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A SOLUTION OF THE MARTINGALE CENTRAL LIMIT PROBLEM, PART II

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SUMMARY. Consider a sequence of triangular arrays $\{(X_{nj}, \mathcal{A}_{n,j-1}), 1 \leqslant j \leqslant k_n\}$ where for each row, the σ-fields $\mathcal{A}_{n0} \subset \mathcal{A}_{n1} \subseteq \cdots \subseteq \mathcal{A}_{nk_n}$ and X_{nj} is $\mathcal{A}_{n,j}$-measurable, $1 \leqslant j \leqslant k_n$, defined on some complete probability space, such that $k_n \uparrow \infty$ and $\mathcal{A}_{nj} \subset \mathcal{A}_{n+1,j}$ for all $j \leqslant k_n$. In Part I of this paper, necessary and sufficient conditions for the stable convergence in distribution of the sequence $S_n = \sum_1^{k_n} X_{nj}$, $n \geqslant 1$, to a mixture of infinitely divisible distributions were investigated under the assumption that the second moments of X_{nj}'s exist. In Part II, the existence of the first or second moments is not assumed and we obtain results analogous to the case of independent summands with not necessarily finite first or second moments. As particular cases, convergences to mixtures of normal, degenerate and Poisson distributions are discussed. It may be noted that, though the title of the paper refers only to martingales, the present set-up goes beyond that of martingales since even the existence of the first moments is not assumed.

Furthermore, it is shown that the class of all limit distributions of $\{S_n\}$ coincides, in some sense, to the class of all mixtures of infinitely divisible distributions.

1. Introduction and the results

Consider a sequence of triangular arrays $\{(X_{nj}, \mathcal{A}_{n,j}), 1 \leqslant j \leqslant k_n\}$ where, for each row, the σ-fields $\mathcal{A}_{n1} \subseteq \ldots \subseteq \mathcal{A}_{n,k_n}$ and X_{nj} is $\mathcal{A}_{n,j}$-mble, $1 \leqslant j \leqslant k_n$, defined on some complete probability space $(\mathscr{X}, \mathcal{A}, P)$, such that

$$\text{(A.1)} \qquad k_n \uparrow \infty \text{ and } \mathcal{A}_{nj} \subseteq \mathcal{A}_{n+1,j} \text{ for all } j \leqslant k_n.$$

In Part I of this paper, necessary and sufficient conditions for the stable convergence in distribution of the sequence $S_n = \sum_1^{k_n} X_{nj}$, $n \geqslant 1$, to a mixture of infinitely divisible distributions were investigated under the assumptions that the second moments of X_{nj}'s exist and that, for each row, X_{nj}'s are martingale differences. The results obtained there were analogous to the case of independent summands with finite variances. In Part II, the existence of the first or second moments is not assumed and we try to obtain results analogous to the case independent summands with not necessarily finite first or second moments.

One reason for having treated the case of bounded variances separatel as Part I of this paper is that it was found simple and convenient to introduc the basic ideas of the proofs. The technical essence of this part remair the same as that of the first part, though some of the computations an arguments are more delicate. For this reason some of the arguments of tl proofs will be either referred to Part I or only briefly indicated. Anoth reason for having a separate treatment of the general case is that the resul presented here are far more complete and that even the sufficiency part the particular normal convergence criterion seems to clarify the existi results to a great extent. Moreover, Poisson and degenerate convergenc are also discussed

A subsequent part will be devoted to a study of limit theorems f cumulative or normed sums, and Part III is devoted to a study of invarian principles with Lévy processes limits.

For the references to earlier results, see Part I. Regarding the nam associated with the formulation and solution of the problem in the case independent summands, we quote the following sentences from Loève (19 p. 290) : "The solution of the problem is due to the introduction, by de Finet of the 'infinitely decomposable' family of laws and to the discovery of th explicit representation by Kolmogorov in the case of finite second momer and by P. Lévy in the general case. It has been obtained, with the help preceding family of laws, by the efforts of Kolmogorov, P. Lévy, Feller, Baw Khintchine, Marcinkiewicz, Gnedenko, and Doblin (1931–1938). The fir form is essentially due to Gnedenko". To these names, Hsu also should added (see Chung (1979)).

As in Part I, we assume in the rest of the paper that

$$\mathscr{A} = \sigma(\bigcup_n \mathscr{A}_{n,k_n}).$$

'$\xrightarrow{P}$ 0' denotes convergence in probability.

Before going into the details of the conditions and the results, we introdu some notations.

Let $F_{nj}(x, w)$ be a regular conditional distribution of X_{nj} given $\mathscr{A}_{n,j}$ Set, for some fixed $\tau > 0$,

$$a_{nj} = \int_{|x| < \tau} x F_{nj}(dx, w).$$

and define

$$\bar{F}_{nj}(x, w) = F_{nj}(x + a_{nj}, w).$$

We set

$$\psi_n(t, w) = it\beta_n + \int \phi(t, x) G_n(dx, w)$$

where

$$\beta_n = \sum_1^{k_n} \left[a_{nj} + \int \frac{x}{1+x^2} \bar{F}_{nj}(dx, w) \right],$$

$$G_n(y, w) = \sum_1^{k_n} \int_{-\infty}^{y} \frac{x^2}{1+x^2} \bar{F}_{nj}(dx, w),$$

and the function $\phi(t, x)$ is defined by

$$\phi(t, x) = \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} \qquad \text{if } x \neq 0$$

$$= -t^2/2 \qquad \text{if } x = 0.$$

Note that the function $\phi(t, x)$ is jointly continuous in t and x. We further set $T_n = G_n(\infty, w)$.

In order to simplify some of the statements we now introduce

Definition 1 : A sequence $\{W_n\}$ of random vectors defined on $(\mathscr{X}, \mathscr{A}, P)$ is said to be *relatively compact in probability* if for every subsequence there exists a further subsequence $\{m\}$ such that the subsequence $\{W_m\}$ converges in probability.

We now introduce the conditions.

(A.2) For every $\varepsilon > 0$

$$\max_{1 \leq j \leq k_n} P(|X_{nj}| > \varepsilon | \mathscr{A}_{n,j-1}) \xrightarrow{p} 0.$$

(A.3) The sequence $\{T_n\}$ is relatively compact in probability.

