Linear Algebra and its Applications 417 (2006) 8–30

LINEAR ALGEBRA
AND ITS
APPLICATIONS

On the optimality of a class of designs with three concurrences

Sunanda Bagchi

Stat.-Math. Unit, Indian Statistical Institute, 8th mile Mysore Road, Bangalore 560059, India

Received 28 April 2004; accepted 8 February 2006
Available online 6 June 2006
Submitted by A. Markiewicz

Abstract

In the present paper we consider a class of unequally replicated designs having concurrence range 2 and spectrum of the form $\mu_1(\mu_2)^{v-3}\mu_3$. Now, Jacroux's [Some sufficient conditions for the type I optimality of block designs, J. Statist. Plann. Inference 11 (1985) 385–396] Proposition 2.4 says that a design with spectrum of the above form, if satisfies some further conditions, is type 1 optimal. Unfortunately, this proposition does not apply to our designs since they have a poor status regarding E-optimality. Yet we are able to prove the A-optimality (in the general class) of these designs using majorisation technique. A method of construction of an infinite series of our A-optimal designs has also been given.

The first and only known infinite series of examples of designs satisfying Jacroux's conditions appears to be the first one in Section 4.1 of Morgan and Srivastav [On the Type-1 optimality of nearly balanced incomplete block designs with small concurrence range, Statist. Sinica 10 (2000) 1091–1116] – hitherto referred to as [MS]. In this paper, we use majorisation technique to prove stronger optimality properties of the above mentioned designs of [MS] as well as to present simpler proof of another optimality result in [MS].

AMS classification: 62K10

Keywords: Majorisation; A-optimality; E-optimality; Type 1 optimality

E-mail address: sbagchi@isibang.ernet.in

1. Introduction

In the present paper, we continue the search for optimal block designs. It is well known that the "best" design (BIBD) must be binary and must have replication numbers as well as the concurrences all equal. It is also well known that these equalities require certain divisibility conditions, which are often not met. So, the following questions arise in one's mind, (a) "what is the best design in a set up where the divisibility conditions are not satisfied?", (b) "if the divisibility conditions are satisfied, but a BIBD does not exist, then what is the best design?" We take a glimpse at the status of our knowledge regarding question (a). For (b) we refer to [12,14].

It is reasonable to believe that in the situations when equal replication is possible but all the concurrences cannot be equal, a binary equireplicate design with concurrences differing by at most one would be optimal. This was conjectured by John and Mitchell [10], who coined the name "regular graph designs" (RGD) for such candidates. While this conjecture has been disproved regarding E-criterion (see [2,3], for instance) it is widely believed to be true for A- and D-criteria. In fact, many RGDs have been proved to satisfy general optimality (see [5,7,1]).

Now suppose equal replication is not possible. Then a likely candidate for optimality is a binary design with replication numbers as well as concurrences differing by at most one. These were termed as semi-RGDs in [9], where many sufficient conditions for the optimality of RGDs and semi-RGDs were provided.

Next, let us consider the situations when neither RGD nor semi-RGD can exist. Morgan and Srivastav [12] considered these. They defined nearly balanced incomplete block designs NBBD(m), which are binary designs with replication numbers differing by at most once and concurrences differing by at most m. They provided sufficient conditions for the optimality of NBBD(2)'s, using which they proved optimality of certain classes of NBBD(2)'s.

The two classes of NBBD(2)'s (say $\bar{d}_1, \bar{d}_2$) considered in Section 4.1 of Morgan and Srivastav [12] caught the attention of the present author for many reasons. Both are unequally replicated, but the spectra of their C-matrices are "very good". Of these $\bar{d}_2$ has spectrum $(\mu_1)^{v-2}\mu_2$ and it turned out to be, not surprisingly, generalised optimal of type 1 like the most balanced group divisible designs (MBGDD) of type 1 (see [5]). On the other hand, $\bar{d}_1$ has spectrum $\mu_1(\mu_2)^{v-3}\mu_3$. Now, in view of Proposition 2.4 of Jacroux [9] many researchers in this area, including the present author, believe that a design with spectrum like this is must be optimal but no example was known. $\bar{d}_1$ seems to be the first example satisfying the hypothesis of above proposition and indeed it is optimal! This observation was so exciting that finding another example like this and verifying its optimality seemed to be very urgent. That led to the birth of the present paper. The design d^* [see after (4.1)] may be thought of "opposite" of $\bar{d}_1$.

In Section 3, we handle existing optimality results: extend one and provide simpler proof of another, both using majorisation technique. In Section 4, we prove A-optimality of d^* in the general class and present a method of construction of it in Section 5.

2. Preliminaries

Notation 2.1. Consider a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

(a) $x \uparrow$ and $x \downarrow$ will denote the vectors obtained by rearranging the coordinates of x in the increasing and decreasing order, respectively.

(b) Suppose x has m distinct entries ($m < n$). Then x will be denoted by $\prod_{i=1}^{m} x_i^{n_i}$, if x_i has multiplicity n_i, $i = 1, \ldots, m$, $\sum_{i=1}^{m} n_i = n$.

Definition 2.1 [11]. For $x, y \in \mathbb{R}^n$, x is said to be weakly majorised from above by y (in symbols, $x \prec^w y$) if

$$\sum_{i=1}^{k} x_i^{\uparrow} \geqslant \sum_{i=1}^{k} y_i^{\uparrow}, \quad k = 1, 2, \ldots, n. \tag{2.1}$$

It is clear that $\prec^w$ is reflexive and transitive.

