Pinching, Trimming, Truncating, and Averaging of Matrices

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To M. G. Nadkarni on his sixtieth birthday.

1. INTRODUCTION. What happens to the norm of a matrix when some of its entries are replaced by zeros? This question leads to some interesting mathematics and the goal of this paper is to describe some of it.

The answer to our question depends on two things: the norm that we choose and the position of the entries that are replaced. Let us illustrate this by examples.

The *operator norm* of an $n \times n$ complex matrix A is its norm as a linear operator on the Euclidean space $\mathbb{C}^n$; i.e., $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}$. The *Frobenius norm* of A is defined as $\|A\|_2 = (\sum_{i,j}|a_{ij}|^2)^{\frac{1}{2}} = (\operatorname{tr} A^*A)^{\frac{1}{2}}$. Both these norms are used frequently in analysis of matrices. They can also be described in terms of the singular values of A—the square roots of the eigenvalues of A^*A enumerated as $s_1(A) \geq \cdots \geq s_n(A)$. We have $\|A\| = s_1(A)$ and $\|A\|_2 = (\sum_j s_j^2(A))^{\frac{1}{2}}$.

Let $M(n)$ be the space of all $n \times n$ (complex) matrices. A norm $|||\cdot|||$ on $M(n)$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for all unitary matrices U,V and for all A in $M(n)$. Since $s_j(UAV) = s_j(A)$, the operator norm and the Frobenius norm are unitarily invariant. Another example of such a norm is the *trace norm* defined as $\|A\|_1 = \sum_j s_j(A)$. More examples, and properties, of these norms may be found in [**2**, Chapter IV].

Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is easy to see that $\|A\| = \sqrt{2}$, while $\|B\| = \frac{1}{2}(1 + \sqrt{5})$. Thus, replacing an entry of a matrix by zero can increase its operator norm. On the other hand, the Frobenius norm of a matrix is diminished if any of its entries is replaced by one with smaller absolute value. It is an interesting fact that among all unitarily invariant norms, the Frobenius norm is the only one that has this property; see [**4**, Prop. 3.1]. The matrix B was obtained from A by *triangular truncation*—wiping out the part below the main diagonal. We say more about this operation later on.

There is an interesting operation on $M(n)$ that reduces all unitarily invariant norms. Let $P_1, \ldots, P_k$ be orthogonal projection operators in $\mathbb{C}^n$ whose ranges are orthogonal to each other and whose sum $P_1 + \cdots + P_k = I$. Let

$$\mathscr{C}(A) = \sum_{j=1}^{k} P_j A P_j. \tag{1}$$

This is called a *pinching* of A. If we choose an orthonormal basis for $\mathbb{C}^n$ whose elements successively span the ranges of P_j, $1 \leq j \leq k$, then the matrix of A can be decomposed into blocks in such a way that the diagonal blocks are square and have sizes dim range P_j and the matrix $\mathscr{C}(A)$ is obtained from this by replacing all the off-diagonal blocks by zeros. Especially interesting is the case when the range of each P_j is 1-dimensional. In this case the pinching replaces all off-diagonal entries of A by zeros. The resulting matrix, the *diagonal part* of A, is written as $\mathscr{D}(A)$.

The sum in (1) is reminiscent of a convex combination, and indeed that is one reason for the interest in pinchings—they describe certain averaging operations on operators. There is another sense in which $\mathscr{C}(A)$ can be obtained from A by averaging. Let $\omega = e^{2\pi i/n}$ and let U be the diagonal matrix with entries $1, \omega, \omega^2, \dots, \omega^{n-1}$ down its diagonal. Using the identity $\sum_{j=0}^{n-1} \omega^j = 0$ and elementary algebra one can see that

$$\mathscr{D}(A) = \frac{1}{n} \sum_{j=0}^{n-1} U^j A U^{*j}. \tag{2}$$

The matrix U is unitary and the expression (2) represents $\mathscr{D}(A)$ as an average of n unitary conjugates of A. Using the same idea one can write any pinching $\mathscr{C}(A)$ corresponding to k orthogonal projections as an average of k unitary conjugates of A.

