* $i \in I$. *The SCf* f: $[\mathscr{D}]^N \to A$ *is strategyproof if and only if for all* $j \in \{1, \ldots, M\}$ *there exists a committee* $C_j = (I, W_j)$ *such that for all* $P \in [\mathscr{D}]^N$ *and for all* $j \in \{1, \ldots, M\}$, $f_j(P) = a_j$ *if and only if* $\{i \in I | a_j P_j^i b_j\} \in W_j$.

REMARK 5.1: (BSZ) prove their result in a different though equivalent setting. In their model, there is a set of voters $I = \{1, \ldots, N\}$ and a set of issues $K = \{1, \ldots, k\}$. Each individual has separable preferences ($>_i$, $>_j$, etc.) over all subsets of K including ϕ. Thus $>$ is separable if for all $B \subseteq K$ and $x \notin B$, $B \cup \{x\} > B$ if and only if $\{x\} > \phi$. A voting scheme associates a subset of K (possibly ϕ) with every profile of separable preferences. The set of maximal elements according to $>$ is denoted by $B(>)$. A voting scheme f is voting by committees if for each $x \in K$, there exists a committee $C_x = (I, W_x)$ such that for all profiles $(>_1, \ldots, >_N)$, $x \in f(>_1, \ldots, >_N)$ if and only if $\{i \in I | x \in B(>_i)\} \in W_x$. The main result is then the following: A nonimposed voting scheme is strategyproof if and only if it is voting by committees.

In order to translate this into our model, let $A_j = \{0, 1\}$ for all issues $j \in K$. The set $A = A_1 \times \cdots \times A_k$ contains a description of all the subsets of K. For example, $G \subseteq K$ can be represented by $a \in A$ as follows: if issue $j \in G$, then $a_j = 1$; otherwise $a_j = 0$. A separable ordering $>$ induces a separable ordering over A. The marginal preference ordering over component j induced by $>$ is given by the following relationship: "1" P_j "0" if and only if $\{j\} > \phi$. Observe also that $j \in B(>)$ if and only if $\{j\} > \phi$. The equivalence of Theorem 5.2 and the (BSZ) result is now transparent.

We turn now to a consideration of the domains described in Example 3.1. We have noted that the domain $\mathscr{D}^{SP}$ is rich. It follows from Theorem 4.1 that in order to characterize strategyproof SCFs over the domain $[\mathscr{D}^{SP}]^N$, we need only to characterize SCF over the single component domain $[\mathscr{D}_j^{SP}]^N$. Following the work of Moulin (1980), BGS provide a complete solution to this problem. They show that a strategyproof SCF over this domain must be a *generalized median voter scheme* (GMVS). The reader is referred to their paper for a precise definition of this concept. An important feature of the SCF is that it is "tops only," i.e. its value depends only on the profile of individual peaks.

THEOREM 5.3: *A SCF defined over the domain* $[\mathscr{D}^{SP}]^N$ *is strategyproof if and only if it is decomposable. Moreover, each marginal SCF must be a GMVS.*

BGS show that the decomposability result extends to SCFs defined over the domain $[\tilde{\mathscr{D}}^{SP}]^N$. We have noted that $\tilde{\mathscr{D}}^{SP}$ contains orderings that are not separable. The domain $[\tilde{\mathscr{D}}^{SP}]^N$ is therefore, not rich and Theorem 4.1 cannot be invoked directly. However, $[\mathscr{D}^{SP}]^N \subset [\tilde{\mathscr{D}}^{SP}]^N$ and we know from Theorem 5.3, the class of strategyproof SCFs over $[\mathscr{D}^{SP}]^N$. Using these facts, it is easy to extend the characterization result to the domain $[\tilde{\mathscr{D}}^{SP}]^N$. The details are to be found in the Appendix.

THEOREM 5.4 (BGS): *A SCF defined over the domain* $[\tilde{\mathscr{D}}^{SP}]^N$ *is strategyproof if and only if it is decomposable. Moreover, each marginal SCF must be a GMVS.*

The results in this subsection illustrate the wide applicability of Theorem 4.1. Several new results can be obtained by imposing specific structure on the marginal domains and appealing to existing results on strategyproofness on these domains.

6. VARIABLE INDIVIDUAL PARTITIONS

In the previous section we analyzed the case where individual preferences were separable with respect to all components. In this section, we turn to the problem of characterizing strategyproof SCFs in a natural generalization of this model. We shall assume that each individual has some partition of the set of components and that her preferences are separable only with respect to each element of the partition.

Let $Q \subset \{1, \dots, M\}$. We shall let A_Q and A_{M-Q} refer to the sets $\prod_{j \in Q} A_j$ and $\prod_{j \notin Q} A_j$ respectively.

Definition 6.1: Let π^i be a partition of the set $\{1,\ldots,M\}$. The ordering $P^i \in P(A)$ is separable with respect to π^i if, for all $Q \in \pi^i$, $a_Q, b_Q \in A_Q$ and $x_{M-Q}, y_{M-Q} \in A_{M-Q}, (a_Q, x_{M-Q}) P^i (b_Q, x_{M-Q})$ implies $(a_Q, y_{M-Q}) P^i (b_Q, y_{M-Q})$.

Let $\mathscr{D}(\pi^i)$ denote the set of all orderings separable with respect to π^i. Let π^0 denote the partition $\{\{1\},\{2\},\ldots,\{M\}\}$.

Remark 6.1: $\mathscr{D} = \mathscr{D}(\pi^0)$. (Recall that $\mathscr{D}$ is the set of separable orderings.) In a manner analogous to Definition 2.2, we can define an ordering that is additively representable with respect to the partition π^i. We shall let $\mathscr{D}^A(\pi^i)$ denote the set of all such orderings.

Remark 6.2: The set of additively separable orderings $\mathscr{D}^A = \mathscr{D}^A(\pi^0)$. The following two properties also hold:
(i) For all partitions π^i, $\mathscr{D}^A(\pi^i) \subset \mathscr{D}(\pi^i)$.
(ii) For all partitions π^i, $\mathscr{D}^A(\pi^0) \subset \mathscr{D}^A(\pi^i)$.
Properties (i) and (ii) play a crucial role in the proof of Theorem 6.1. It is important to note that $\mathscr{D}(\pi^0) \subset \mathscr{D}(\pi^i)$ does not hold.

We now introduce another domain restriction.

Definition 6.2: The ordering P^i is *tops separable with respect to* π^i if, for all $Q \in \pi^i$, $a_Q \in A_Q$, $x_{M-Q}, y_{M-Q} \in A_{M-Q}, (a_Q, x_{M-Q}) P^i (b_Q, x_{M-Q})$ for all $b_Q \in A_Q$ implies that $(a_Q, y_{M-Q}) P^i (b_Q, y_{M-Q})$ for all $b_Q \in A_Q$.

Let $\mathscr{D}^T(\pi)$ denote the set of all tops separable orderings with respect to π^i.

Remark 6.3: For all partitions $\pi^i, \mathscr{D}(\pi^i) \subset \mathscr{D}^T(\pi^i)$.

Remark 6.4: For any $Q \in \pi^i$ and $P^i \in \mathscr{D}(\pi^i)$ or $\mathscr{D}^i(\pi^i)$, we can extract marginal orderings over the set A_Q as before. We shall denote this marginal ordering by P^i_Q. Likewise, if $P^i \in \mathscr{D}^T(\pi^i)$ and $Q \in \pi^i$, we can unambiguously define a maximal element $\tau(P^i, A_Q)$ over the set A_Q.

We observe that if individual partitions differ, it is no longer possible to define marginal preference profiles unambiguously. Consequently, decomposability is no longer well-defined and the results of the previous section cannot be extended. In order to make progress we need to impose more structure on the marginal preference domains. We assume that these domains, defined appropriately according to individual partitions, are unrestricted. This allows us to obtain sharp characterization results.