We begin with a trivial but useful result.

Theorem 2.1. *Consider an $n \times 1$ vector x. Let $\bar{x} = \left(\sum_{i=1}^{n} x_i\right)/n$. Then*

$$\bar{x}^n \prec^w \prod_{i=1}^{n} x_i.$$

We shall now state Tomic's theorem and derive a few results from it. For the proof of this theorem and other results on weak majorisation see [11].

Theorem 2.2 (Tomic). *$x \prec^w y$ if and only if*

$$\sum_{i=1}^{n} g(x_i) \leqslant \sum_{i=1}^{n} g(y_i)$$

for every convex decreasing function $g : \mathbb{R} \to \mathbb{R}$.

Theorem 2.3. *Suppose $x^{(1)}$, $y^{(1)}$ are $m \times 1$ and $x^{(2)}$, $y^{(2)}$ are $n \times 1$ vectors such that*

$$x^{(i)} \prec^w y^{(i)}, \quad i = 1, 2.$$

Then,

$$x = \left(x^{(1)} | x^{(2)}\right) \prec^w y = \left(y^{(1)} | y^{(2)}\right).$$

Here $(p|q)$ is the juxtaposition of the vectors p and q.

Theorem 2.4. *For an $n \times 1$ vector x, let $\tilde{x}(t)$ denote the $t \times 1$ vector $(x_1, x_2, \ldots, x_t)$, $t \leqslant n$. Consider two $n \times 1$ vectors x and y with entries arranged in ascending order and satisfying the following conditions:*

(i) $\displaystyle\sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} y_i$

and

(ii) $\tilde{x}(t) \prec^w \tilde{y}(t)$, *for some $t < n$.*

Then, each of the following is a sufficient condition for $x \prec^w y$:

(a) $x_{t+1} = x_{t+2} = \cdots = x_n$

(b) $x_{t+1} = x_{t+2} = \cdots = x_{n-1}$, $y_n \geqslant x_n - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.

Proof. (a) Take $q^{(1)} = \tilde{q}(t)$, $q^{(2)} = (q_{t+1}, \ldots, q_n)$, $q = x$ or y. By assumption,

$$\sum_{i=1}^{t} x_i \geqslant \sum_{i=1}^{t} y_i.$$

If equality holds in the above relation, then we are done by Theorems 2.1 and 2.3. So, we assume strict inequality. Let $\delta = \sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i$. Clearly $x^{(2)}$ is not majorised by $y^{(2)}$. We define a vector y^* as follows. $y_t^* = y_t + \delta$, $y_{t+1}^* = y_{t+1} - \delta$, $y_i^* = y_i$, $i \neq t, t+1$. Then, clearly,

$$\sum_{i=1}^{u} y_i^* = \sum_{i=1}^{u} y_i, \quad u = t, n.$$

Thus, by Theorem 2.3

$$x \prec^w y^*.$$

But it is clear from the definition of y^* that $y^* \prec^w y$. Hence, the result follows from the transitivity of $\prec^w$.

(b) is proved by applying (a) on $\bar{x}(n-1)$ and $\bar{y}(n-1)$. □

Notation 2.2. Consider an $n \times n$ real symmetric matrix A.

(a) The principal submatrix bordered by the set of rows $i, j, \ldots, l$ of A will be denoted by $A(i, j, \ldots, l)$.
(b) $\mu(A)$ will denote the vector of eigenvalues of A, arranged in ascending order. If A is nonnegative definite, then $\mu(A)$ will denote the vector of positive eigenvalues of A.

We now present a few inequalities on the eigenvalues of real symmetric matrices. The first one is a well-known result called Ky Fan's maximum principle (see Problem I.6.15 of Bhatia [4], for instance), from which the others can be derived easily.

Theorem 2.5. *Consider a symmetric matrix A of order n. Suppose $x_1, x_2, \ldots, x_k$ $(k < n)$ are orthonormal vectors $\in \mathbb{R}^n$. Then,*

$$\sum_{j=1}^{k} \mu_j^{\uparrow}(A) \leqslant \sum_{j=1}^{k} x_j^{\mathrm{T}} A x_j \leqslant \sum_{j=1}^{k} \mu_j^{\downarrow}(A). \tag{2.2}$$

Theorem 2.6. *Consider a symmetric matrix A of order n and constant row sum s. If the average row sum of a principal submatrix B of order t is p then*

$$\mu_1^{\uparrow}(A) \leqslant (np - ts)/(n - t) \leqslant \mu_1^{\downarrow}(A).$$

Proof. W.l.g., let $B = A(1, 2, \ldots, t)$. Let x denote the normalised version of the vector $(n-t)^t$. $(-t)^{n-t}$. Now apply Theorem 2.5 with $k = 1$. □

Putting $s = 0$ and $t = 1$ in the theorem above we get the following well-known (see [8], for instance) and very useful result.