(A.4) There exists a countable dense subset D of R such that for all $t \in D$, the sequence $\{\psi_n(t)\}$ is relatively compact in probability. Furthermore, for every subsequence there exists a further subsequence $\{m\} \subset \{n\}$ such that, for every $\varepsilon > 0$,

$$\lim_{a \to \infty} \limsup_{m \to \infty} P[G_m(|x| > a) > \varepsilon] = 0. \qquad (*)$$

Note that (*) is equivalent to the more meaningful condition : for every $\varepsilon > 0$,

$$\lim_{a \to \infty} \limsup_{m \to \infty} P\left[\sum_1^{k_m} P(|X_{mj} - a_{nj}| > a | \mathscr{A}_{m,j-1}) > \varepsilon \right] = 0.$$

In order to avoid confusion, relative compactness under weak convergence of distribution functions will be referred as relative compactness in distribution.

Remark 1 : As is well known, when the summands are independent and when it is given that the sequence $\{S_n\}$ is relatively compact in distribution, the conditions (A.3) and (A.4) are automatically satisfied under the condition (A.2).

Following Loève (1963, p. 381), the condition (A.2) will be called Conditional Uniform Asymptotic Negligibility (CUAN) condition.

In what follows by couple (β, G) we mean that β is an $\mathscr{A}$-mble r v and G is an $\mathscr{A}$-mble kernel as defined in Part I (Definition 2).

Definition 2 : The sequence $\{S_n\}$ is said to converge stably in distribution to the distribution of a mixture of infinitely divisible distributions with the couple (β, G) if, for every $\mathscr{A}$-mble function g,

$$E(e^{iug+itS_n}) \to E(e^{iug+\psi(t)})$$

for all real u and t, where

$$\psi(t) = it\beta + \int \phi(t, w)\, G\,(dx, w).$$

Henceforth "mixture of infinitely divisible distributions with couple (β, G)" will be abbrevited as "MIDD with (β, G)".

We next state the results. It is assumed, without further mentioning, that (A.1) is satisfied in the statements of all the results stated below.

Theorem 1 : (i) *Suppose that the conditions* (A.3) *and* (A.4) *are satisfied. Then for every subsequence there exists a further subsequence* $\{m\} \subseteq \{n\}$, *a P-null set N and a couple* (β, G) *such that, for each* $w \in \mathscr{X} - N$, $\beta_m \to \beta$ *and* G_m *converges weakly to* G, *i.e.*,

$$\int e^{itx} G_m(dx) \to \int e^{itx} G(dx)$$

for all real t and $w \in \mathscr{X} - N$.

(ii) *Suppose that the CUAN condition* (A.2) *is satisfied. Let the subsequence* $\{m\}$ *and the couple* (β, G) *be as in the above statement* (i). *Then the sequence* $\{S_m\}$ *converges stably in distribution to the MIDD with* (β, G).

Theorem 2 : (i) *The condition,*

(A.5) : *there exists a couple* (β, G) *such that*

$$\beta_n \xrightarrow{p} \beta$$

and, for all real t,

$$\int e^{itx} G_n(dx) \xrightarrow{p} \int e^{itx} G(dx),$$

is equivalent to the conditions (A.3) *and*

(A.6) : *For all real t, the sequence $\{\psi_n(t)\}$ converges in probability. Furthermore, for every subsequence there exists a further subsequence $\{m\} \subset \{n\}$ such that the condition* (*) *of* (A.4) *holds.*

(ii) *Suppose that the conditions* (A.2) *and* (A.5) *are satisfied. Then the sequence $\{S_n\}$ converges stably distribution to the MIDD with (β, G).*

Theorem 3 : *Under the conditions* (A.2)–(A.4), *the class of all distributions which are the limits, in the sense of stable convergence in distribution, of the sequence $\{S_n\}$ coincides with the class of all MIDD with some couple (β, G).*

Theorem 4 : *Suppose that the conditions* (A.2)–(A.4) *are satisfied. Further suppose that the sequence $\{S_n\}$ converges stably in distribution. Then the condition* (A.5) *holds.*

Remark 2 : It may happen that the sequence $\{\psi_n(t)\}$ is not relatively compact in probability, yet the sequence

$$\psi'_n(t) = it(\beta_n - \gamma_n) + \int \phi(t, x) G_n(dx)$$

is relatively compact in probability for suitably chosen sequence of r.v's γ_n. In such a case necessary and sufficient conditions for the stable convergence in distribution of the sequence $S_n - \gamma_n$, $n \geqslant 1$, can be formulated. Since the details of the statements will be clear from the proofs of the results (Section 3) and from the known results for the case of independent summands (cf. Loève (1963, pp. 310 and 314), they are omitted.

Analogous to the case of independent summands, we next state the central convergence criterion which is very useful in applying the results to particular cases.

In the statements of this criterion, when we say that a kernel G has a fixed point $\tau > 0$ of continuity we mean that there is a P-null set N such that $\pm\tau$ are continuity points of $G(., w)$ for all $w \in \mathcal{X} - N$.

Remark 3 : It may be noted that a kernel G has always a fixed point of continuity. To see this first note that there is no loss of generality in assuming $G(\infty, w) \leqslant 1$ a.s. Then

$$K(x) = E(G(x, w))$$

is a non-decreasing, right continuous function of bounded variation. Hence there always exists a point τ such that $\pm\tau$ is a continuity point of $K(x)$. Now note that, since $G(x, w) \leqslant 1$ a.s.

$$K(\tau-)-K(\tau) = E[G(\tau-)-G(\tau)].$$

Hence, since τ is a continuity point of K,

$$E[G(\tau-)-G(\tau)] = 0.$$

This implies, since $G(\tau)-G(\tau-) \geqslant 0$,

$$G(\tau-) = G(\tau) \text{ a.s.}$$

Similarly, it follows that, since $-\tau$ is also a continuity point of K,

$$G(\tau'-) = G('\tau) \text{ a.s.}$$

where $\tau' = -\tau$.