Definition 6.3: *A Rights Assignment Map* (RAM) is a function $\sigma: \{1,\ldots,M\} \to I$.

For all partitions π^i and components j, let $t(j, \pi^i)$ denote the element of π^i to which j belongs.

Given an N-tuple of partitions $(\pi^1, \ldots, \pi^N)$, a RAM σ is *consistent* with $(\pi^1, \ldots, \pi^N)$ if for all $j, k \in \{1, \ldots, M\}$, $k \in t(j, \pi^{\sigma(j)})$ implies $\sigma(k) = \sigma(j)$.

In other words, if $\sigma(j) = l$ and $k \in \{1, \ldots, M\}$ is such that j and k belong to the same element of the partition π^l, then it must be the case that $\sigma(k) = l$. The definition will be further clarified by the example below.

EXAMPLE 6.1: Let $I = \{1, 2, 3\}$, $M = 4$, $\pi^1 = (\{1\}, \{2\}, \{3\})$, $\pi^2 = (\{1, 2\}\{3, 4\})$ and $\pi^3 = (\{1, 3\}, \{2\}, \{4\})$. Then $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = \sigma(4) = 2$ is a RAM consistent with (π^1, π^2, π^3). Another one is $\sigma(1) = \sigma(2) = \sigma(3) = \sigma(4) = 1$.

DEFINITION 6.4: Let σ be a RAM. The SCF f is *σ-maximal* if for all P in the admissible domain, $f(P) = a$ implies that $a_j = b_j$ where $b = \tau(P^{\sigma(j)}, A)$ for all $j \in \{1, \ldots, M\}$.

If f is σ-maximal then the jth component $(j = 1, \ldots, M)$ of $f(P)$ is the jth component of the maximal element of $P^{\sigma(j)}$ in the set A.

THEOREM 6.1: *Assume that A is finite and $|A_j| \geq 3$ for all $j \in \{1, \ldots, M\}$. Let $(\pi^1, \ldots, \pi^N)$ be an N-tuple of partitions of $\{1, \ldots, M\}$. Let f be a nonimposed SCF over $\prod_{i \in I} \mathscr{D}(\pi^i)$. Then f is strategyproof if and only if it is σ-maximal for some RAM σ consistent with $(\pi^1, \ldots, \pi^N)$.*

The next result says that the same characterization of strategyproof SCFs remains valid when the domain is extended to include tops separable preferences.

THEOREM 6.2: *Assume that $|A_j| \geq 3$ for all $j \in \{1, \ldots, M\}$. Let $(\pi^1, \ldots, \pi^N)$ be an N-tuple of partitions of $\{1, \ldots, M\}$. Let f be a nonimposed SCF over $\prod_{i \in I} \mathscr{D}^T(\pi^i)$. Then f is strategyproof if and only if it is σ-maximal for some RAM σ consistent with $(\pi^1, \ldots, \pi^N)$.*

We note that a sufficient condition for a RAM σ to be consistent with an N-tuple of partitions $(\pi^1, \ldots, \pi^N)$ is the constancy of σ over the partition $\pi \equiv \Lambda_{i \in I} \pi^i$.[4] Furthermore, this condition is necessary, when all the partitions are the same. Thus, in particular if $\pi^i, \pi^j = \pi^0$ for all $i, j \in I$, then $\pi = \pi^0$ and we obtain Theorem 5.1; in that case every RAM is consistent with $(\pi^1, \ldots, \pi^N)$. More generally, if $\pi^i = \pi^j = \pi^*$ for all $i, j \in I$ then a RAM σ is consistent if $\sigma(j) = \sigma(k)$ for all j, k belonging to the same element of the partition π^*. So, in the extreme case where π^i is the trivial partition $\{1, \ldots, M\}$ for all $i \in I$, the only consistent RAMs are the constant ones. Of course, a SCF which is σ-maximal where σ is a constant RAM, must be dictatorial. Observe finally that constant RAMs are consistent with every N-tuple of partitions $(\pi^1, \ldots, \pi^N)$.

[4] $\Lambda_{i \in I} \pi^i$ denotes the infimum of the partitions $(\pi^i)_{i \in I}$ in the usual lattice of partitions.

This is a restatement of the obvious result that dictatorial SCFs are always strategy proof. In sum, the characterization of strategyproof SCFs in separable and tops separable domains is closely related to a particular combinatorial issue in the lattice of partitions.

DEFINITION 6.5: Let f be a SCF over some admissible domain. Then f is *efficient* if for all admissible profiles P and for all $a \in A$, if $f(P) \neq a$, then there exists $i \in I$ such that $f(P) P^i a$.

What is the class of SCFs that is both strategyproof and efficient? The next result addresses this question directly.

THEOREM 6.3: *Assume that A is finite and $|A_j| \geq 3$ for all $j \in \{1, \ldots, M\}$. Let $(\pi^1, \ldots, \pi^N)$ be an N-tuple of partitions of $\{1, \ldots, N\}$. Let f be a nonimposed SCF over $\prod_{i \in I} \mathscr{D}(\pi^i)$. Then f is strategyproof and efficient if and only if it is dictatorial.*

REMARK 6.5: Theorem 6.3 remains valid when the SCF f under consideration is defined over the domain $\prod_{i \in I} \mathscr{D}^T(\pi^i)$. This is completely obvious in view of Theorem 6.3 and the definition of efficiency.

Theorem 6.3 and Remark 6.5 confirm a recurrent feature of incentive theory: "first-best" efficiency cannot be achieved when there are incentive constraints. In our model, the incentive constraints are of the strongest variety, viz. truthtelling must be a dominant strategy. It is perhaps not surprising therefore, that the incentive constraints and efficiency cannot be simultaneously satisfied except by dictatorship.

We now consider an important application of the results of this section.

DEFINITION 6.6: Assume $M = N$ and let f be a SCF defined over some admissible domain of preferences. Then, f is *libertarian* if for all admissible profiles P and for all $a \in A$, if $f(P) = a$ then $a_i = b_i$ where $b = \tau(P^i, A)$, for all $i \in I$.

Consider a model where A_i is the set of personal issues of individual i. A social state is merely a description of the personal issues of each individual. Individuals are not assumed to be selfish and they may be sensitive to the specification of the social state for other people in society. A SCF is libertarian if the ith component of the optimal social state is the ith component of the maximal element in A according to individual i's preferences. Loosely speaking, each individual gets to "choose" his own personal issue.

It was pointed out in Sen (1970) that the libertarian SCF defined over the set of separable preference profiles with respect to the common partition π^0 is inefficient. Since then several papers have been written on the subject. The primary objective of this literature has been to reconcile the desirable ethical features of libertarianism on the one hand and its inefficiency on the other. Our aim is to show that the imposition of the axiom of strategyproofness leads to a very natural characterization of the libertarian SCF.

For all $i \in I$, let $\hat{\pi}^i$ be the partition $(\{N-\{i\}\},\{i\})$. If individual i's preferences are separable with respect to $\hat{\pi}^i$, it means that his preference over his personal issues are independent of what happens to the others.

COROLLARY 6.1: *Assume that A is finite and $|A_j| \geq 3$ for all $j \in \{1,\dots,M\}$. Let f be a nonimposed, nondictatorial SCF defined over the domain $\prod_{i \in I} \mathscr{D}(\hat{\pi}^i)$. Then f is strategyproof if and only if it is libertarian.*

The proof of Corollary 6.1 is an immediate consequence of Theorem 6.1—just observe that the only RAMs consistent with $(\hat{\pi}^1,\dots,\hat{\pi}^N)$ are either constant or of the form $\sigma(1)=i$ for all $i \in I$. Since f is assumed to be nondictatorial only the latter RAM is admissible. However, Theorem 6.1 now implies that f is libertarian. We note that Corollary 6.1 would go through if f were defined on the larger domain $\prod_{i \in I} \mathscr{D}^T(\hat{\pi}^i)$.