Corollary 2.1. *For a symmetric matrix A of order n with row sums zero the following equation holds for every i, $1 \leqslant i \leqslant n$:*

$$\mu_1^{\uparrow}(A) \leqslant (n/(n-1))a_{ii} \leqslant \mu_1^{\downarrow}(A).$$

Consider a nonnegative definite matrix A with each row sum zero. Let B_i, $0 \leqslant i \leqslant m$, be disjoint principal submatrices of A; t_i denotes the order of B_i. For every i, $1 \leqslant i \leqslant m$, let there exist an integer u_i, $1 \leqslant u_i \leqslant t_i$ and a set $y_{i1}, y_{i2}, \ldots, y_{iu_i}$ of orthonormal vectors in $\mathbb{R}^{t_i}$, each of which is orthogonal to the all-one vector in $\mathbb{R}^{t_i}$. Also, let $z_i \in \mathbb{R}^{u_i}$ be a vector with nonnegative entries in the increasing order, $1 \leqslant i \leqslant m$. Further, let $z_0 = np/(n - t_0)$, where p is the average row sum of B_0. Finally, let $z = (z_0 \mid z_1 \mid \cdots \mid z_m) \in \mathbb{R}^h$ where $h = 1 + \sum_{i=1}^m u_i$. For such a data set, we have the following result.

Theorem 2.7

(a) *Suppose the following inequalities are satisfied:*

$$\sum_{j=1}^{l} y_{ij}^T B_i y_{ij} \leqslant \sum_{j=1}^{l} z_{ij}, \quad 1 \leqslant l \leqslant u_i, \; 1 \leqslant i \leqslant m. \tag{2.3}$$

Then we can say the following about the eigenvalues of A:

$$\sum_{j=1}^{l} \mu_j^{\uparrow}(A) \leqslant \sum_{j=1}^{l} z_j^{\uparrow}, \quad 1 \leqslant l \leqslant h. \tag{2.4}$$

(b) *If $\geqslant$ holds in place of $\leqslant$ in (2.3), then the following hold:*

$$\sum_{j=1}^{l} \mu_j^{\downarrow}(A) \geqslant \sum_{j=1}^{l} z_j^{\downarrow}, \quad 1 \leqslant l \leqslant h. \tag{2.5}$$

Theorem 2.8 *Consider a $v \times v$ matrix A. For some m, n, $m + n < v$, suppose there are real numbers z_i, $1 \leqslant i \leqslant m$ and w_i, $1 \leqslant i \leqslant n$, such that*

(a) (2.4) *holds with $h = m$,*
(b) (2.5) *holds with n for h and w_j for z_j, $j = 1, 2, \ldots, n$ and*
(c) $\sum_{j=1}^m z_j + \sum_{j=1}^n w_j < \mathrm{tr}(A)$.

Then

$$\prod_{i=1}^{m} z_i \cdot (\bar{z})^{v-m-n} \cdot \prod_{j=1}^{n} w_j \prec^w \mu(A).$$

Here $\bar{z} = \left(\mathrm{tr}(A) - \sum_{j=1}^m z_j - \sum_{j=1}^n w_j\right)/(v - m - n)$.

Let us now consider a block design set up. All designs in this paper are connected block designs with constant block size. We present a set of notations, which are commonly used.

Notation 2.3

(i) $\mathscr{D} = \mathscr{D}_{b,k,v}$ denotes the class of all connected block designs with v treatments and b blocks of size k each.
(ii) $\mathscr{D}_{b,k,v}^B$ denotes the class of binary designs in $\mathscr{D}_{b,k,v}$.

(iii) $r := [bk/v]$. $\lambda = [r(k-1)/(v-1)]$. Here $[x]$ is the smallest integer $\geqslant x$.
(iv) The replication number of the ith treatment in a design $d \in \mathscr{D}$ will be denoted by r_{di} $(1 \leqslant i \leqslant v)$. $R(d)$ will denote the diagonal matrix $\mathrm{diag}(r_{d1}, \ldots, r_{dv})$.
(v) For a design $d \in \mathscr{D}$, $N(d)$ is the usual $(v \times b)$ treatment-block incidence matrix of d. $C(d)$ will denote the information matrix of d : $C(d) = R(d) - k^{-1}N(d)N(d)^{\mathrm{T}}$. $\mu(d)$ will denote the vector of positive eigenvalues of $kC(d)$. $\lambda_{i,j}$ will denote the (i, j)th entry of $N(d)N(d)^{\mathrm{T}}$.

We shall drop d from the notations in (iv) and (v) when there is no scope of confusion as to which design is meant.

Next, we present a few known definitions.

Definition 2.2 [1]. A design $d_1 \in \mathscr{D}$ is said to be better than another design $d_2 \in \mathscr{D}$ in the sense of majorisation (in short M-better) if

$$\mu(d_1) \prec^{w} \mu(d_2).$$

$d^* \in \mathscr{D}_{b,k,v}$ is said to be optimal in the sense of majorisation in a subclass of $\mathscr{D}_{b,k,v}$ (or, in short, d^* is M-optimal in this subclass) if it is M-better than every member of this subclass.

Definition 2.3 [5]. Let M be a number larger than all the eigenvalues of $C(d)$ for all $d \in \mathscr{D}$. Then, a thrice differentiable function $f : (0, M) \to \mathbb{R}$ is said to a (generalised) optimality criterion of type 1 (respectively, type 2) if (i) $f(0+) = \infty$, (ii) $f' < 0$, (iii) $f'' > 0$, (iv) $f''' < 0$ (respectively, $f''' > 0$). If f is such a function, then define $\Psi_f : \mathscr{D} \to \mathbb{R}$ by $\Psi_f(d) = \sum_{i=2}^{v} f(\mu(d)_i^{\uparrow})$, $d \in \mathscr{D}$. We say that the design d_1 is better than the design d_2 with respect to the criterion f (in short f-better) if $\Psi_f(d_1) \leqslant \Psi_f(d_2)$. A design d^* is said to be type 1 (respectively, type 2) optimal in a subclass of $\mathscr{D}$ if it is f-better than all the designs in this subclass for all type 1 (respectively, type 2) optimality criteria f.