It follows from (2) that $|||\mathscr{D}(A)||| \le |||A|||$ for *every* unitarily invariant norm. This is a consequence of the triangle inequality and the fact that each of the summands in (2) has the same norm as A. More generally, we have $|||\mathscr{C}(A)||| \le |||A|||$ for every pinching $\mathscr{C}$. For the sake of brevity, we write $|||X||| \le |||Y|||$ to mean that for two given matrices X, Y, every unitarily invariant norm of X is bounded by the corresponding norm of Y.

This idea of representing diagonals as averages can be carried much further, as we will soon see. We will replace the sum in (2) by an integral and the roots of unity by trigonometric polynomials. Such representations lead to interesting bounds for norms of matrices obtained by trimming A in different ways.

2. DIAGONALS AS AVERAGES. In addition to the main diagonal of A we consider other diagonals as well. Let σ be any permutation of the indices $\{1, 2, \dots, n\}$ and let $\mathscr{G}(A)$ be the matrix obtained from A by replacing all its entries except $a_{j\sigma(j)}$ by zeros. This is a *generalized diagonal* of A obtained by retaining exactly one entry from each row and each column. When σ is the identity permutation $\mathscr{G}(A) = \mathscr{D}(A)$. To each σ corresponds a permutation matrix P_σ that permutes the basis vectors. One can see that $\mathscr{G}(A)$ is just the main diagonal of the matrix AP_σ. Hence $|||\mathscr{G}(A)||| \le |||AP_\sigma|||$. So, by unitary invariance $|||\mathscr{G}(A)||| \le |||A|||$, another instance when norms are diminished when some entries are replaced by zeros.

Now for $1 \le j \le n-1$, let $\mathscr{D}_j(A)$ be the matrix obtained from A by replacing all its entries except those on the jth superdiagonal by zeros. Likewise, let $\mathscr{D}_{-j}(A)$ be the matrix obtained by retaining only the jth subdiagonal of A. (The superdiagonals are the diagonals above the main diagonal and parallel to it; the subdiagonals are the ones below the main diagonal.) To be consistent with this notation, put $\mathscr{D}_0(A) = \mathscr{D}(A)$.

How big are the norms of $\mathscr{D}_j(A)$? Note that for each $1 \le j \le n-1$, the sum $\mathscr{D}_j(A) + \mathscr{D}_{j-n}(A)$ is a generalized diagonal $\mathscr{G}(A)$. Hence $|||\mathscr{D}_j(A) + \mathscr{D}_{j-n}(A)||| \le |||A|||$. Once again, by a permutation we can bring all the nonzero entries of $\mathscr{D}_j(A) + \mathscr{D}_{j-n}(A)$ to the main diagonal. This does not change norms. The norm of a diagonal matrix is certainly reduced if any of its entries is replaced by a zero. (Hint: express the new matrix as an average of the original matrix and another one of equal norm.) Thus

$$|||\mathscr{D}_j(A)||| \le |||A||| \quad \text{for all } j. \tag{3}$$

Using the triangle inequality, we obtain from this

$$|||\mathscr{D}_j(A) + \mathscr{D}_{-j}(A)||| \le 2|||A||| \quad \text{for all } j. \tag{4}$$

By the same argument, if $\mathscr{T}_3(A) = D_{-1}(A) + \mathscr{D}_0(A) + \mathscr{D}_1(A)$ is the *tridiagonal part* of A, then

$$|||\mathscr{T}_3(A)||| \le 3|||A|||. \tag{5}$$

A slightly cleverer argument gives a better inequality. We can write $\mathscr{T}_3(A) = \mathscr{C}(A) + \Gamma(A)$, where $\mathscr{C}(A)$ is a pinching of A by 2×2 blocks and $\Gamma(A)$ is a part of a generalized diagonal of A. For example, if we write the tridiagonal part of a 5×5 matrix A as

$$\begin{bmatrix} \bigcirc & \bigcirc & & & \\ \bigcirc & \bigcirc & \star & & \\ & \star & \bigcirc & \bigcirc & \\ & & \bigcirc & \bigcirc & \star \\ & & & \star & \bigcirc \end{bmatrix}$$

then all the entries represented by circles together constitute a pinching of A into blocks of sizes 2, 2, and 1. The remaining entries represented by stars constitute a part of a generalized diagonal of A. This shows that

$$|||\mathscr{T}_3(A)||| \le 2|||A|||. \tag{6}$$

Can one improve this further? How about the inequality (4)?