The implication of Corollary 6.1 is that the only SCF that satisfies incentive constraints over an appropriate domain (and subject to some mild ethical conditions) is the libertarian SCF. This can also be reformulated according to an implementation theory perspective. The libertarian SCF is the only SCF whose outcomes can be achieved in a decentralized fashion in the presence of informational asymmetries between the individuals and the planner.

7. MAXIMAL DOMAINS

We know from Theorem 6.2 that tops separable domains of preference profiles admit nondictatorial strategyproof SCFs. In this section we investigate whether such SCFs exist over larger domains. Of course, the Gibbard-Satterthwaite Theorem tells us that we cannot go all the way to the unrestricted domain of all strict orderings. But can we extend the tops separable domain "a little further" without losing either the strategyproofness or the nondictatorship property? We shall, in fact answer this question in the negative.

DEFINITION 7.1: Let π^i be a partition of $\{1,\dots,M\}$. The domain $\mathscr{D}^i \subseteq P(A)$ is *composite* with respect to π^i if it satisfies the following two properties:

(i) $\mathscr{D}^T(\pi^i)$ is a strict subset of $\mathscr{D}^i$.

(ii) Let Q be a strict subset of $\{1,\dots,M\}$ such that $Q = Q^1 U Q^2 \cup \cdots \cup Q^K$ where $Q^1, Q^2, \dots, Q^K \in \pi^i$. There exists $a_Q, b_Q \in A_Q, (a_Q \neq b_Q) c_{M-Q}, d_{M-Q} \in A_{M-Q}$ and $P^i \in \mathscr{D}^i$ such that $(a_Q, c_{M-Q}) P^i (x_Q, c_{M-Q})$ for all $x_Q \in A_Q$ and $(b_Q, d_{M-Q}) P^i (a_Q, d_{M-Q})$.

Thus, $\mathscr{D}^i$ is composite with respect to π^i if it is a strict superset of the set of all tops separable orderings with respect to π^i. Moreover, it is not tops separable with respect to any nontrivial partition that is coarser than π^i. We illustrate this notion by means of a simple example.

EXAMPLE 7.1: $M=2$, $A_1=\{a_1,b_1\}$, $A_2=\{a_2,b_2\}$, and $\pi=(\{1\},\{2\})$. In this case the set of separable orderings and the set of tops separable orderings coincides. This is the set of the following eight orderings:

(1) $a_1a_2P^ia_1b_2P^ib_1a_2P^ib_1b_2$, (2) $a_1a_2P^ib_1a_2P^ia_1b_2P^ib_1b_2$,
(3) $b_1b_2P^ib_1a_2P^ia_1b_2P^ia_1a_2$, (4) $b_1b_2P^ia_1b_2P^ib_1a_2P^ia_1a_2$,
(5) $a_1b_2P^ib_1b_2P^ia_1a_2P^ib_1a_2$, (6) $a_1b_2P^ia_1a_2P^ib_1b_2P^ib_1a_2$,
(7) $b_1a_2P^ia_1a_2P^ib_1b_2P^ia_1b_2$, (8) $b_1a_2P^ib_1b_2P^ia_1a_2P^ia_1b_2$.

An example of a composite domain is the domain consisting of these eight orderings together with the ordering $a_1b_2P^ib_1b_2P^ia_1a_2P^ib_1a_2$. We note that this domain is strictly smaller than the domain consisting of all strict orderings. (The former has nine orderings and the latter, twenty-four.)

REMARK 7.1: It is always possible to construct a composite domain by adding a single additional ordering to the set of tops separable orderings. More formally, for all partitions π, there exists a composite domain $\mathscr{D}^i$ with respect to π such that $|\mathscr{D}^i|=|\mathscr{D}^T(\pi)|+1$. We omit the proof of this claim.

THEOREM 7.1: *Assume that A is finite $|A_j|\geq 3$ for all $j\in\{1,\dots,M\}$. Let $(\pi^1,\dots,\pi^N)$ be an N-tuple of partitions of $\{1,\dots,M\}$. For all $i\in I$, let $\mathscr{D}^i$ be a domain composite with respect to π^i. Let f be a nonimposed SCF over $\prod_{i\in I}\mathscr{D}^i$. Then f is strategyproof if and only if it is dictatorial.*

Theorem 7.1 establishes that the tops separable domain is the largest domain that admits nondictatorial strategyproof SCFs. An alternative interpretation of Theorem 7.1 is that a necessary condition for the existence of a nondictatorial strategyproof SCF in a domain that includes separable orderings is that it has a minimal degree of "separability" between components.

We conclude this section by making a brief comment on a certain feature of the definition of a composite domain $\mathscr{D}^i$ with respect to a partition π^i. We require that $\mathscr{D}^i$ include orderings that are tops separable with respect to any nontrivial partition coarser than π^i. Suppose $\mathscr{D}^i$ was not tops separable with respect to π^i but was so with respect to some nontrivial partition $\hat{\pi}^i$ coarser than π^i. The domain of profiles would then remain tops separable although with respect to a different N-tuple of partitions. Theorem 6.1 would still apply and the existence of a nondictatorial strategyproof SCF cannot be ruled out.

8. THE PROOFS

In this section we gather the proofs of the main results of the paper.

PROOF OF THEOREM 4.1: *Sufficiency*—Let f be a decomposable SCF, i.e., for all $j\in\{1,\dots,M\}$, there exists a strategyproof SCF f_j: $\prod_{i\in I}\mathscr{D}^i_j\to A_j$ with the following property: for all $P\in\prod_{i\in I}\mathscr{D}^i$ and $a\in A$, $f(P)=a$ implies that $a_j=f_j(P_j)$. To show that f is strategyproof, consider $P\in\prod_{i\in I}\mathscr{D}^i$, $i\in I$, and $\bar{P}^i\in\mathscr{D}^i$ and let us show that $f(P)P^if(P|\bar{P}^i)$. Suppose on the contrary we have $a=$

$f(P|\bar{P}^i) \neq f(P) = b$ and aP^ib. Since P^i is separable with respect to $\{1, \dots, M\}$, there must exist $j \in \{1, \dots, M\}$ such that $a_jP_j^ib_j$. Since f is decomposable, $f_j(P_j|\bar{P}_j^i) = a_j$ and $f_j(P_j) = b_j$. However, this contradicts the assumption that f_j is strategyproof. Since P, $\bar{P}^i$ and i were chosen arbitrarily, the proof is complete.

Necessity—We proceed in a sequence of lemmata. According to Lemma 1 rich domains permit a certain "lifting property" that is used repeatedly in subsequent arguments.

LEMMA 1: *Let $\mathscr{D}^i$ be rich. For all $P^i \in \mathscr{D}^i$, for all $j = 1, \dots, M$ and $b \in A$, there exists $\bar{P}^i \in \mathscr{D}^i$ such that*
(i) $\bar{P}_j^i = P_j^i$,
(ii) $b_k = \tau(\bar{P}_k^i, A_k)$ *for all* $k \neq j$,
(iii) $bP^ic \to b\bar{P}^ic$ *for all* $c \in A$.

PROOF: Since $\mathscr{D}^i$ satisfies Property A, there exists $\tilde{P}^i$ such that $\tau(\tilde{P}_k^i, A_k) = b_k$ for all $k \neq j$. From part (ii) of Property B, it follows that there exists $\bar{P}^i \in \mathscr{D}^i$ such that $\bar{P}_k^i = \tilde{P}_k^i$ for all $k \neq j$, $\bar{P}_j^i = P_j^i$, and such that for all $x, y \in A$, $x \neq y$, $[x\bar{P}^iy$ and $x_j\bar{P}_j^iy_j] \to$ [either $x_k \neq y_k$ and $x_k\bar{P}_k^iy_k$ for some $k \neq j$ or $x_{M-j} = y_{M-j}$]. Pick $c \in A$ such that $b \neq c$ and bP^ic. Since $b_k\bar{P}_k^ic_k$ for all $k \neq j$, $b_j\bar{P}_j^ic_j$ implies that $b\bar{P}^ic$. Suppose that $c_j\bar{P}_j^ib_j$ and $c\bar{P}^ib$. Therefore, $c_{M-j} = b_{M-j}$. But then bP^ic implies that $b_jP_j^ic_j$ which contradicts the assumption that $\bar{P}_j^i = P_j^i$. Therefore, $b\bar{P}^ic$ and $\bar{P}^i$ satisfies properties (i)–(iii). *Q.E.D.*

Lemma 2 expresses a positive association property.