Definition 2.4. A design d^* is said to be E-optimal in $\mathscr{D}$ if $\mu(d^*)_1^{\uparrow} \geqslant \mu(d)_1^{\uparrow}$ $\forall d \in \mathscr{D}$.

Definition 2.5. A design d^* is said to be A-optimal in $\mathscr{D}$ if $\sum_{i=1}^{v-1}(\mu_i(d^*))^{-1} \leqslant \sum_{i=1}^{v-1}(\mu_i(d))^{-1}$ $\forall d \in \mathscr{D}$.

Extending the notion of A-optimality criterion to vectors, we define the following.

Definition 2.6. An $n \times 1$ vector x is said to be A-better than another $n \times 1$ vector y if

$$\psi_{x,y} = \sum_{i=1}^{n} y_i^{-1} - \sum_{i=1}^{n} x_i^{-1} > 0. \tag{2.6}$$

Remark 1. As noted in Remark 3.1 of Bagchi and Bagchi [1], if d_1 is M-better than d_2, then d_1 is A-better than d_2, apart from being better with regard to many other (convex) optimality criteria. In view of this, we have the following result.

Corollary 2.2. *Suppose the C-matrix of a design d satisfies Theorem 2.8 for certain z's and w's. If further $\prod_{i=1}^{m} z_i(\bar{z})^{v-m-n} \prod_{j=1}^{n} w_j$ is A-worse than $\mu(d^*)$, then d is A-worse than d^*.*

3. A review of known results

We shall refer to the paper Morgan and Srivastav [12] as MS throughout this paper.

First we present a result which is a direct consequence of Proposition 2.4 of Jacroux [9].

Theorem 3.1. *Suppose d^* is a design in $\mathscr{D}^B_{b,k,v}$ satisfying the following properties:*

(i) *C^*_d has spectrum of the form $\mu_1(\mu_2)^{v-3}\mu_3$,*
(ii) *d^* is E-optimal in $\mathscr{D}^B_{b,k,v}$,*
(iii) *d^* minimises $\mathrm{tr}[(C_d)^2]$ over $d \in \mathscr{D}^B_{b,k,v}$,*

Then d^* is type 1 optimal in $\mathscr{D}^B_{b,k,v}$.

Next, we state a well-known result of Cheng [6].

Theorem 3.2. *Suppose d^* is a design in $\mathscr{D}^B_{b,k,v}$ such that C_{d^*} has spectrum of the form $\mu_1(\mu_2)^{v-2}$ and further, d^* satisfies Property (iii) of Theorem 3.1. Then d^* is type 1 optimal in $\mathscr{D}^B_{b,k,v}$.*

We now consider two series of optimal NBBD(2)'s of MS. Using majorisation technique we now prove stronger optimality property for one series. For the other series, we provide a simpler proof for the known result. The parameters of both the series satisfy

$$k = 3, \quad v \equiv 2 \pmod 3, \quad r(k-1)/(v-1) \text{ is an integer (which is, of course, } \lambda). \tag{3.1}$$

Here r, λ are as defined in Notation 2.3(iii).

At first, we consider the set up satisfying $bk = vr + 1$. More precisely, the parameters are

$$v = 3t + 2, \ \ \lambda = 2 \quad \text{and hence } b = 3t^2 + 3t + 1 \ \text{ and } \ r = v - 1. \tag{3.2}$$

Here t is an integer $\geqslant 1$.

An NBBD(2) $\bar{d}_1$ and a non-binary design $\hat{d}_1$ with completely symmetric C-matrix co-exist in this set up. Both of them are optimal with regard to some optimal criteria or other. (For the description, construction and other details see MS and [15].)

Let us define

$$a = r(k-1) + \lambda. \tag{3.3}$$

We note that in a set up where $r(k-1)/(v-1)$ is an integer (which is the case here),

$$a = v\lambda. \tag{3.4}$$

Now, we express the spectrums of $kC_{\bar{d}_1}$ and $kC_{\hat{d}_1}$ in terms of a:

$$\text{spectrum}[kC_{\bar{d}_1}] = (a-1)a^{v-3}(a+3), \tag{3.5}$$

$$\text{spectrum}[kC_{\hat{d}_1}] = a^{v-1}. \tag{3.6}$$

Morgan and Uddin [13] proved that $\hat{d}_1$ is E-optimal in $\mathscr{D}_{b,k,v}$. Here we show that

Theorem 3.3. *$\bar{d}_1$ is E-optimal in $\mathscr{D}_{b,k,v}\backslash\{\hat{d}_1\}$.*

Throughout the remaining part of the paper, d will denote a competing design. Further, we follow the notations below.

Notation 3.1

(i) $A = kC_d$, $\mu = \mu(d)$.
(ii) While using Theorem 2.7, the vector y_{ij} is the normalised version of the vector x_{ij} presented. For instance, y_{11} of Lemma 4.4 is $(1/\sqrt{(6)})x_{11} = (1/\sqrt{(6)})(2, -1, -1)^{\mathrm{T}}$.