Central convergence criterion. Suppose that the CUAN-condition (A.2) holds. Then the condition (A.3) and (A.4) holds and that the sequence $\{S_n\}$ converges stably in distribution to MIDD with (β, G) if, and only if, for every subsequence there exists a further subsequence $\{m\} \subseteq \{n\}$ and a P-null set N such that

(i) whenever $w \in \mathscr{X}-N$ and $x(w)$ is a continuity point of $G(., w)$

$$\sum_1^{k_m} F_{mj}(x(w)) \to \int_{-\infty}^{x(w)} \frac{1+y^2}{y^2} G(dy, w) \qquad \text{for } x(w) > 0,$$

$$\sum_1^{k_m} [1-F_{mj}(x(w))] \to \int_{x(w)}^{\infty} \frac{1+y^2}{y^2} G(dy, w) \quad \text{for } x(w) < 0$$

(ii) whenever $w \in \mathscr{X}-N$

$$\lim_{\epsilon \to 0} \limsup_{m \to \infty} W_m(\epsilon, w) = \lim_{\epsilon \to 0} \liminf_{m \to \infty} W_m(\epsilon, w)$$
$$= G(+0, w)-G(-0, w)$$

where

$$W_n(\epsilon, w) = \sum_1^{k_n} \left\{ \int_{|x| < \epsilon} x^2 F_{nj}(dx, w) - \left[\int_{|x| < \epsilon} x F_n(dx, w) \right]^2 \right\}$$

(iii) For a fixed $\tau > 0$ of fixed point of continuity of G, with the exceptional set N',

$$\sum_1^{k_m} \int_{|x| < \tau} x F_{mj}(dx\ w) \to \beta + \int_{|x| < \tau} x G(dx, w) - \int_{|x| \geqslant \tau} x^{-1} G(dx, w)$$

whenever $w \in \mathscr{X}-N \cup N'$.

As corollaries to this criterion we obtain the following particular cases. In what follows we set

$$\sigma^2_{nj}(\alpha) = \int_{|x| < \alpha} x^2 F_{nj}(dx) - \left[\int_{|x| < \alpha} x F_{nj}(dx) \right]^2$$

and

$$a_{nj}(\alpha) = \int_{|x| < \alpha} x F_{nj}(dx).$$

Normal convergence criterion. The conditions (A.2)−(A.4) hold and that the sequence $\{S_n\}$ converges stably in distribution to the distribution whose characteristic function is given by $E[\exp(it\beta - t^2T/2)]$ if, and only if, for every $\varepsilon > 0$ and for some $\alpha > 0$

(i) $$\sum_1^{k_n} P(|X_{nj}| > \varepsilon \mid \mathscr{A}_{n,\,j-1}) \xrightarrow{p} 0$$

(ii) $$\sum_1^{k_n} \sigma^2_{nj}(\alpha) \xrightarrow{p} T \text{ and } \sum_1^{k_n} a_{nj}(\alpha) \xrightarrow{p} \beta.$$

Remark 4 : The sufficiency part of this result with the usual weak convergence, for the particular case $T = 1$ and $\beta = 0$, already occurs in Helland (1981).

An important thing that remains is to verify that the above conditions are implied by the variety of sufficient conditions introduced by several authors. This has been done to some extent by Helland (1981) for the above mentioned particular case, and it is clear that his argument can be applied to the general case also.

Remark 5 : One might be interested to know whether the usual invariance principles with Brownian motion limit hold under the above conditions. Helland (1981) has shown that this is true for the above mentioned particular case, and it is easy to see that this is true for the general case also. But the more interesting problem seems to be the following. In his important work, Skorokhod (1957) has considered the invariance principles with limits Infinitely Divisible Processes also, of which Brownian motion limit is only a particular case. One might be interested in knowing whether his results extend to the general case treated in the present paper. A detailed investigation of this case is presented as Part III.

Convergence in probability criterion. The conditions (A.2)–(A.4) hold and that the sequence $\{S_n\}$ converges in probability to an rv β if, and only if, for every $\varepsilon > 0$ and for some $\alpha > 0$,

(i) $$\sum_1^{k_n} P(|X_{nj}| > \varepsilon \mid \mathcal{A}_{n, j-1}) \xrightarrow{p} 0,$$

(ii) $$\sum_1^{k_n} \sigma_{nj}^2(\alpha) \xrightarrow{p} 0 \text{ and } \sum_1^{k_n} a_{nj}(\alpha) \xrightarrow{p} \beta.$$

Poisson convergence criterion. Suppose that CUAN-condition (A.2) holds. Then the conditions (A.3) and (A.4) holds and that the sequence $\{S_n\}$ converges stably in distribution to the mixtures of Poisson distributions with ch.f. $E[\exp(\lambda(e^{it}-1))]$ if, and only if, for every $\varepsilon \in (0, 1)$ and for some $\alpha \in (0, 1)$

(i) $$\sum_1^{k_n} P(|X_{nj}| \geqslant \varepsilon, |X_{nj}-1| \geqslant \varepsilon \mid \mathcal{A}_{n, j-1}) \xrightarrow{p} 0$$

and

$$\sum_1^{k_n} P(|X_{nj}-1)| < \varepsilon \mid \mathcal{A}_{n, j-1}) \xrightarrow{p} \lambda$$

(ii) $$\sum_1^{k_n} \sigma_{nj}^2(\alpha) \to 0 \text{ and } \sum_1^{k_n} a_{nj}(\alpha) \xrightarrow{p} 0.$$

Remark 6 : As a final remark we note that the following fact will be implicitely used in several places of the paper : For some kernel G

$$\int e^{itx} G_m(dx) \xrightarrow{p} \int e^{itx} G(dx)$$

for all real t if, and only if,

$$\int f(x) G_m(dx) \xrightarrow{p} \int f(x)\, G\,(dx)$$

for all $f \in C(R)$.

This can be easily seen to be true using standard arguments, see e.g., Billingsley (1968, pp. 46 and 47).

2. Some prelemmas and reduction arguments

Lemma 1 : *Suppose that* (β_1, G_1) *and* (β_2, G_2) *be two couples such that*

$$\psi_1(t) = \psi_2(t) \text{ a.s.}$$

for all real t, *where*

$$\psi_i(t) = it\beta_i + \int \phi(t, x) G_i(dx), \quad i = 1, 2.$$

Then there exists a P-null set N such that

$$\psi_1(t) = \psi_2(t)$$

and

$$\int e^{itx} G_1(dx) = \int e^{itx} G_2(dx)$$

for all real t *and* $w \in \mathscr{X} - N$, *and* $\beta_1 = \beta_2$ *a.s.*

Proof: Using a simple continuity argument it is easy to see, under the given supposition, that there exists a P-null set N such that

$$\psi_1(t) = \psi_2(t)$$

for all real t and $w \in \mathscr{X} - N$. Hence the result follows since for each fixed $w \in \mathscr{X}$, there is a one-to-one correspondence between $\psi(t)$ and the couple (β_i, G_i), $i = 1, 2$, (cf. Loève (1963, p. 300)).