LEMMA 2: *For all $P \in \prod_{i \in I}\mathscr{D}^i$ and $a \in A$, if $f(P) = a$, then $f(\bar{P}) = a$ for all $\bar{P} \in \prod_{i \in I}\mathscr{D}^i$ such that $aP^ib \to a\bar{P}^ib$ for all $i \in I$ and for all $b \in A$.*

PROOF: We first claim that $f(P|\bar{P}^1) = a$. Suppose on the contrary that $f(P|\bar{P}^1) = c \neq a$. Then, either cP^1a or aP^1c. In the first case individual 1 manipulates at P. If aP^1c, then $a\bar{P}^1c$ by hypothesis and 1 manipulates at $P|\bar{P}^1$. Thus, $f(P|\bar{P}^1) = a$. Applying this argument repeatedly we conclude that $f(\bar{P}) = a$. *Q.E.D.*

Lemma 3 asserts that f must satisfy a conditional unanimity property.

LEMMA 3: *Let $j \in \{1, \dots, M\}$ and let $P \in \prod_{i \in I}\mathscr{D}^i$ be such that $\tau(P_j^i, A_j) = a_j$ for all $i \in I$. Then $f_j(P) = a_j$.*

PROOF: Observe first that the nonimposition of f and Lemma 2 together imply that f satisfies unanimity; i.e. if $a = \tau(P^i, A)$ for all $i \in I$, then $f(P) = a$.

Suppose Lemma 3 is false. It follows that there exists $j \in \{1, \dots, M\}$, and $P \in \prod_{i \in I}\mathscr{D}^i$ such that $\tau(P_j^i, A_j) = a_j$ for all $i \in I$ but $f(P) = b$ where $b_j \neq a_j$. Applying Lemma 1, there exists $\bar{P}^i \in \mathscr{D}^i$ such that
(i) $\bar{P}_j^i = P_j^i$,
(ii) $\tau(\bar{P}^i, A) = (a_j, b_{M-j})$, and
(iii) $bP^ic \to b\bar{P}^ic$ for all $c \in A$.

Applying unanimity, we have $f(\bar{P}) = (a_j, b_{M-j})$. However, since $f(P) = b$, it follows from (iii) and Lemma 2 that $f(\bar{P}) = b$. Since $b_j \neq a_j$ by assumption, we have a contradiction. *Q.E.D.*

Lemma 4 says the following: pick $j \in \{1, \ldots, M\}$ and consider two profiles that induce the same marginal preference profile over j and that are unanimous over the components $M - j$. Then f must take the same value at the two profiles.

LEMMA 4: *Let* $j \in \{1, \ldots, M\}$. *For all* $i \in I$, *let* $\hat{P}^i, \bar{P}^i \in \mathscr{D}^i$ *be such that* (i) $\hat{P}^i_j = \bar{P}^i_j$ *and* (ii) $x^i_{M-j} = y^i_{M-j} = a_{M-j}$ *where* $x^i = \tau(\hat{P}^i, A)$ *and* $y^i = \tau(\bar{P}^i, A)$. *Then* $f(\hat{P}) = f(\bar{P})$.

PROOF: Let $f(\hat{P}) = b$ and $f(\bar{P}) = c$. It follows from Lemma 2 that $b_{M-j} = c_{M-j} = a_{M-j}$. We first claim that $f(\hat{P}|\bar{P}^1) = f(\hat{P})$. Suppose on the contrary that $f(\hat{P}|\bar{P}^1) = (d_j, a_{M-j})$ where $d_j \neq b_j$. Either $b_j \bar{P}^1_j d_j$ or $d_j \bar{P}^1_j b_j$ must hold. In the first case 1 manipulates at $\hat{P}|\bar{P}^1$ since $(b_j, a_{M-j})\bar{P}^1(d_j, a_{M-j})$. In the second case, we must have $d_j \hat{P}^1_j b_j$ since $\hat{P}^1_j = \bar{P}^1_j$ and 1 manipulates at $\hat{P}$ since $(d_j, a_{M-j})\hat{P}^1(b_j, a_{M-j})$. Therefore $d_j = b_j$. Repeated application of this result yields $f(\hat{P}) = f(\bar{P})$. *Q.E.D.*

Let $a \in A$ and $j \in \{1, \ldots, M\}$. We denote by $\mathscr{D}^i_{a, M-j}$ the set of P^i in $\mathscr{D}^i$ that $\tau(P^i, A) = d$ implies that $d_{M-j} = a_{M-j}$. We define the function $f_{a,j}: \prod_{i \in I} \mathscr{D}^i_j \to A_j$ as follows: for all $P \in \prod_{i \in I} \mathscr{D}^i_{a, M-j}, f(P) = b$ implies that $f_{a,j}(P_j) = b_j$. We describe this construction more informally. Consider $P_j \in \prod_{i \in I} \mathscr{D}^i_j$. Now, pick a profile $\bar{P} \in \prod_{i \in I} \mathscr{D}^i$ such that

(i) $\bar{P}^i_j = P_j$ for all $i \in I$ and

(ii) $\tau(P^i, A) = (d^i_j, a_{M-j})$ where $d^i_j = \tau(\bar{P}_j, A_j)$ for all $i \in I$.

Suppose $f(\bar{P}) = b$. Then, $f_{a,j}(P_j) = b_j$.

We make two important observations regarding the function $f_{a,j}$. First, it follows from Lemma 3 that the function is well-defined. Second, since $\mathscr{D}^i$ satisfies A it must be the case that for all P^i_j in $\mathscr{D}^i_j$, there exists $\bar{P}^i \in \mathscr{D}^i_{a, M-j}$ such that $\bar{P}^i_j = P^i_j$ so that the domain of the function is indeed $\prod_{i \in I} \mathscr{D}^i_j$.

We now establish a crucial property of the function $f_{a,j}$.

LEMMA 5: *For all* $a \in A$ *and* $j \in \{1, \ldots, M\}$, *the function* $f_{a,j}$ *is strategyproof*.

PROOF: Pick $P \in \prod_{i \in I} \mathscr{D}^i_j$ and $\bar{P}^i_j \in \mathscr{D}^i_j$. Let $f_{a,j}(P_j) = b_j$ and $f_{a,j}(P_j|\bar{P}^i_j) = c_j$. Suppose $c_j P^i_j b_j$, i.e. $f_{a,j}$ is manipulable. Since $\mathscr{D}^l$ is rich for all $l \in I$, there exists $\tilde{P} \in \prod_{i \in I} \mathscr{D}^i$ such that:

(i) $\tilde{P}^l_j = P^l_j$ for all $l \in I$, and

(ii) $\tau(\tilde{P}^l, A) = (d^l_j, a_{M-j})$ where $d^l_j = \tau(P^l_j, A_j)$ for all $l \in I$.

Also since $\mathscr{D}^i$ is rich, there exists $\hat{P}^i \in \mathscr{D}^i$ such that:

(i) $\hat{P}^i_j = \bar{P}^i_j$ and

(ii) $\tau(\hat{P}^i, A) = (d^i_j, a_{M-j})$ where $d^i_j = \tau(\bar{P}^i_j, A_j)$.