Proof of Theorem 3.3. Consider a design d in $\mathscr{D}_{b,k,v}\setminus\{\hat{d}_1\}$.

Case 1: $a_{ii} < r(k-1)$ for some i, say $i = 1$.

In this case, $a_{11} \leqslant r(k-1) - 2$ and so applying (a) of Theorem 2.7 with $B = A(1)$, we get $\mu_1^{\uparrow} < a - 1$.

Case 2: $a_{ii} = r(k-1)$ for every i.

Since $d \neq \hat{d}_1$, $\exists(i, j)$, such that $\lambda_{i,j} \leqslant \lambda - 1$. So, taking $B = A(i, j)$ and applying (b) of Theorem 2.7 we get $\mu_1^{\uparrow} \leqslant a - 1$ and the proof is complete. □

In fact $\hat{d}_1$ satisfies stronger optimality as it is shown below.

Theorem 3.4. *$\hat{d}_1$ is*

(a) *type 1 optimal in $\mathscr{D}^B_{b,k,v}$ and*
(b) *M-better than every non-binary design in $\mathscr{D}_{b,k,v}$ other than $\hat{d}_1$.*

Proof. (a) follows from the fact that $\hat{d}_1$ satisfies all the conditions of Theorem 3.1.

(b) Let d be an arbitrily fixed non-binary design other than $\hat{d}_1$. By (b) of Theorems 2.4 and 3.3 it is enough to show that

$$\mu_1^{\downarrow} \geqslant a. \tag{3.7}$$

Let $u = \max\{a_{ii}, 1 \leqslant i \leqslant v\}$.

Case 1: $u > r(k-1)$.

In this case, $\exists i$ such that $a_{ii} \geqslant r(k-1) + 2$, so that $\mu_1^{\downarrow} \geqslant a + 2$. Hence we are done.

Case 2: $u \leqslant r(k-1)$.

Case 2.1: $a_{ii} = r(k-1)$ for all i.

Since $d \neq \hat{d}$, $\exists(i, j)$, such that $\lambda_{i,j} \geqslant \lambda + 1$. So, taking $B = A(i, j)$ and applying (b) of Theorem 2.7 we get $\mu_1^{\downarrow} \geqslant a + 1$.

Case 2.2: $a_{ii} < r(k-1)$ for at least one i.

Let m be the number of i's such that $a_{ii} < r(k-1)$, i.e., $a_{ii} \leqslant r(k-1) - 2$. Then,

$$\sum_{i=1}^{v-1} \mu_i \leqslant vr(k-1) - 2m = (v-1)a - 2m. \tag{3.8}$$

Case 2.2.a: $m = v$.

In this case $\mu_i \leqslant a - 2\ \forall i$ and we are done.

Case 2.2.b: $m < v$.

In this case, $\exists i$, such that $a_{ii} = r(k-1)$. Thus, $\mu_1^{\downarrow} \geqslant a$. □

As an immediate corollary, we have

Corollary 3.1. *$\bar{d}_1$ is type 1 optimal in $\mathscr{D}_{b,k,v} \setminus \{\hat{d}_1\}$.*

We now consider another set up satisfying (3.1) and having $bk = vr + 2$. This was first considered by Roy and Shah [15] who provided the first example of a type 1 optimal unequally replicated design, referred to $\bar{d}_2$ here. $\bar{d}_2$ is an NBBD(2), according to the definition of MS. The set up of Roy and Shah [15] is of the following nature:

$$v \equiv 5 \,(\mathrm{mod}\ 6), \quad r = (v-1)/2, \quad \lambda = 1. \tag{3.9}$$

In MS, a very similar set up is considered. This has

$$v \equiv 2 \,(\mathrm{mod}\ 3), \quad r = 2(v-1), \quad \lambda = 2. \tag{3.10}$$

MS constructed an NBBD(2) having the form of C-matrix as well as its spectrum similar to that of $\bar{d}_2$ and proved the same optimality property. They also found a non-binary design, termed $\hat{d}_2$ here, which do not seem to satisfy any optimality property like $\hat{d}_1$. It is not known whether a design corresponding to $\hat{d}_2$ exists in the set up (3.9). The spectrums of these are as follows:

$$\mathrm{spectrum}[\bar{d}_2] = a^{v-2}(a+4), \tag{3.11}$$

$$\mathrm{spectrum}[\hat{d}_2] = a^{v-1}. \tag{3.12}$$

Here a is as in (3.3). Looking at the spectrums, the following result is clear.

Theorem 3.5. *$\bar{d}_2$ is M-better than $\hat{d}_2$.*

In MS, the optimality property of $\bar{d}_2$ has been derived from general lemmas. However, if we restrict to this particular set up and also use majorisation techniques, then the proofs becomes considerably simpler and transparent. This is what is done below. Henceforth, $\bar{d}_2$ would refer to both the designs of MS and [15]. We shall also refer to $\hat{d}_2$, which may be a hypothetical design in the set up (3.9).

Let us first state a well-known result.