Lemma 2 : *Statement* (i) *of Theorem* 1.

Proof: Using the diagonal argument, one can find for every subsequence a further subsequence $\{m\}$ and a P-null set N such that $T_m \to T$ for all $w \in \mathscr{X} - N$ and $\{\psi_m(t)\}$ converges for all $t \in D$ and $w \in \mathscr{X} - N$, where the countable dense subset D is the one occurring in the condition (A.4). Also suppose without loss of generality, that this subsequence $\{m\}$ is such that, for every $\varepsilon > 0$,

$$\lim_{a \to \infty} \limsup_{m \to \infty} \; P[G_m(|x| > a) > \varepsilon] = 0. \qquad \ldots \quad (1)$$

We now show that, setting $B_M = \{t : |t| \leqslant M\}$, $M > 0$,

$$\lim_{\delta \to 0} \limsup_{m \to \infty} P[\sup_{|t-s| < \delta} |\psi_m(s) - \psi_m(t)| > \varepsilon;\; t,\; s \in B_M] = 0 \quad \ldots \quad (2)$$

for every $\varepsilon > 0$ and $M > 0$. First observe that

$$|t| \; |\beta_m| \leqslant |\psi_m(t)| + K(t) T_m$$

for some constant $K(t) > 0$, and hence $\{\beta_m\}$ is bounded in probability. Hence it is enough to show that, setting

$$\psi'_m(t) = \int \phi(t, x)\, G_m\,(dx),$$

$$\lim_{\delta \to 0} \limsup_{m \to \infty} P[\sup_{|t-s| < \delta} |\psi'_m(t) - \psi'_m(s)| > \varepsilon\,;\; t, s \in B_M] = 0$$

for every $\varepsilon > 0$. Now note that

$$\sup_{|t| < M} \sup_{x} |\phi(t, x) < \infty$$

and so, in view of (1),

$$\lim_{\alpha \to \infty} \limsup_{m \to \infty} P[\sup_{|t| < M} \int_{|x| > \alpha} |\phi(t, x)|\, G_m\,(dx) > \varepsilon] = 0$$

for every $\varepsilon > 0$ and $M > 0$. Hence it is enough to show that

$$\lim_{\delta \to 0} \limsup_{m \to \infty} P[\sup_{|t-s| < \delta} \int_{|x| < \alpha} |\phi(t, x) - \phi(s, x)|\, G_m(dx) > \varepsilon : t, s \in B_M] = 0$$

for every $\epsilon > 0$, $\alpha > 0$ and $M > 0$. This is true since, for every $w \in \mathscr{X} - N$,

$$\sup_{\substack{|t-s| < \delta \\ t, s \in B_M}} \sup_{|x| < \alpha} |\phi(t, x) - \phi(s, x)| \to 0$$

as $\delta \to 0$ since on the compact interval $\{(t, x) : |t| \leqslant M,\ |x| \leqslant \alpha\}$ $\phi(t, x)$ is uniformly continuous, and $T_m \to T$ for all $w \in \mathscr{X} - N$. Thus (2) holds.

Note that since the set D is dense in R and since $\{\psi_m(t)\}$ converges in probability for every $t \in D$, (2) entails that $\{\psi_m(t)\}$ converges in probability for every real t. We now show that

$$\int_0^1 [\psi_m(t+h) + \psi_m(t-h)]\, dh$$

converges in probability for every real t. Fix t and set

$$\psi^*_m(h) = \psi_m(t+h) + \psi_m(t-h).$$

Let $h_0 = 0$, $h_1 = 1/k$, $h_2 = 2/k, \ldots, h_k = 1$. It is easy to see from (2), that

$$\lim_{k \to \infty} \limsup_{m \to \infty} P\left[\left| \int_0^1 \psi^*_m(h)dh - \sum_0^k (h_{j+1} - h_j)\psi^*_m(h_j) \right| > \varepsilon \right] = 0$$

for every $\epsilon > 0$. Hence, using the fact that $\{\psi_m(t)\}$ mutually converges in probability for every real t, it follows easily that the sequence

$$\int_0^1 \psi^*_m(h)\, dh$$

mutually converges in probability and hence converges in probability for every real t.

Thus

$$g_m(t) = \psi_m(t) - \tfrac{1}{2} \int_0^1 [\psi_m(t+h) + \psi_m(t-h)]\, dh$$

converges in probability for every real t. Now observe that, (cf. Loève, 1963, p. 300)

$$g_m(t) = \int e^{itx} G^*_m\, (dx)$$

where

$$G^*_m\, (dx) = \left[1 - \frac{\sin x}{x}\right] \frac{1+x^2}{x^2}\, G\, (dx)$$

with

$$0 < c' \leqslant \left[1 - \frac{\sin x}{x}\right] \frac{1+x^2}{x^2} \leqslant c'' < \infty \qquad \ldots \quad (3)$$

where c' and c'' are independent of x. Also observe that (1) entails, in view of (3), that

$$\lim_{a \to \infty} \limsup_{m \to \infty} P[G^*_m(|x| > a) > \epsilon] = 0 \qquad \ldots \quad (4)$$

for every $\epsilon > 0$.

Using (4) and the fact that $\{g_m(t)\}$ converges in probability it can be easily shown, via Weierstrass approximation theorem, (see e.g. Billingsley (1968, p. 46))), that

$$\int f(x)\, G^*_m\, (dx).$$

Converges in probability for every $f \in C_{00}(R)$, where C_{00} denotes the class of all continuous functions vanishing outside compacts. We have thus shown that the sequence $\{ \int f(x)\, G^*_n\, (dx)\}$ is relatively compact in probability for all $f \in C_{00}$. Hence, using the diagonal argument again one can find (see the proof of Lemma 1 of Part I) for every subsequence a further subsequence $\{m\} \subset \{n\}$, a P-null set N and a kernel G^* such that

$$\int f(x)\, G^*_m\, (dx) \to \int f(x) G^*\, (dx)$$

for all $w \in \mathscr{X}-N$ and $f \in C_{00}$. Assume further, without loss of generality, that (4) holds. Then it is easy to show that

$$G_m^*(\infty) \xrightarrow{p} G^*(\infty).$$

Now let $\{r\} \subset \{m\}$ be a further subsequence such that, for some P-null set N',