From the definition of $f_{a,j}$, we deduce that $f(\tilde{P})=(b_j, a_{M-j})$ and $f(\tilde{P}|\hat{P}^i)=(c_j, a_{M-j})$. Since $c_j\tilde{P}_j^i b_j$, it follows immediately that $f(\tilde{P}|\tilde{P}^i)\tilde{P}^i f(\tilde{P})$, i.e. f is manipulable. Therefore, it cannot be the case that $c_j P_j^i b_j$. Since P_j, $\bar{P}_j^i$ and i were chosen arbitrarily, $f_{a,j}$ is strategyproof. *Q.E.D.*

LEMMA 6: *For all* $P\in\prod_{i\in I}\mathscr{D}^i$, *for all* $a\in A$, *for all* $j\in\{1,\dots,M\}$, $f(P)=a \rightarrow f_{a,j}(P_j)=a_j$.

PROOF: Suppose not. Let $P\in\prod_{i\in I}\mathscr{D}^i$, $a\in A$ and $j\in\{1,\dots,M\}$ be such that $f(P)=a$, but $f_{a,j}(P_j)\neq a_j$. Since $\mathscr{D}^i$ is rich for all $i\in I$, there exists $\bar{P}\in\prod_{i\in I}\mathscr{D}^i$ such that:

(i) $\bar{P}_j^i=P_j^i$ for all $i\in I$,

(ii) $a_{M-j}=d_{M-j}^i$ where $\tau(\bar{P}^i, A)=d^i$ for all $i\in I$, and

(iii) $aP^ic\rightarrow a\bar{P}^ic$ for all $c\in A$ and for all $i\in I$.

Since $f(P)=a$, we deduce from (iii) and an application of Lemma 2 that $f(\bar{P})=a$. It now follows from (i) and (ii) and the definition of $f_{a,j}$ that $f_{a,j}(P_j)=a_j$. We have a contradiction. *Q.E.D.*

LEMMA 7: *For all* $j\in\{1,\dots,M\}$ *and* $a\in A$, *the function* $f_{a,j}$ *is independent of* a.

PROOF: Suppose not. Let $j\in\{1,\dots,M\}$ and $a,b\in A$ be such that $f_{a,j}(P_j)\neq f_{b,j}(P_j)$ for some $P_j\in\prod_{i\in I}\mathscr{D}_j^i$. Since $\mathscr{D}^i$ is rich there exist $\hat{P},\bar{P}\in\prod_{i\in I}\mathscr{D}^i$ such that, for all $i\in I$,

(i) $\hat{P}_j^i=\bar{P}_j^i=P_j^i$,

(ii) $a_{M-j}=u_{M-j}^i$ where $u^i=\tau(\hat{P}^i, A)$, and

(iii) $b_{M-j}=v_{M-j}^i$ where $v^i=\tau(\bar{P}^i, A)$,

(iv) for all $x,y\in A$ such that $x_j\neq y_j$, $x\hat{P}^iy$ if and only if $x_j\hat{P}_j^iy_j$, and

(v) for all $x,y\in A$ such that $x_j\neq y_j$, $x\bar{P}^iy$ if and only if $x_j\bar{P}_j^iy_j$.

From (i), (ii), and (iii) above, we obtain that $f(\hat{P})=(c_j, a_{M-j})$ and $f(\bar{P})=(d_j, b_{M-j})$ where $f_{a,j}(P_j)=c_j$ and $f_{b,j}(P_j)=d_j$. Let l be the smallest value of $i\in I$ such that $c_j\neq z_j$ where $z=f(\hat{P}|\bar{P}^0,\dots,\bar{P}^i)$. Here we adopt the convention that $f(\hat{P}|\bar{P}^0)=f(\hat{P})$. Such an l must exist since $f(\hat{P}|\bar{P}^0,\dots,\bar{P}^N)=f(\bar{P})=(d_j, b_{M-j})$ and $c_j\neq d_j$ by assumption. Either $z_j\hat{P}_j^lc_j$ or $c_j\hat{P}_j^lz_j$ must hold. Suppose that it is the case that $z_j\hat{P}_j^lc_j$. Applying (iv), we have $z=f(\hat{P}|\bar{P}^0,\dots,\bar{P}^l)$ $\hat{P}^lf(\hat{P}|\bar{P}^0,\dots,\bar{P}^{l-1})=(c_j,\cdot)$. Therefore, l manipulates at $(\hat{P}|\bar{P}^0,\dots,\bar{P}^{l-1})$. Suppose that $c_j\hat{P}_j^lz_j$ is true. From (i), we have $c_j\bar{P}_j^lz_j$. Using (v), we deduce that $(c_j,\cdot)=f(\hat{P}|\bar{P}^0,\dots,\bar{P}^{l-1})\bar{P}^lf(\hat{P}|\bar{P}^0,\dots,\bar{P}^l)=z$. Therefore, l manipulates at $(\hat{P}|\bar{P}^0,\dots,\bar{P}^l)$. In both cases, we contradict the assumption that f is strategyproof. *Q.E.D.*

We are now in a position to conclude the proof of Theorem 4.1. Consider $j\in\{1,\dots,M\}$ and two profiles $P,\bar{P}\in\prod_{i\in I}\mathscr{D}^i$ such that $P_j=\bar{P}_j$. Suppose $f(P)=a$ and $f(\bar{P})=b$. We have to show that $a_j=b_j$. From Lemma 5 we deduce that $f_{a,j}(P_j)=a_j$ and $f_{b,j}(\bar{P}_j)=b_j$. But since $P_j=\bar{P}_j$ we deduce from Lemma 7 that $f_{a,j}(P_j)=f_{b,j}(\bar{P}_j)$. The proof is complete. *Q.E.D.*

Proof of Theorem 5.4: We prove only the necessity part of the Theorem. Let $f\colon [\tilde{\mathscr{D}}^{SP}]^N \to A$ be a strategyproof SCF. Since $\mathscr{D}^{SP} \subset \tilde{\mathscr{D}}^{SP}$, we can define the restriction of f on the domain $[\mathscr{D}^{SP}]^N$. This SCF must be strategyproof; applying Theorem 5.3 we conclude that it must be decomposable with each marginal SCF, $f_j\colon [\mathscr{D}_j^{SP}]^N \to A_j$, $j = 1, \ldots, M$ being a GMVS. We have noted that a GMVS is a "tops only" SCF. The proof is completed by showing that, for all $P \in [\tilde{\mathscr{D}}^{SP}]^N$, there exists $\bar{P} \in [\mathscr{D}^{SP}]^N$ with $\tau(P^i, A) = \tau(\bar{P}^i, A)$ for all $i \in I$, such that $f(P) = f(\bar{P})$.

Let $P \in [\tilde{P}^{SP}]^N$ and denote $\tau(P^i, A)$ by m^i. Let $\tilde{P} \in [\mathscr{D}^{SP}]^N$ be such that $\tau(\tilde{P}^i, A) = m^i$ for all $i \in I$. Let $f(\tilde{P}) = a$. Assuming w.l.o.g. that $a_j >_j m_j$ for all $j = 1, \ldots, M$. Let J be a set of those components for which $a_j \neq m_j$. Pick $\delta, 0 < \delta < (M-1)/M$. Let $U_j^i\colon\ A_j \to [0,1]$, $j = 1, \ldots, M$, represent a single-peaked ordering on A_j normalized so that the maximum and minimum elements are assigned values 1 and 0 respectively. Assume, in addition, that $U_j^i(a_j) = \delta/(M-1)$ and for all $j \in J$, $U_j^i(r(a_j)) > \delta M/(M-1)$ where $r(a_j)$ is the element immediately preceding a_j according to $>_j$. Let $\bar{P}^i$ be the strict ordering represented by the utility function $\sum_{j \in J} U_j^i$ with ties broken lexicographically according to some fixed ordering of the components. It is easy to verify that $\bar{P}^i$ is separable.