For each real number θ, let U_θ be the diagonal matrix with entries $e^{ir\theta}$, $1 \le r \le n$, down its diagonal. Then the (r, s) entry of the matrix $U_\theta A U_\theta^*$ is $e^{i(r-s)\theta} a_{rs}$. Hence, we have

$$\mathscr{D}_k(A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} U_\theta A U_\theta^* \, d\theta. \tag{7}$$

When $k = 0$, this gives another representation of $\mathscr{D}_0(A)$ as an average over unitary conjugates of A. For other values of k, this expresses $\mathscr{D}_k(A)$ as a "twisted average" over unitary conjugates of A. From this expression we can again derive the inequality (3). We can also use it to write

$$\mathscr{D}_k(A) + \mathscr{D}_{-k}(A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2\cos k\theta) U_\theta A U_\theta^* \, d\theta.$$

Hence

$$|||\mathscr{D}_k(A) + \mathscr{D}_{-k}(A)||| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |2\cos k\theta| \, d\theta |||A|||.$$

It is easy to evaluate this integral. One gets

$$|||\mathscr{D}_k(A) + \mathscr{D}_{-k}(A)||| \le \frac{4}{\pi} |||A|||. \tag{8}$$

This is an improvement over the inequality (4). Using the same argument we see that

$$|||\mathscr{T}_3(A)||| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 + 2\cos\theta| \, d\theta \, |||A|||.$$

Once again, it is easy to evaluate the integral. We now get

$$|||\mathscr{T}_3(A)||| \le \left(\frac{1}{3} + \frac{2\sqrt{3}}{\pi} \right) |||A|||. \tag{9}$$

This is an improvement over (6). The constant factor in the inequality (9) is smaller than 1.436, that in (8) is smaller than 1.274.

More generally, consider the *trimming* of A obtained by replacing all its diagonals outside the band $-k \leq j \leq k$ by zeros; i.e., consider the matrices

$$\mathscr{T}_{2k+1}(A) = \sum_{j=-k}^{k} \mathscr{D}_j(A), \quad 1 \leq k \leq n. \tag{10}$$

Then, from (7) we get

$$\mathscr{T}_{2k+1}(A) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_k(\theta) U_\theta A U^*_\theta \, d\theta, \tag{11}$$

where

$$D_k(\theta) = \sum_{j=-k}^{k} e^{ij\theta} \tag{12}$$

is the *Dirichlet kernel*, a familiar object related to Fourier series. See [**3**, Sec. 2.2] or [**9**, p. 174]. The numbers

$$L_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_k(\theta)| \, d\theta \tag{13}$$

are called the *Lebesgue constants*. It is known that for large values of k, L_k are like $\log k$. For example, one knows that

$$L_k \leq \log k + \log \pi + \frac{2}{\pi}\left(1 + \frac{1}{2k}\right),$$

and that

$$L_k = \frac{4}{\pi^2}\log k + O(1).$$

From (11) we see that

$$|||\mathscr{T}_{2k+1}(A)||| \leq L_k |||A|||. \tag{14}$$

Naive arguments would have shown only that $|||\mathscr{T}_{2k+1}(A)||| \leq (2k+1)|||A|||$. The inequality (14) is a striking improvement, when n, k, and $n-k$ are large.

The trimming operation we have introduced here has an interesting connection with the triangular truncation that we talked of in the Introduction. Let Δ_U be the linear map on the space of matrices (of a fixed size) that takes a matrix B to its upper triangular part; i.e., Δ_U acts by replacing all entries of a matrix below the main diagonal by zeros. Given a $k \times k$ matrix B, consider the matrix $A = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}$. The singular values of A are the singular values of B counted twice as often. Hence, $\|A\| = \|B\|$. Note also that

$$\mathscr{T}_{2(k+1)+1}(A) = \begin{bmatrix} 0 & \Delta_U(B)^* \\ \Delta_U(B) & 0 \end{bmatrix}.$$

So, it follows from (14) that

$$\|\Delta_U(B)\| \leq L_{k+1}\|B\|. \tag{15}$$

In other words, the norm of the triangular truncation operator (on the space of $k \times k$ matrices equipped with the operator norm) is bounded by the Lebesgue constant L_{k+1}. We have remarked earlier that $L_{k+1} \approx 4\pi^{-2}\log k$.