$$G_r^*(\infty) \rightarrow G^*(\infty)$$

for all $w \in \mathscr{X}-N'$, we thus have

$$\int f(x) G_r^* (dx) \rightarrow \int f(x)\, G^*(dx)$$

for all $w \in \mathscr{X}-N$ and $f \in C_{00}$, and

$$G_r^*(\infty) \rightarrow G^*(\infty)$$

for all $w \in \mathscr{X}-N'$. Hence we have

$$\int f(x) G_r^* (x) \rightarrow \int f(x) G^*(dx)$$

for all $f \in C(R)$ and $w \in \mathscr{X}-NUN'$, where $C(R)$ denotes the class of all bounded continuous functions. In view of (3), we then have

$$\int f(x) G_r (dx) \rightarrow \int f(x)\, G\, (dx)$$

for all $f \in C(R)$, and $w \in \mathscr{X}-NUN'$, where

$$G(dx) = G^*(dx) / \left(1 - \frac{\sin x}{x}\right) \; \frac{1+x^2}{x^2}.$$

Now assume without loss of generality, that $\{\psi_r(t)\}$ converges almost surely for some $t \neq 0$. Since $\phi(t, x) \in C(R)$ for all t, we then have $\{\beta_r\}$ converges almost surely.

This completes the proof of the lemma.

Lemma 3 : *Statement* (1) *of Theorem* 2.

Proof : In view of Lemmas 1 and 2, the proof is quite analogous to that of Lemma 3 of Part I, and so it is omitted.

As in Part I, we denote the next few steps to reduce the problems to simpler cases in order to overcome certain technical difficulties.

Consider the CUAN-condition (A.2), i.e., for every $\epsilon > 0$

$$b_n = \max_{1 < j < k_n} P(|X_{nj}| > \epsilon \mid \mathscr{A}_{n,j-1}) \rightarrow 0. \qquad \dots (5)$$

Set $X^*_{nj} = \min(|X_{nj}|, 1)$. Then (5) in particular entails that

$$b'_n = E\left[\max_{1 < j < k_n} E(X^{*2}_{nj} | \mathcal{A}_{n,j-1})\right] \to 0$$

and

$$b''_n = E\left[\max_{1 < j < k_n} E^{1/2}(X^{*2}_{nj} | \mathcal{A}_{n,j-1})\right] \to 0.$$

Also, following the arguments used for the case of independent summands, it follows from (5) that

$$d_n = \max_{1 < j < k_n} |a_{nj}| \xrightarrow{p} 0$$

and

$$e_n = \max_{1 < j < k_n} \int \frac{x^2}{1+x^2} \overline{F}_{nj}(dx) \xrightarrow{p} 0.$$

We now choose an increasing sequence $0 < \alpha_n \uparrow \infty$, $\alpha_n < k_n$, such that

$$\alpha_n b'_n \to 0,\ \alpha_n b''_n \to 0,\ \alpha_n d_n \xrightarrow{p} 0 \text{ and } \alpha_n e_n \xrightarrow{p} 0. \qquad \ldots \quad (6)$$

One implication of these conditions is the following lemma.

Lemma 4 : *Suppose that* (6) *holds. Then*

(i) $\sum_{1}^{\alpha_n} X_{nj} \xrightarrow{p} 0,$

(ii) $\sum_{1}^{\alpha_n} \int \frac{x^2}{1+x^2} \overline{F}_{nj}(dx) \xrightarrow{p} 0,$

and

(iii) $\sum_{1}^{\alpha_n} a_{nj} \xrightarrow{p} 0.$

Proof : The proof of (i) can be found in Part I. The proofs of (ii) and (iii) are immediate.

Now suppose that $\{m\} \subsetneq \{n\}$ be a subsequence such that $T_m \xrightarrow{p} T$. In view of Lemma 4 of Part I there is a non-negative sequence $\{T'_n\}$ adapted to $\{\mathcal{A}_{n,\alpha_n}\}$ such that $T'_n \xrightarrow{p} T$. In particular $T_m \xrightarrow{p} T$. In the rest of this section we suppose that this subsequence $\{m\} \subseteq \{n\}$ is fixed.

We define, for a constant $\delta > 1$, (δ is fixed throughout),

$$U_{mj} = (T'_m + \delta)^{-1} \sum_{\alpha_m+1}^{j} \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx),$$

and

$$B_{mj} = \{U_{mj} \leqslant \delta, \ |a_{mj}| < \tau/2\}$$

where the positive constant τ is the one occuring in the definition of a_{nj}'s. Note that both the rv U_{mj} and the set B_{mj} are $\mathscr{A}_{m,\,j-1}$-mble for $j = \alpha_m + 1, \ldots, k_m$. Now define

$$\xi_{mj} = T^*_m I(B_{mj})(X_{mj} - a_{mj}),$$

$j = \alpha_m + 1, \ldots, k_m$, where

$$T^*_m = [T'_m + (T'_m + \delta)^2]^{-1/2}.$$

To simplify the writing we make the convention, valid for the rest of this Part, that the *summations* $\sum_j$ *are over* $j = \alpha_m + 1, \ldots, k_m$ *only, unless otherwise stated explicitely.*

The reason for defining the ξ_{mj}'s in this rather complicated way is just to obtain the following crucial inequalities.

Lemma 5 : *For* $j = \alpha_m + 1, \ldots, k_m$ *and* $\delta > 1$,

$$|E(e^{it\xi_{mj}} - 1 \,|\, \mathscr{A}_{m,\,j-1})| \leqslant I(U_{mj} \leqslant \delta) C(t)(T'_m + \delta)^{-1} \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx) \ \text{a.s.} \quad \ldots \ (7)$$

where $C(t)$ *is a positive constant depending only on* t *and* τ, *and*

$$\sum_j I(U_{mj} \leqslant \delta)(T'_m + \delta)^{-1} \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx) \leqslant \delta \ \text{a.s.} \quad \ldots \ (8)$$

for all m.