Lemma 3.1. *Suppose x_i, $1 \leqslant i \leqslant n$ are integers satisfying $\sum_{i=1}^{n} x_i = a$. Let u be the greatest integer $\leqslant a/n$ and $g = a - nu$. Then $\sum_{i=1}^{n}(x_i)^2$ is minimum if x_i's are "as nearly equal as possible". More precisely,*

(a) $\displaystyle\sum_{i=1}^{n}(x_i)^2 \geqslant (n-g)a^2 + g(a+1)^2 = m(u)$ *say.*

(b) *Further, if $\sum_{i=1}^{n}(x_i)^2 > m(u)$, then $\sum_{i=1}^{n}(x_i)^2 \geqslant m(u) + 2$.*

Now we present a proof of the crucial property of $\bar{d}_2$.

Lemma 3.2. *$\bar{d}_2$ minimises* $\mathrm{tr}[(C_d)^2]$ *over* $d \in \mathcal{D}^B_{b,k,v}$.

Proof. Fix an arbitrary design $d \in \mathcal{D}^B_{b,k,v}$. Now, $\mathrm{tr}[(C_d)^2] = \sum (c_{dij})^2 = T_1(d) + 2T_2(d)$, where

$$T_1(d) = (k-1)^2 \left[\sum_{i=1}^{v} (r_i)^2\right] \quad \text{and} \quad T_2(d) = \sum_{i<j} (\lambda_{ij})^2. \tag{3.13}$$

Since $\sum_{i<j} \lambda_{i,j} = (1/2)v(v-1)\lambda + 2$, applying Lemma 3.1 on $\lambda_{i,j}$'s, we find that $m(\lambda) = (1/2)v(v-1)(\lambda)^2 + 4\lambda + 2 = T_2(\bar{d}_2) - 2$.

From this and the expression for $T_1(d)$, it is clear that if the replication vector of d is different from that of $\bar{d}_2$, then $\mathrm{tr}[(C_d)^2] > \mathrm{tr}[(C_{\bar{d}_2})^2]$. Hence, we assume

$$r_i = r, \quad 1 \leqslant i \leqslant v-2, \quad r_{v-1} = r_v = r+1.$$

In view of (b) of Lemma 3.1, all we have to show is $T_2(d) > m(\lambda)$. But to show that it is enough to show the following.

Claim. *The expression for $T_2(d)$ always contain at least a term $(\lambda - 1)^2$.*

Proof of the claim. Recall that

$$\sum_{j \neq i} \lambda_{i,j} = r_i(k-1). \tag{3.14}$$

So, $\sum_{j \neq v} \lambda_{i,j} = (v-1)\lambda + 2$, $i \in \{v, v-1\}$. So, $\sum_{j \neq v} (\lambda_{v,j})^2$ is minimum if $\lambda_{v,m} = \lambda_{v,l} = \lambda + 1$ for some $m, l < v$ and for all other j's, $\lambda_{v,j} = \lambda$. Clearly, one of m, l has to be $\leqslant v-2$. W.l.g., let $l = 1$. Then, $\lambda_{1,v} = \lambda + 1$, so that $\exists j$ such that $\lambda_{1,j} = \lambda - 1$. This completes the proof of the claim and hence the proof of the lemma. □

A direct consequence of the preceding lemma, in view of Theorem 3.2, is the following.

Corollary 3.2. *$\bar{d}_2$ is type* 1 *optimal in $d \in \mathcal{D}^B_{b,k,v}$.*

We shall now consider the general class and prove the following result.

Theorem 3.6. *$\bar{d}_2$ is type* 1 *optimal in $d \in \mathcal{D}_{b,k,v}$.*

Proof. Fix an arbitrary non-binary design $d \in \mathcal{D}_{b,k,v}$. In view of Corollary 3.2, it is enough to show that $\bar{d}_2$ is M-better than d.

Case 1: d has at least two non-binary blocks.

In this case, $\mathrm{tr}[C_d] \leqslant \mathrm{tr}[C_{\hat{d}_2}]$ [for the description of $\hat{d}_2$ see Section 4 of MS]. Since $C_{\hat{d}_2}$ is completely symmetric, d is M-worse than $\hat{d}_2$ and hence M-worse than $\bar{d}_2$ by Theorem 3.5.

Case 2: d has exactly one non-binary block.

Let β denote the non-binary block. Since $k = 3$, only one treatment (say i_0) can appear more than once in β. Since β is the only non-binary block, $n_{i,j} \leqslant 1\ \forall i \neq i_0, \forall j$.

W.l.g., let $v \neq i_0$. Applying (a) of Theorem 2.7 with $A = kC_d$, $B = A(v)$, $\mu = \mu(A)$, we get $\mu_1^{\downarrow} \geqslant v(r+1)(k-1)/(v-1) \geqslant a + 2$. Therefore, $\sum_{i=1}^{v-2} \mu_i^{\uparrow} \leqslant (v-2)a$. Hence the result follows from (b) of Theorem 2.4. □

4. A new optimality result

We consider a set up where $k = 3$ and $bk + 1$ is divisible by v. Thus, $bk = vr - 1$. We further assume $r = (v - 1)/2$, so that $\lambda = 1$. Thus, the parameters are of the following form:

$$v = 6s + 5, \quad r = 3s + 2 \quad \text{and hence } b = 6s^2 + 9s + 3. \tag{4.1}$$

Here s is an integer $\geqslant 1$.

Let d^* denote the design with the following parameters. $r_1 = r - 1, r_i = r, 2 \leqslant i \leqslant v; \lambda_{1,2} = \lambda - 1 = \lambda_{1,3}, \lambda_{2,3} = \lambda + 1, \lambda_{i,j} = \lambda$ for all other (i, j)'s.