It is remarkable that arguments from Fourier series lead to interesting bounds (8), (9), (14), and (15) for norms of matrices. The unexpected appearance of the number π makes them especially attractive. Of course, this appeal would be lost if better bounds were to be found. As it turns out, the bounds (8) and (9) are sharp, as are the bounds (14) and (15) in an asymptotic sense. This is discussed in the next section.

3. EXAMPLES.

We show that the bounds (8) and (9) are sharp for the trace norm and, therefore, by a duality argument they are sharp for the operator norm.

Example 3.1. Let B be the tridiagonal $n \times n$ matrix with each entry on its first superdiagonal and the first subdiagonal equal to 1, and all other entries equal to 0. It is not difficult to see that the eigenvalues of B are $2\cos(j\pi/n+1)$, $1 \le j \le n$; see [**2**, p. 60]. The matrix $B - 2I$ is the familiar second difference matrix that is used in numerical analysis to discretize the second derivative operator.

Example 3.2. Let $A = E$, the matrix with all entries equal to 1. Then

$$\mathscr{D}_1(A) + \mathscr{D}_{-1}(A) = B,$$

where B is the tridiagonal matrix in Example 3.1. Here

$$\frac{\|B\|_1}{\|A\|_1} = \frac{1}{n}\sum_{j=1}^{n}\left|2\cos\frac{j\pi}{n+1}\right|. \tag{16}$$

Let $f(\theta) = |2\cos\theta|$. The sum

$$\frac{1}{n+1}\sum_{j=1}^{n+1}\left|2\cos\frac{j\pi}{n+1}\right|$$

is a Riemann sum for the function $\pi^{-1}f(\theta)$ over a subdivision of the interval $[0, \pi]$ into $n+1$ equal parts. As $n \to \infty$, this sum and the one in (16) tend to the same limit. This limit is equal to

$$\frac{1}{\pi}\int_0^{\pi}|2\cos\theta|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|2\cos\theta|\,d\theta = \frac{4}{\pi}.$$

This shows that the inequality (8) can not be improved if it has to be valid for all dimensions n and for all unitarily invariant norms.

The same example shows that the inequality (9) is also sharp.

Note that in this example A was Hermitian, so the inequalities (8) and (9) are sharp even on the space of all Hermitian matrices.

The duality principle we need says that if T is a linear operator from a Banach space X to another Banach space Y, then its adjoint T^* (a linear map from the dual Y^* to X^*) has the same norm as T.

Let X be the space $M(n)$, or the space $H(n)$ of all Hermitian $n \times n$ matrices. The space X has a natural inner product defined as $\langle A, B\rangle = \operatorname{tr} A^*B$. By the Riesz Representation Theorem every linear functional φ on X is of the form $\varphi(A) = \operatorname{tr} A\Phi$, where Φ is some element of X. The norms $\|\cdot\|$ and $\|\cdot\|_1$ on X are dual to each other; i.e., if X is equipped with the operator norm $\|\cdot\|$, then its dual space X^* is the space X equipped with the trace norm $\|\cdot\|_1$, and vice versa. The adjoint of a linear operator $\mathscr{L}: X \to X$, is the linear operator $\mathscr{L}^*: X \to X$ that satisfies the relation

$$\langle A, \mathscr{L}(B)\rangle = \langle \mathscr{L}^*(A), B\rangle \quad \text{for all } A, B.$$

It is easy to verify that for each k the operator taking a matrix A to $\mathscr{D}_k(A) + \mathscr{D}_{-k}(A)$ is its own adjoint. Hence, its norm as a linear operator on the space $(X, \|\cdot\|)$ is the same as its norm as a linear operator on the space $(X, \|\cdot\|_1)$. Thus the inequalities (8) and (9) are sharp for the operator norm as well. Further, they are sharp even when A is Hermitian.