Proof : According to the Central Inequalities given in Loève (1963, p. 304)

$$\left| \int (e^{itx} - 1) \bar{F}_{mj}(dx) \right| \leqslant C_1(t) \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx)$$

where

$$C_1(t) = [1 + (\tau + |a_{mj}|)^2] \left[\frac{2 + |a_{mj}|\,|t|}{(\tau - |a_{mj}|)^2} + \frac{|t|^2}{2} \right].$$

Now note that

$$E(e^{it\xi_{mj}} - 1 \,|\, \mathscr{A}_{m,\,j-1}) = \int (e^{it_{mj}x} - 1) \bar{F}_{mj}(dx) \ \text{a.s.}$$

where $t_{mj} = tT_m^* I(B_{mj})$. Observe that $T_m^* I(B_{mj})$ is $\mathcal{A}_{m,\, j-1}$-mble for $j = \alpha_m+1, \ldots, k_m$ and therefore in the above integral it can be treated as a constant for these values of j. Since one of the factors in t_{mj} is $I(|a_{mj}| < \tau/2)$, we then have

$$C_1(t_{mj}) \leqslant I(U_{mj} \leqslant \delta) K_1 \left[K_2 |t| T_m^* + \frac{|t|^2}{2} \; T_m^{*2} \right]$$

where K_1 and K_2 are positive constants depending only on τ. Note that, since $\delta > 1$,

$$T_m^* = [T_m' + (T_m' + \delta)^2]^{-1/2} \leqslant (T_m' + \delta)^{-1}$$

and

$$T_m^{*2} = [T_m' + (T_m' + \delta)^2]^{-1} \leqslant (T_m' + \delta)^{-1}.$$

This proves (7) and (8) can be easily verified using the definition of U_{mj}'s. The proof is complete.

In view of the above relations (7) and (8) the proof of the following lemma is essentially identical to the proof of Lemma 6 of Part I and so it is omitted.

Lemma 6 : *Suppose that the CUAN-condition* (A.2) *is satisfied. Let the subsequence* $\{m\} \subseteq \{n\}$ *be as above. Then for every* $\mathcal{A}$*-mble k-vector* g, *the difference*

$$E\{exp[iu'g + itS_m' + \sum_j E(1 - e^{it\xi_{mj}} | \mathcal{A}_{m,\, j-1})]\} - E[exp(iu'g)]$$

converges to zero for every $u \in R^k$ *and* $t \in R$, *where* $S_m' = \sum_j \xi_{mj}$.

Lemma 7 : *Suppose that the CUAN condition* (A.2) *is satisfied. Let* $\{m\}$ *be a subsequence such that, for some couple* (β, G),

(i) $\beta_m \xrightarrow{p} \beta$

and for all real t

(ii) $\int e^{itx} G_m(dx) \xrightarrow{p} \int e^{itx} G(dx)$.

Then the sequence $\{S_m\}$ *is relatively compact in distribution, where* $S_m = \sum_1^{k_m} X_{mj}$.

Proof : First note that the given conditions entails that the sequences $\{\psi_m(t)\}$ and $\{T_m\}$ are convergent in probability. Further note that

$$\psi_m(t) = it \sum_1^{k_m} a_{mj} + \sum_1^{k_m} \int (e^{itx} - 1) \bar{F}_{mj}(dx).$$

Hence it follows, from the condition $\max_{1 \leqslant j \leqslant k_n} |a_{nj}| \xrightarrow{p} 0$ and the Central Inequality mentioned in the proof of Lemma 5, that the sequence $\sum_{1}^{k_m} a_{mj}$, $m \geqslant 1$, is relatively compact, and so it is enough to show that the sequence $\sum_{1}^{k_m} (X_{mj}-a_{mj})$, $m \geqslant 1$, is relatively compact.

Using Lemma 4 it follows then that it is enough to show that the sequence $\sum_{j} (X_{mj}-a_{mj})$, $m \geqslant 1$, is relatively compact. To proceed further, observe that

$$U_{mk_m} \xrightarrow{p} T/T+\delta \leqslant 1. \qquad \ldots \quad (9)$$

Now let $\{P_m^*\}$ be a sequence of measures such that

$$dP_m^* = (T_m'+\delta)^{-1}\, dP.$$

Note that

$$\int dP_m^* \leqslant \delta^{-1} \text{ for all } m.$$

One can further easily check that the relative compactness under P is equivalent to the relative compactness under the sequence $\{P_m^*\}$.

Define, for $M > 0$,

$$X_m' = (X_{mj}-a_{mj})I(|X_{mj}-a_{mj}| < M),$$

$j = \alpha_m+1, \ldots, k_m$. Then

$$P_m^*(U_{mk_m}^1 \leqslant \delta,\ X_{mj}' \neq (X_{mj}-a_{mj}) \quad \text{for some } j = \alpha_m+1, \ldots, k_m)$$

$$\leqslant \sum_j P_m^*(U_{mj} \leqslant \delta,\ |X_{mj}-a_{mj}| > M) \text{ (Since } U_{mj} \leqslant U_{mk_m})$$

$$= E\left[(T_m'+\delta)^{-1} \sum_j I(U_{mj} \leqslant \delta) \int_{|x|>M} \bar{F}_{mj}(dx)\right]$$

$$\leqslant \frac{1+M^2}{M^2} E\{(T_m'+\delta)^{-1}[G_m'(\infty)-G_m'(M)+G_m'(-M)]\}$$

where

$$G_m'(y) = \sum_j I(U_{mj} \leqslant \delta) \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, \bar{F}_{mj}(dx).$$

Since on the set $\{U_{mk_m} \leqslant \delta\}$, $G_m(x)$ and $G_m'(x)$ coincides, it is easy to see, using (9) and Lemma 4, that the condition (ii) holds if and only if for all real t

$$\int e^{itx} G_m'(dx) \xrightarrow{p} \int e^{itx} G\,(dx).$$

Also note that $(T'_m+\delta)^{-1}G'_m(\infty) \leqslant \delta$ for all m, by (8). Hence it follows easily that the quantity

$$E\{(T'_m+\delta)^{-1}[G'_m(\infty)-G'_m(M)+G'_m(-M)]\}$$

Converges to zero by first letting $m \to \infty$ and then $M \to \infty$. Thus it follows from the above inequalities that it is enough to show that the sequence $\Sigma X'_{mj}$, $m \geqslant 1$, which depends on M, is relatively compact under $\{P^*_m\}$ for each fixed M. Equivalently, we shall show that the sequence $\sum_j I(U_{mj} \leqslant \delta)X'_{mj}$, $m \geqslant 1$, is relatively compact. Now consider

$$\int_{|x| \leqslant M} x\bar{F}_{mj}(dx) = \int_{|x| \leqslant \tau} x\bar{F}_{mj}(dx) + \int_{\tau < |x| \leqslant M} x\bar{F}_{mj}(dx),$$

and so

$$\sum_j \int_{|x| \leqslant M} x\bar{F}_{mj}(dx) \leqslant \sum_j \int_{|x| \leqslant \tau} x\bar{F}_{mj}(dx) + \sum_j \int_{\tau < |x| \leqslant M} x\bar{F}_{mj}(dx).$$