Lemma 4.1. *The spectrum of kC_{d^*} is as follows:*

$$\mu(d^*) = (a - 3)^1 a^{v-3} (a + 1)^1. \tag{4.2}$$

Proof. By straightforward verification we find that the spectrum of kC_{d^*} in terms of a is as above. [Recall (3.3).] □

Remark 2. Even though a happens to be equal to v in the present set up [see (3.4)], we prefer to continue with the symbol a, so that the magnitudes of the eigenvalues are not mixed up with the multiplicities.

Remark 3. It is easy to verify that d^* satisfies conditions (i) and (iii) of Theorem 3.1. But it appears that it does not satisfy condition (ii), although we have not yet found a design E-better than d^*. Because of this, Theorem 3.1 could not be applied and general optimality of d^* could not be proved. We believe that d^* is also D-optimal, but the proof would be more involved.

We now present our main result.

Theorem 4.1. *d^* is A-optimal in $\mathscr{D}_{b,k,v}$ with b, k, v as in* (4.1) *provided $a \geqslant 11$.*

We prove this in two steps. First, we show that

Theorem 4.2. *d^* is A-better than any non-binary design in $\mathscr{D}_{b,k,v}$, whenever $a \geqslant 11$.*

Next, we prove

Theorem 4.3. *d^* is A-optimal in $\mathscr{D}^B_{b,k,v}$ if the parameters satisfy $a \geqslant 11$.*

An outline of the proofs of Theorems 4.2 *and* 4.3: We fix an arbitrary design d: a non-binary design in $\mathscr{D}_{b,k,v}$ for the former and a design in $\mathscr{D}^B_{b,k,v}$ for the later theorem with b, k, v as in (4.1). We need to show that d is A-worse than d^* whenever $a \geqslant 11$. To do this, we proceed as follows. In the Appendix, we have listed vectors vi, $1 \leqslant i \leqslant 12$ and proved in Theorems A4 and A5 that each of them is A-worse than $v0 = \mu(d^*)$, if $a \geqslant 11$. Therefore by Corollary 2.2, it is enough to show that $\mu(d)$ is A-worse than vi for some i, $1 \leqslant i \leqslant 12$. This is what is done here. Now the proof for Theorem 4.3 is quite involved. We first rule out the possibility of d having the replication vector different from d^*. [See Theorem 4.4.] Next we take up $\lambda_{i,j}$'s. We show that if these are too small or too big, then d is A-worse than d^*. Explicitly, we find that $\lambda_{i,j}$'s must satisfy (4.8) and (4.10). Thus, there are two possibilities for $\lambda_{2,3}$: λ or $\lambda - 1$. These two cases are handled in Theorems 4.5 and 4.6, respectively. For proving each of these theorems, we have to handle several cases separately. Usually, to show that $\mu(d)$ is A-worse than some vl, we apply Theorem 2.6 or 2.7 on submatrices of $A = kC_d$. We shall also use the result of Corollary A3 of Appendix often.

Proof of Theorem 4.2. Consider a non-binary design $d \in \mathscr{D}_{b,k,v}$. Let

$$\delta = \max_{1 \leqslant i \leqslant v} a_{ii}.$$

Claim. *If $\delta < r(k-1)$, then d is M-worse than d^*.*

Proof of the claim. Suppose the hypothesis is true. Then, $a_{ii} < r(k-1) - 2$, for each i. This implies the following statements.

(a) $\operatorname{tr}(A) < (v-1)(a-2)$ and
(b) $\exists (i, j)$ such that $\lambda_{i,j} \leqslant \lambda - 1$.

Now, (a) implies

$$\sum_{i=1}^{v-2} \mu_i^{\uparrow} \leqslant (v-2)(a-2). \tag{4.3}$$

Further, in view of (b), applying (a) of Theorem 2.7 with $B = A(i, j)$, we get $\mu_1^{\uparrow} \leqslant a - 3$. This, together with (4.3) and (c) of Theorem 2.4 proves the claim.

So, we assume $\delta \geqslant r(k-1)$. This means $a_{ii} \geqslant r(k-1) + 2$, for some i. Thus, by Corollary 2.1, $\mu_1^{\downarrow} \geqslant a$. Again, as $r_1 \leqslant r - 1$, $\mu_1^{\uparrow} < a - 2$, by the same corollary. These, in view of Theorem 2.8 implies that μ is M-worse than the vector $v12$ of Appendix. Hence, the proof is complete by Lemma A5 and Corollary 2.2. □

Before going to the proof of Theorem 4.3, we obtain a few useful results, the first of which is trivial.

Lemma 4.2. *Consider a design d. Fix a treatment i.*

(a) *If $r_i < r$, then either $\lambda_{i,j} \leqslant \lambda - 2$ for some $j \neq i$, or there exist j_1, j_2, such that $\lambda_{i,j_u} \leqslant \lambda - 1$, $u = 1, 2$.*
(b) *If $r_i = r$, and $\lambda_{i,j} >$ (respectively, $<$) λ for some j, then there exists l such that $\lambda_{i,l} <$ (respectively, $>$) λ.*

Lemma 4.3. *For any d, $\mu_1^{\downarrow} \geqslant a + 1$.*

Proof. W.l.g., let us assume that the replication numbers of d are in the increasing order.