Example 3.3. Let A be the $n \times n$ matrix with entries $a_{ij} = (i-j)^{-1}$ if $i \neq j$, and $a_{ii} = 0$. With some work [**6**, p. 39] it can be seen that $\|A\| \leq \pi$ and $\|\Delta_U(A)\| \geq \frac{4}{5}\log n$ for large values of n. See the delightful article [**5**] for several examples related to this. This example shows that the norm of the operator Δ_U on $(M(n), \|\cdot\|)$ or $(H(n), \|\cdot\|)$ grows like $\log n$. More elaborate analysis shows that the norm of Δ_U approaches $\pi^{-1}\log n$ as n increases; see [**1**]. Our inequality (15) gives just a little larger number $4\pi^{-2}\log n$ as an asymptotic bound for this norm.

Example 3.4. Let B be a Hermitian matrix of a large order k for which $\|\Delta_U(B)\|$ is close to $\pi^{-1}\log k$; such a matrix exists, as we have remarked in our discussion of Example 3.3. Let $A = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$. Then $\|A\| = \|B\|$.

Note that

$$\mathscr{T}_{2k+3}(A) = \begin{bmatrix} 0 & \Delta_U(B)^* \\ \Delta_U(B) & 0 \end{bmatrix}$$

and the norm of this matrix is $\|\Delta_U(B)\|$. So, the ratio of $\|\mathscr{T}_{2k+3}(A)\|$ and $\|A\|$ is approximately $\pi^{-1}\log k$, again showing that the bound (14) is almost exact for the operator norm (and by duality for the trace norm).

4. MORE ON AVERAGES.

For I, a subset of $\{1, 2, \dots, n\}$, let X_I be the diagonal matrix whose diagonal entry x_{ii} is 1 if $i \in I$, and -1 if $i \notin I$. It can be seen easily that

$$\mathscr{D}(A) = \frac{1}{2^n}\sum_I X_I A X_I. \tag{17}$$

This expression has the advantage of using real diagonal matrices X_I instead of the complex matrices U^k used in (2). Further, these matrices have only ± 1 on the diagonal. So, an analogous expression can be used for matrices over other fields. On the other hand, now there are far more terms involved. This difference is crucial. In [**4**] it was shown that (2) leads to the bound

$$|||A - \mathscr{D}(A)||| \leq 2\left(1 - \frac{1}{n}\right)|||A|||$$

for the off-diagonal part of A, and that this inequality is sharp for the norms $\|\cdot\|$ and $\|\cdot\|_1$. Using (17) one would obtain a weaker inequality here.

Is it possible to obtain a representation for $\mathscr{D}(A)$ using real diagonal conjugations as in (17), but with fewer terms? It has been shown [**4**] that we could not have fewer than n terms in any case.

This question has an amusing connection with a famous problem in the theory of design of experiments. A matrix all of whose entries are ± 1, and whose columns are mutually orthogonal, is called a *Hadamard matrix*. Do such matrices exist? The 2×2 matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a Hadamard matrix. Taking m-fold tensor products of this matrix with itself gives Hadamard matrices of order $n = 2^m, m = 1, 2\dots$. It is not difficult to see that for $n > 2$, a necessary condition for the existence of a Hadamard matrix of order n is that $n = 4k$ for some k. It has been conjectured that this condition is sufficient as well. This conjecture has been proved for $k \leq 106$; see [**10**].

Suppose n is such that a Hadamard matrix of order n exists. Let Y_j, $1 \leq j \leq n$, be the diagonal matrix whose diagonal is the jth column of this Hadamard matrix. Then,

$$\mathscr{D}(A) = \frac{1}{n} \sum_{j=1}^{n} Y_j A Y_j.$$

So, for such values of n, one does have a representation of $\mathscr{D}(A)$ as an average of n real diagonal conjugates of A.

If the conjecture on Hadamard matrices were to have a positive solution, then for all n we could find such a representation for $\mathscr{D}(A)$ with at most $n + 3$ terms.

Although we have concentrated on norms in this article, there is a long tradition in matrix theory of comparing eigenvalues, determinants, and singular values of a matrix to those of its diagonal. Several famous results due to Schur, Hadamard, Mirsky, Fan, Thompson, and others belong to this tradition. The interested reader could find them in the books [**2**], [**7**], and [**8**].

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