One can easily check that, (cf. Loève, 1963, p. 314),

$$\sum_j \int_{|x| \leqslant \tau} x\bar{F}_{mj}(dx) \leqslant 3\tau \sum_j \int_{|x| > \tau} \bar{F}_{mj}(dx) \leqslant 3\tau^{-1}(1+\tau^2) \sum_j \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx).$$

Also

$$\sum_j \int_{\tau < |x| \leqslant M} x\bar{F}_{mj}(dx) \leqslant \frac{(1+\tau^2)M}{\tau^2} \sum_j \int \frac{x^2}{1+x^2} \bar{F}_{mj}(dx).$$

From these inequalities it follows that

$$\sum_j \int_{|x| \leqslant M} x\bar{F}_{mj}(dx)$$

is relatively compact, and hence

$$\sum_j I(U_{mj} \leqslant \delta) \int_{|x| \leqslant M} x\bar{F}_{mj}(dx)$$

is relatively compact. Hence it follows that, denoting the expectation with respect to P^*_m by E^*_m and setting

$$X^*_{mj} = I(U_{mj} \leqslant \delta)X'_{mj},$$

$$\sum_j E^*_m(X^*_{mj} \mid \mathscr{A}_{n,j-1})$$

is relatively compact.

Hence, to complete the proof it is enough to show that $\sum_j [X^*_{mj} - E^*_m(X^*_{mj} \mid \mathcal{A}_{n.j-1})]$, $m \geqslant 1$, is relatively compact. The expectation (with respect to P^*_m) of the square of this sum is less than or equal to

$$\sum_j E^*_m(X^{*2}_{mj}) = E[(T'_m+\delta)^{-1} \sum_j I(U_{mj} \leqslant \delta) \int_{|x| \leqslant M} x^2 \bar{F}_{mj}(dx)]$$

$$\leqslant (1+M^2)E[(T'_m+\delta)^{-1}G'_m(\infty)] \leqslant (1+M^2)\delta \quad \text{(by (8))}.$$

Hence the proof is complete.

3. The proofs of the results

Proof of Theorem 1 : In view of Lemma 2, it remains only to prove the statement (ii). Let $\{m\}$ be a subsequence such that, for some couple (β, G),

$$\beta_m \xrightarrow{P} \beta$$

and for all real t

$$\int e^{itx}\, G_m(dx) \xrightarrow{P} \int e^{itx}\, G\,(dx).$$

It is easy to show that this entails $T_m \xrightarrow{P} T$ and

$$\psi_n(t) = it\beta_m + \int \phi(t, x)\, G_m\,(dx)$$

$$\xrightarrow{P} it\beta + \int \phi(t, x)\, G\,(dx) = \psi(t).$$

Now let, with $B_{mj} = \{U_{mj} \leqslant \delta,\ |a_{mj}| \leqslant \tau/2\}$, (recall the convention that the sum $\sum_j$ is over $j = \alpha_m+1, \ldots, k_m$),

$$\psi^*_m(t) = it \sum_j I(B_{mj})a_{mj} + \sum_j \int (e^{itI(B_{mj})x} - 1)\bar{F}_{mj}\,(dx)$$

$$= it \sum_j I(B_{mj})a_{mj} + \sum_j I(B_{mj}) \int (e^{itx} - 1)\bar{F}_{mj}\,(dx).$$

(Since $I(.)$ takes the value either 0 or 1).

Note that, on the set $\{U_{mk_m} \leqslant \delta,\ \sup_{1 \leqslant j \leqslant k_m} |a_{mj}| < \tau/2\}$, $\psi^*_m(t)$ coincides with the sum

$$it \sum_j a_{mj} + \sum_j \int (e^{itx} - 1)\bar{F}_{mj}\,(dx).$$

Hence, using Lemma 4, the relation (9) (of Sec. 2) and the fact

$$\sup_j |a_{mj}| \xrightarrow{P} 0,$$

it follows that, for every $M > 0$,

$$\sup_{t \leq M} |\psi_m(t) - \psi^*_m(t)| \xrightarrow{P} 0.$$

Hence it follows easily that the difference

$$\psi_m(t_m) - \psi^*_m(t_m) \xrightarrow{P} 0$$

where $t_m = tT^*_m$, since

$$T^*_m \xrightarrow{P} T^* = [T + (T+\delta)^2]^{-1/2}.$$

(T^*_m is as defined in Sec. 2). Also, it is not difficult to check that, (cf. Statement (iii) of Lemma 5 of Part I),

$$\psi_m(t_m) \xrightarrow{P} \psi(tT^*)$$

and hence

$$\psi^*_m(t_m) \xrightarrow{P} \psi(tT^*).$$

Now observe that, ($S'_m = \sum_j \xi_{mj}$),

$$itS'_m + \sum_j E(1 - e^{it\xi_{mj}} \mid \mathscr{A}_{m,\,j-1}) = itS^*_m - \psi^*_m(t_m)$$

where we set

$$S^*_m = T^*_m \sum_j I(B_{mj}) X_{mj}.$$

Therefore, in view of Lemmas 5, 6 and 7, it follows by following the arguments of the proof of Theorem 1 of Part I that

$$E[\exp(iug + itS^*_m)] \to E[\exp(iug + \psi(t)T^*)]$$

for all real u and t and for every $\mathscr{A}$-mble g.

Now note that the difference

$$S_m - T^{*-1} S^*_m \xrightarrow{P} 0.$$

Hence, in view of the last part of the proof of Theorem 1 of Part I, the proof follows.

This completes the proof of the Theorem.

Proof of Theorem 2 : The proof follows by repeating the arguments of the proof of Theorem 1 for the sequence $\{n\}$.

Proof of Theorem 4 : In view of Lemma 1, the proof is essentially identical to the proof of Theorem 3 of Part I.

Proofs of the convergence criterions : It follows from the statement (i) of Theorem 1 and Lemma 1 that there exists a couple (β, G) such that the condition (A.5) is satisfied if, and only if, for every subsequence there exists a further subsequence $\{m\} \subseteq \{n\}$ and a P-null set N such that $\beta_m \to \beta$ for all $w \in \mathscr{X} - N$ and

$$\int e^{itx} G_m(dx) \to \int e^{itx} G(dx)$$

for all $w \in \mathscr{X} - N$ and for all real t. Hence the proof of the Central Convergence Criterion follows from the known results for the case of independent summands (cf. Loève, 1963, p. 311).