Let $W(a) = \{b \in A | b_j >_j c_j \text{ for all } j \in J\}$. We claim that $b\bar{P}^i a$ for all $b \notin W(a)$. Pick $b \notin W(a)$ and observe that there must exist $k \in J$ such that $a_k >_k b_k$. Now,

$$\begin{aligned}
\sum_{j \in J} U_j^i(b) - \sum_{j \in J} U_j^i(a) &= U_k^i(b_k) - U_k^i(a_k) + \sum_{j \in J - \{k\}} \left(U_j^i(b_j) - U_j^i(a_j)\right) \\
&\geq U_k^i\left(r(a_j)\right) - U_k^i(a_k) - \sum_{j \in J - \{k\}} U_j^i(a_j) \\
&> \frac{\delta M}{M-1} - \frac{\delta}{M-1} - \delta \\
&= 0.
\end{aligned}$$

Since $\bar{P}^i$ is separable, $a\bar{P}^i c$ for all $c \in W(a)$. Since P^i is multi-dimensional, single-peaked, $aP^i c$ for all $c \in W(a)$. Therefore, $a\bar{P}^i c \to aP^i c$ for all $c \in A$ and $i \in I$. Since $f\colon [\mathscr{D}^{SP}]^N \to A$ is "tops only," $f(\tilde{P}) = f(\bar{P}) = a$. Applying Lemma 2, we conclude that $f(P) = a$. *Q.E.D.*

Proof of Theorem 6.1: *Sufficiency*—This is an easy consequence of the sufficiency argument of Theorem 6.2 which we will describe in detail later. Let f be a SCF over $\prod_{i \in I} \mathscr{D}^T(\pi^i)$ and let f be σ-maximal where σ is a RAM consistent with $(\pi^1, \ldots, \pi^N)$. In the proof of Theorem 6.2, we show that f is strategyproof. Since $\mathscr{D}(\pi^i) \subset \mathscr{D}^T(\pi^i)$ for all i, the restriction of f to the domain $\prod_{i \in I} \mathscr{D}(\pi^i)$ must also be strategyproof. To complete the proof, observe that this restriction is also σ-maximal.

Necessity—We proceed once again in a sequence of lemmata, building on Theorem 4.1. Let f be a SCF. For all $P \in \prod_{i \in I} \mathscr{D}(\pi^i)$ and $i \in I$, we denote by $B^i(P)$ the set $\{x \in A | f(P|\bar{P}^i) = x \text{ for some } \bar{P}^i \in \mathscr{D}^A(\pi^0)\}$. We note that the set

$B^i(P)$ differs from individual i's option set as defined in Barberá and Peleg (1990) since individual i's preferences are restricted to lie in the set $\mathscr{D}^A(\pi^0)$, rather than the individual preference domain $\mathscr{D}(\pi^i)$.

In all subsequent lemmas used in this proof, f and f^A are strategyproof SCFs over the domains $\prod_{i\in I}\mathscr{D}(\pi^i)$ and $\prod_{i\in I}\mathscr{D}^A(\pi^i)$.

LEMMA 7: *For all* $P\in\prod_{i\in I}\mathscr{D}^A(\pi^i)$ *and* $i\in I$, $f^A(P)=\tau(P^i,B^i(P))$.

PROOF: Suppose the lemma is false. Let $P\in\prod_{i\in I}\mathscr{D}^A(\pi^i)$ and $i\in I$ be such that $a=f(P)\neq\tau(P^i,B^i(P))=b$. Either bP^ia or aP^ib must hold. Suppose that the former case applies. Since $b\in B^i(P)$, there exists $\bar{P}^i\in\mathscr{D}^A(\pi^0)$ such that $f(P|\bar{P}^i)=b$. Since $\mathscr{D}^A(\pi^0)\subset\mathscr{D}^A(\pi^i)$, individual i will manipulate at P. Suppose aP^ib holds. Let $\hat{P}^i\in\mathscr{D}^A(\pi^0)$ such that $a_j=\tau(\hat{P}^i_j,A_j)$ for all $j\in\{1,\dots,M\}$. Clearly, $a=\tau(\hat{P},A)$. It follows from Lemma 1 that $f^A(P|\hat{P}^i)=a$. According to the definition of the set $B^i(P)$, $a\in B^i(P)$. However, this contradicts the initial supposition that $b=\tau(P^i,B^i(P))$ since aP^ib. *Q.E.D.*

We denote by f_{π^0} the restriction of f to the domain $[\mathscr{D}^A(\pi^0)]^N$. This is meaningful because of Remark 6.2.

LEMMA 8: *There exists a RAM* σ *consistent with the N-tuple* $(\pi^1,\dots,\pi^N)$ *such that for all* $P\in[\mathscr{D}^A(\pi^0)]^N$, $f_{\pi^0}(P)=a$ *implies that* $a_j=b_j$ *where* $b=\tau(P^{\sigma(j)}_j,A_j)$ *for all* $j=1,\dots,M$.

PROOF: Since f is strategyproof, f_{π^0} must be strategyproof. It follows that there exists a RAM σ such that for all $P\in[\mathscr{D}^A(\pi^0)]^N$ we have: $f_{\pi^0}(P)=a$ implies that $a_j=b_j$ where $b=\tau(P^{\sigma(j)}_j,A_j)$ for all $j=1,\dots,M$. To conclude, it remains to prove that σ is consistent with $(\pi^1,\dots,\pi^N)$. Assume on the contrary that it is not, i.e. without loss of generality there exists $j,k\in\{1,\dots,M\}$ such that $\sigma(j)=1$, $k\in t(j,\pi^1)$, and $\sigma(k)=2$. Let $Q=\{1,\dots,M\}\setminus\{j,k\}$ and let $\tilde{\pi}^1=(\{1\},\dots,\{j,k\},\dots,\{M\})$. Observe that $\mathscr{D}^A(\tilde{\pi}^1)\subset\mathscr{D}^A(\pi^1)$ (Remark 6.2). Pick $a_j,b_j\in A_j$ and $a_k,b_k\in A_k$ such that $a_j\neq b_j$ and $a_k\neq b_k$.

Now let $P\in\mathscr{D}^A(\tilde{\pi}^1)\times[\mathscr{D}^A(\pi^0)]^{N-1}$ be such that:

(i) $\tau(P^l_r,A_r)=c_r$ for all $r\in Q$ when $l=1,2$ and for all $r\in\{1,\dots,M\}$ when $l=3,\dots,N$;

(ii) $(b_j,a_k,x_Q)P^2(b_j,b_k,x_Q)P^2(a_j,a_k,x_Q)P^2(a_j,b_k,x_Q)$ for all $x_Q\in A_Q$; and

(iii) $(a_j,a_k,x_Q)P^1(b_j,b_k,x_Q)P^1(a_j,b_k,x_Q)P^1(b_j,a_k,x_Q)$ for all $x_Q\in A_Q$.

(Observe that $P^1\notin\mathscr{D}(\pi^0)$.)