Using the arguments used for the case of independent summands (cf. Loève, 1963, pp. 315–317), the proofs of the normal, Poisson and 'degenerate' convergence criterions follows easily from the Central Convergence Criterion.

Proof of Theorem 3 : First note that it follows from Theorem 1 that the limit distribution must be a MIDD with some couple (β, G), whenever the conditions (A.1)–(A.4) hold. We now show that for any given couple (β, G) on some probability space $(\mathscr{X}_1, \mathscr{A}_1, P)$, one can construct a sequence of triangular arrays $\{(\mathscr{X}_{nj}, \mathscr{A}_{nj}), 1 \leqslant j \leqslant n\}$, $\mathscr{A}_{n1} \subseteq \ldots \subseteq \mathscr{A}_{nn}$, $n = 1, 2, \ldots$, such that the conditions (A.1)–(A.4) are satisfied and that the sequence $\{S_n\}$ converges stably to MIDD with (β, G).

To this end, let

$$\psi(t) = it\beta + \int \phi(t, x) G(dx).$$

Note that $\exp\left(\frac{\psi(t)}{n}\right)$ is a ch.f. for each fixed $w \in \mathscr{X}_1$, and is an $\mathscr{A}_1$-mble function for each fixed real t. Therefore, for each $n \geqslant 1$, one can construct a stochastic kernel $P_n(w, \cdot)$ such that

$$\int e^{itx} P_n(w, dx) = \exp\left(\frac{\psi(t)}{n}\right)$$

for all $w \in \mathscr{X}_1$. Let, then, $P^*(w, \cdot)$ be the stochastic kernel on $(\mathscr{X}_2, \mathscr{A}_2)$ where

$$P^*(w, \cdot) = \overset{\infty}{\underset{n=1}{\times}} \overset{n}{\underset{j=1}{\times}} P_{n \cdot j}(w, .)$$

with $P_{nj}(w, \cdot) = P_n(w, \cdot)$ for all $1 \leqslant j \leqslant n < \infty$,

$$\mathscr{X}_2 = \overset{\infty}{\underset{n=1}{\times}} \overset{n}{\underset{j=1}{\times}} R_{nj} \quad \text{with } R_{nj} = R, \quad 1 \leqslant j \leqslant n \leqslant \infty,$$

and

$$\mathscr{A}_2 = \overset{\infty}{\underset{n=1}{\times}} \overset{n}{\underset{j=1}{\times}} \mathscr{B}_{nj} \quad \text{with } \mathscr{B}_{nj} = \mathscr{B}, \quad 1 \leqslant j \leqslant n \leqslant \infty.$$

($\mathscr{B}$ is the σ-field of Borel subsets of the real line R). Let P^* be the probability measure on $(\mathscr{X}_1 \times \mathscr{X}_2, \mathscr{A}_1 \times \mathscr{A}_2)$ defined by

$$P^* = \int P^*(w, dx)P(dx).$$

Now define the rv's X_{nj}'s on $(\mathscr{X}_1 \times \mathscr{X}_2, \mathscr{A}_1 \times \mathscr{A}_2, P^*)$ as follows

$$X_{nj}(w, x) = x_{nj}$$

for all $w \in \mathscr{X}_1$ and $x \in \mathscr{X}_2$, where x_{nj} is the nj-th co-ordinate of x. Then define

$$\mathscr{A}_{n,j} = \sigma(Z, X_{mj}, j = 1, 2, \ldots, m, \ m = 1, 2, \ldots, n-1, X_{n1}, \ldots, X_{nj})$$

where Z is the identity map :

$$Z : (\mathscr{X}_1 \times \mathscr{X}_2, \mathscr{A}_1 \times \mathscr{A}_2) \to (\mathscr{X}_1, \mathscr{A}_1).$$

Obviously, the σ-fields $\mathscr{A}_{nj}$'s satisfy the condition (A.1) with $k_n = n$. $\mathscr{A}_{n1} \subseteq \ldots \subset \mathscr{A}_{nn}$ for each $n \geqslant 1$, and X_{nj} is $\mathscr{A}_{nj}$-mble for $1 \leqslant j \leqslant n < \infty$. Note that for any integrable function f

$$E(f(X_{nj}) \mid \mathscr{A}_{n,j-1}) = E(f(X_{nj}) \mid \sigma(Z)) \text{ a.s.} \qquad \ldots \quad (10)$$

where the expectation is with respect to P^*, as is easily seen from the construction. In what follows fix $w \in \mathscr{X}_1$ and let E_w denotes the expectation with respect to $P^*(w, \cdot)$. From the construction it follows that

$$E_w(e^{itS_n}) = \prod_{j=1}^{n} E_w(e^{itX_{nj}}) = \prod_{j=1}^{n} \exp\left(\frac{\psi(t)}{n}\right) = \exp(\psi(t)). \qquad \ldots \quad (11)$$

This is true for all $n \geqslant 1$. Also note that, for every $\varepsilon > 0$,

$$\max_{1 \leqslant j \leqslant n} E_w[I(|X_{nj}| > \varepsilon)] \to 0. \qquad \ldots \quad (12)$$

Hence from the known necessary conditions for the case of independent summands it follows that

$$\beta_n(w) \to \beta(w) \qquad \dots \quad (13)$$

and

$$\int e^{itx} G_n(dx, w) \to \int e^{itx} G(dx, x) \qquad \dots \quad (14)$$

for all real t, where

$$\beta_n(w) = \sum_1^n a_{nj}(w) + E_w \left[\frac{(X_{nj} - a_{nj})}{1 + (X_{nj} - a_{nj})^2} \right]$$

with

$$a_{nj}(w) = E_w[X_{nj} I(|X_{nj}| < \tau)], \quad \tau > 0,$$

and

$$G_n(x, w) = E_w \left[\frac{(X_{nj} - a_{nj})^2}{1 + (X_{nj} - a_{nj})^2} I((X_{nj} - a_{nj}) < x) \right].$$

Now observe that, in view of (10), a regular conditional distribution (with the underlying prob. measure P^*) of X_{nj} given $\mathcal{A}_{n, j-1}$ can be taken as

$$F_{nj}(x, w) = E_w[I(X_{nj} \leqslant x)].$$

Hence, in view of (12)–(14) and Theorem 2, the desired conclusion follows.

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