Let f^A be the restriction of f to the domain $[\mathscr{D}^A(\pi^0)]^N$. Let $\hat{P}^1\in\mathscr{D}^A(\pi^0)$ be such that $a_j=\tau(\hat{P}^1_j,A_j)$ and $a_k=\tau(\hat{P}^1_k,A_k)$. Since $\sigma(k)=2$, $a_k=\tau(P^2_k,A_k)$, and $\sigma(j)=1$, we conclude that $f^A(P|\hat{P}^1)=(a_j,a_k,c_Q)$. Therefore, $(a_j,a_k,c_Q)\in B^1(P)$. Since $(a_j,a_k,c_Q)=\tau(P^1,B^1(P))$ it follows from Lemma 7 that $f(P)=(a_j,a_k,c_Q)$. Let $\bar{P}^2\in\mathscr{D}^A(\pi^0)$ be such that $b_k=\tau(\bar{P}^2_k,A_k)$. It follows that for all $\tilde{P}^1\in\mathscr{D}^A(\pi^0)$, $f^A(P|\tilde{P}^1,\bar{P}^2)=x$ implies that $x_k=b_k$. Clearly, $\tau(P^1,B^1(P|\bar{P}^2))=(b_j,b_k,c_Q)$. From Lemma 7, we infer that $f^A(P|\bar{P}^2)=(b_j,b_k,c_Q)$. Therefore,

$(b_j, b_k, c_Q) \in B^2(P)$. Since $(b_j, b_k, c_Q)P^2(a_j, a_k, c_Q)$ we deduce from Lemma 7 that $f^A(P) = \tau(P^2, B^2(P))$ is different from (a_j, a_k, c_Q). We have a contradiction. *Q.E.D.*

We now conclude the proof of the Theorem. From Lemma 8 we know that there exists a RAM σ consistent with $(\pi^1, \dots, \pi^N)$ such that for all $P \in [\mathscr{D}^A(\pi^0)]^N$, $f(P) = a$ implies that $a_j = \tau(P_j^{\sigma(j)}, A_j)$ for all $j \in \{1, \dots, M\}$. It remains to show that f satisfies the same property when P is chosen arbitrarily from $\prod_{i \in I} \mathscr{D}(\pi^i)$. For all $i \in I$, let $a^i = \tau(P^i, A)$ and let $T^i \equiv \{k \in t(j, \pi^i)\}$. Pick $\bar{P}^i \in \mathscr{D}^A(\pi^i)$ such that:

(i) $\tau(\bar{P}^i, A) = c$ where $c_{T^i} = a^i_{T^i}$;

(ii) the minimal element in A according to $\bar{P}^i$ is d where $d_{M-T^i} = a^i_{M-T^i}$;

(iii) for all $x, y \in A$ such that $x_{M-T^i} \neq y_{M-T^i}$, $x\bar{P}^i y$ iff $x_{M-T^i} P_{M-T^i} y_{M-T^i}$.

Since $\mathscr{D}^A(\pi^i)$ is rich, such an ordering can be found. From Lemma 8, $f(\bar{P}) = (a^1_{T^1}, a^2_{T^2}, \dots, a^N_{T^N}) \equiv a$. Now, suppose $a\bar{P}^i z$ for some $z \in A$. It follows from (iii) that $z_{M-T^i} = a_{M-T^i}$. If $zP^i a$, then there must exist $Q \subset M$ such that $T^i \supset Q \in \pi^i$ and $z_Q P^i_Q a_Q$. However, this is impossible since $a_Q \equiv a^i_Q$ and $a^i = \tau(P^i, A)$ by assumption. Applying Lemma 1, we have $f(P) = a$. This implies that f is σ-maximal. The proof is complete. *Q.E.D.*

PROOF OF THEOREM 6.2: *Sufficiency*—Let σ be a RAM consistent with the N-tuple of partitions $(\pi^i, \dots, \pi^N)$. Pick $i \in I$ and let $Q \equiv \{j \in \{1, \dots, M\} | \sigma(j) = i\}$. Since σ is consistent with $(\pi^i, \dots, \pi^n)$, $Q = Q^1 \cup Q^2 \cup \cdots \cup Q^K$ where $Q^1, Q^2, \dots, Q^K \in \pi^i$. Pick $P \in \prod_{i \in I} \mathscr{D}^T(\pi^i)$ and let $a_{Q^l} = \tau^T(P^i, A_{Q^l})$ for all $l = 1, \dots, K$. Let f be σ-maximal. Then $f(P) = (a_Q, x_{M-Q})$ for some $x_{M-Q} \in A_{M-Q}$ where $a_Q \equiv (a_{Q^1}, \dots, a_{Q^l})$. Let $\bar{P}^i \in \mathscr{D}^T(\pi^i)$. It follows from the definition of f that $f(P|\bar{P}^i) = (b_Q, x_{M-Q})$ for some $b_Q \in A_Q$. Since $a_{Q^l} = \tau(P^i, A_{Q^l})$ for all $l = 1, \dots, K$, therefore, $(a_Q, x_{M-Q})P^i(b_Q, x_{M-Q})$ and i does not manipulate at P. Since P, i and $\bar{P}^i$ were chosen arbitrarily, the proof is complete.

Necessity—Let $f\colon \prod_{i \in I} \mathscr{D}^T(\pi^i) \to A$ be strategyproof. Let g be the restriction of f to the domain $\prod_{i \in I} \mathscr{D}(\pi^i)$. This is meaningful in view of Remark 6.3. Clearly, g must be strategyproof. It is easy to verify that since f is nonimposed, so is g. Applying Theorem 6.1 we deduce that there exists σ consistent with $(\pi^i, \dots, \pi^N)$ such that g is σ-maximal. We show that f is σ-maximal as well.

Suppose not. Assume that there exists $P \in \prod_{i \in I} \mathscr{D}^T(\pi^1)$ and $i \in I$ such that $f(P) = a$, $\tau(P^i, A) = b$, but $a_Q \neq b_Q$ where $Q = \{j \in \{1, \dots, M\} | \sigma(j) = i\}$. Since σ is consistent with $(\pi^1, \dots, \pi^N)$, $b_Q = \tau(P^i, A_Q)$. Let $\bar{P}^i \in \mathscr{D}^A(\pi^1)$ be such that:

(i) $b_Q \bar{P}^i_Q a_Q \bar{P}^i_Q c_Q$ for all $c_Q \in A_Q \setminus \{a_Q, b_Q\}$,

(ii) $a_{M-Q} \bar{P}^i_{M-Q} c_{M-Q}$ for all $c_{M-Q} \in A_{M-Q} \setminus \{a_{M-Q}\}$, and

(iii) for all $x, y \in A$ such that $x_{M-Q} \neq y_{M-Q}$, $x\bar{P}^i y$ if and only if $x_{M-Q} \bar{P}^i_{M-Q} y_{M-Q}$.

Consider $x \in A \setminus \{a\}$, such that $x\bar{P}^i a$. In view of (i)–(iii), it must be the case that $x_Q = b_Q$ and $x_{M-Q} = a_{M-Q}$. Since $b_Q = \tau(P^i, A_Q)$, $x = (b_Q, a_{M-Q})$

$P^i(a_Q, a_{M-Q}) = a$. Therefore, $aP^ix \to a\bar{P}^ix$ for all $x \in A \setminus \{a\}$. For all $l \in I \setminus \{i\}$, let $\bar{P}^l \in \mathscr{D}^A(\pi^l)$ be such that $\tau(P^l, A) = a$. Now consider the profile $\bar{P} \in \prod_{l \in I} \mathscr{D}^A(\pi^l)$. Since $f(P) = a$ and $aP^lx \to a\bar{P}^lx$, it follows from Lemma 1 that $f(\bar{P}) = a$. But $f(\bar{P}) = g(\bar{P})$ and g is σ-maximal. Therefore, $a_Q = b_Q$ and we have a contradiction. *Q.E.D.*

PROOF OF THEOREM 6.3: The sufficiency part of the theorem is obvious. We consider only the necessity part. Let f be a strategyproof SCF over the domain $\prod_{i \in I} \mathscr{D}(\pi^i)$. We know from Theorem 6.1 that there exists a RAM σ consistent with $(\pi^1, \ldots, \pi^N)$ such that f is σ-maximal. Suppose f is nondictatorial, i.e. there exists $j, k \in \{1, \ldots, M\}$ such that $\sigma(j) \neq \sigma(k)$. We will show that f is not efficient.

Consider $P \in [\mathscr{D}(\pi^0)]^N$ such that:

(i) for all $l = I \setminus \{\sigma(j)\}$, $\tau(P^l_r, A_r) = a_r$ for all $r \in \{1, \ldots, M\} \setminus \{j, k\}$ and $(a_j, a_k)P^l_{j,k}(a_j, b_k)P^l_{j,k}(b_j, a_k)P^l_{j,k}(b_j, b_k)$ for arbitrary $a, b \in A$ such that $a_k \neq b_k$.

(ii) $\tau(P^{\sigma(j)}_r, A_r) = a_r$ for all $r \in \{1, \ldots, M\} \setminus \{j, k\}$ and $(b_j, b_k)P^{\sigma(j)}_{j,k}(a_j, b_k)P^{\sigma(j)}_{j,k}(b_j, a_k)P^{\sigma(j)}_{j,k}(a_j, a_k)$.

Since f is σ-maximal, $f(P) = (b_j, a_k, a_{M-\{j,k\}})$. However, $(a_j, b_k, a_{M-\{j,k\}})$ $P^l(b_j, a_K, a_{M-\{j,k\}})$ for all $l \in I$. Therefore, f is not efficient. *Q.E.D.*

PROOF OF THEOREM 7.1: Once again, we consider only necessity since sufficiency is obvious.

Let f be a nonimposed strategyproof SCF over $\prod_{i \in I} \mathscr{D}^i$ where $\mathscr{D}^i$ is composite with respect to π^i for all $i \in I$. By assumption, $\mathscr{D}(\pi^i) \subset \mathscr{D}^T(\pi^i) \subset \mathscr{D}^i$. Let g be the restriction of f to the domain $\prod_{i \in I} \mathscr{D}(\pi^i)$. It follows from Theorem 6.1 that g is σ-maximal for some RAM σ consistent with $(\pi^1, \ldots, \pi^n)$.

LEMMA 9: *The RAM σ is constant, i.e. g is dictatorial.*

PROOF: Suppose the Lemma is false. Assume that there exists $i \in I$ such that the set $Q = \{j \in \{1, \ldots, M | \sigma(j) = i\}$ is a nonempty strict subset of $\{1, \ldots, M\}$. It follows from the consistency of σ with $(\pi^1, \ldots, \pi^N)$ that $Q = Q^1 \cup Q^2 \cup \cdots \cup Q^K$ where $Q^1, Q^2, \ldots, Q^K \in \pi^i$. From (ii) of Definition 7.1, we deduce that there exists $a_Q, b_Q \in A_Q$ $(a_Q \neq b_Q)$, $c_{M-Q}, d_{M-Q} \in A_{M-Q}$ and $P^i \in \mathscr{D}^i$ such that $(a_Q, c_{M-Q})P^i(x_Q, c_{M-Q})$ for all $x_Q \in A_Q$ and $(b_Q, d_{M-Q})P^i(a_Q, d_{M-Q})$. For all $j \neq i$, let $\bar{P}^j, P^j \in \mathscr{D}(\pi^0)$ be such that:

(i) $P^j_Q = \bar{P}^j_Q$,

(ii) $c_{M-Q} = u_{M-Q}$ where $u = \tau(P^j, A)$,

(iii) $d_{M-Q} = v_{M-Q}$ where $v = \tau(\bar{P}^j, A)$,

(iv) for all $x, y \in A$ such that $x_Q \neq y_Q$, xP^jy if and only if $x_QP^j_Qy_Q$,

(v) for all $x, y \in A$ such that $x_Q \neq y_Q$, $x\bar{P}^jy$ if and only if $x_Q\bar{P}^j_Qy_Q$.

We first claim that $f(P) = (a_Q, c_{M-Q})$. Suppose $f(P) = w \neq (a_Q, c_{M-Q})$. Either $(a_Q, c_{M-Q})P^iw$ or $wP^i(a_Q, c_{M-Q})$ must hold. Suppose the former is true.

Let $\tilde{P}^i \in \mathscr{D}(\pi^0)$ be such that $\tau(\tilde{P}^i, A) = a$. It follows from Theorem 6.1 that $f(P|\tilde{P}^i) = (a_Q, c_{M-Q})$ so that i will manipulate at P. Suppose $wP^i(a_Q, c_{M-Q})$. Let $\hat{P}^i \in \mathscr{D}(\pi^0)$ be such that $\tau(\hat{P}^i, A) = w$. It follows from Lemma 1 that $f(P|\hat{P}^i) = w$. Since $\hat{P}^i \in \mathscr{D}(\pi^0)$, Theorem 6.1 applies to the profile $P|\hat{P}^i$, so that $w_{M-Q} = c_{M-Q}$. Therefore, $(w_Q, c_{M-Q})P^i(a_Q, c_{M-Q})$, which contradicts an assumption regarding P^i.

Let $\bar{P}^i = P^i$. Our second claim is that if $f(\bar{P}) = z$, then $z_{M-Q} = d_{M-Q}$ and $z_Q \neq a_Q$. Let $f(\bar{P}) = z$ and let $\tilde{P}^i \in \mathscr{D}(\pi^0)$ be such that $\tau(\tilde{P}^i, A) = z$. From Lemma 1, we have $f(\bar{P}|\tilde{P}^i) = z$. Applying Theorem 6.1 to the profile $(\bar{P}|\tilde{P}^i)$, we deduce that $z_{M-Q} = d_{M-Q}$. Suppose $f(\bar{P}) = (a_Q, d_{M-Q})$. Let $\hat{P}^i \in \mathscr{D}(\pi^0)$ be such that $b = \tau(\hat{P}, A)$. Applying Theorem 6.1 again, $f(\bar{P}|\hat{P}^i) = (b_Q, d_{M-Q})$. Since $(b_Q, d_{M-Q})P^i(a_Q, d_{M-Q})$ by assumption, i will manipulate at $\bar{P}$.

Let $f(\bar{P}) = (z_Q, d_{M-Q})$. Let r be the first integer in the sequence $\{i, 1, 2, \ldots, i-1, i+1, \ldots, M\}$ such that $f(P|\bar{P}^i, \tilde{P}^1, \ldots, \tilde{P}^r) = w$ and $w_Q \neq a_Q$. Since by proceeding to the end of the sequence we obtain (z_Q, d_{M-Q}) and since we have proved $z_Q \neq a_Q$, we are assured of the existence of r. Moreover, $r \neq i$ since $P^i = \bar{P}^i$. Therefore, $P^r, \bar{P}^r \in \mathscr{D}(\pi^0)$. Either $w_Q P^r_Q a_Q$ or $a_Q P^r_Q w_Q$ must hold. Suppose the former is true. Applying (iv), we have $wP^r a$. Therefore, r manipulates at the profile $(P|\bar{P}^i, \bar{P}^1, \ldots, \bar{P}^{r-1})$. In the latter case, we have $a_Q \bar{P}^r_Q w_Q$ using (i) and $a\bar{P}^r w$ using (v). Therefore, r manipulates at $(P|\bar{P}^i, P^1, \ldots, \bar{P}^r)$. This proves the lemma. *Q.E.D.*

We now complete the proof of Theorem 7.1. Assume without loss of generality that $\sigma(j) = 1$ for all $j \in \{1, \ldots, M\}$. We claim that individual 1 is the dictator over the entire domain $\prod_{i \in I} \mathscr{D}^i$. Suppose the claim is false. Assume that there exists $P \in \prod_{i \in I} \mathscr{D}^i$ such that $a = f(P) \neq \tau(P^1, A) = b$. For all $i \neq 1$, let $\bar{P}^i \in \mathscr{D}(\pi^0)$ be such that $\tau(\bar{P}^i, A) = a$. It follows from Lemma 1 that $f(P^1, \bar{P}^2, \ldots, \bar{P}^N) = a$. Let $\bar{P}^1 \in \mathscr{D}(\pi^0)$ be such that $\tau(\bar{P}^1, A) = b$. Since $\bar{P} \in [\mathscr{D}(\pi^0)]^N \subset \prod_{i \in I} \mathscr{D}(\pi^i)$, we have $f(\bar{P}) = g(\bar{P}) = b$, from Lemma 9. Since $bP^1 a$ by assumption, individual 1 will manipulate at $(P^1, \bar{P}^2, \ldots, \bar{P}^N)$. This completes the proof. *Q.E.D.*

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Manuscript received 1992; final revision received September, 1997.

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