Case 1: The replication vector of d is different from that of d^*.
Then, $r_v > r$, that is $r_v \geqslant r + 1$. Now applying Corollary 2.1 with $i = v$, we get $\mu_1^{\downarrow} \geqslant a + 2$.

Case 2 (*Remaining case*): The replication vector of d is same as that of d^*.

By (a) of Lemma 4.2, $\lambda_{1,j} < \lambda$ for some j. W.l.g., let $j = 2$. Then, by (b) of Lemma 4.2, there exist l, such that $\lambda_{2,l} \geqslant \lambda + 1$. Now we apply (a) of Theorem 2.7 with B_1 as $A(2, l)$ and get the required result. □

Corollary 4.1. *If* $\mu_1^{\uparrow} \leqslant a - 3$, *then* d *is M-worse than* d^*.

Proof. By (c) of Theorem 2.4. □

Theorem 4.4. *If the replication vector of* d *is not the same as that of* d^* *then* d *is A-worse than* d^*.

Proof. It is enough to show that if $r_1 + r_2 \leqslant 2r - 2$, then d is A-worse than d^*.

Case 1: $r_1 \leqslant r - 2$.

In this case, applying Corollary 2.1 with $i = 1$, we get $\mu_1^{\uparrow} \leqslant a - 4$ and so the result follows from Corollary 4.1.

Case 2: $r_1 = r_2 = r - 1$. Clearly, $r_v \geqslant r + 1$.

First we take $B_1 = A(1, 2)$. Applying (a) of Theorem 2.7 if $\lambda_{1,2} \leqslant \lambda - 1$ and (a) of the same theorem if $\lambda_{1,2} \geqslant \lambda + 1$, we get $\mu_1 \leqslant a - 3$ and we are done by Corollary 4.1. Hence, we assume $\lambda_{1,2} = \lambda$.

Now, we take $i = v$ and apply Corollary 2.1. We get

$$\mu_1^{\downarrow} \geqslant a + 2v/(v - 1). \tag{4.4}$$

Again, by Lemma 4.2, there exist $j \neq 1, 2, l \neq 1, 2$ such that $\lambda_{1,j} \leqslant \lambda - 1$ and $\lambda_{2,l} \leqslant \lambda - 1$. We choose $B_1 = A(1, j)$, $B_2 = A(2, l)$. Now applying Theorem 2.7(a) and using (4.4) we find that $v1 \prec^w \mu(d)$. □

In view of the preceding theorem, henceforth we assume that d has the same replication vector as d^*.

Next we obtain bounds on $\lambda_{i,j}$'s.

Lemma 4.4. *If one of the following conditions holds, then* d *is M-worse than* d^*:

(a) $|\lambda_{i,j} - \lambda| \geqslant 3$, *for some* (i, j), $i, j > 1$.
(b) $|\lambda_{1,j} - \lambda| \geqslant 2$, *for some* $j > 1$.
(c) $\lambda_{1,j} = \lambda_{1,l} = \lambda - 1$, $\lambda_{l,j} > \lambda$; *for some* $l, j > 1$.

Proof. In view of Corollary 4.1, it is enough to show that $\mu_1^{\uparrow} \leqslant a - 3$.

Suppose condition (a) holds. Taking B_1 (respectively, B_0) $= A(i, j)$ if $\lambda_{i,j} <$ (respectively, $>$) λ and applying (a) (respectively, (b)) of Theorem 2.7, we get the required condition.

Now suppose condition (b) holds. Recall that $r_1 = r - 1$, so that $a_{1,1} = r(k - 1) - 2$. Proceeding on the line as above with $A(1, j)$ instead of $A(i, j)$, we get the result.

Finally, assume condition (c). We take $B_1 = A(1, j, l)$, $x_{1,1} = (2, -1, -1)^{\mathrm{T}}$. Now, applying (a) of Theorem 2.7 we get the required result. □

In view of our findings above and Lemma 4.2, we can assume w.l.g., that

$$\lambda_{1,2} = \lambda_{1,3} = \lambda - 1, \quad \lambda_{2,3} \leqslant \lambda. \tag{4.5}$$

W.l.g. let $j = 2, l = 3$. Then,

Lemma 4.5. *If $|\lambda_{i,j} - \lambda| \geqslant 2$, for some (i, j), $i, j > 1$, then d is A-worse than d^*.*

Proof. We shall show that if the condition holds then, $\mu(d)$ is M-worse than $v\mathbf{1}$. Now, suppose the condition holds. Then, by Lemma 4.4, $\lambda_{i,j} = \lambda - 2$ or $\lambda + 2$.

Case 1: $\lambda_{i,j} = \lambda - 2$.

Applying (b) of Theorem 2.7 on $C = A(i, j)$ we get

$$\mu_1^{\downarrow} \geqslant a + 2v/(v - 2). \tag{4.6}$$

In view of Lemma A4, it is enough to show

$$\sum_{j=1}^{l} \mu_j \leqslant l(a - 2), \quad l = 1, 2. \tag{4.7}$$

Case 1.1: $\{i, j\} = \{2, 3\}$.

Then we choose $B_1 = A(1, 2, 3)$, $x_{1,1} = (2, -1, -1)^{\mathrm{T}}$ and $x_{1,2} = (0, 1, -1)^{\mathrm{T}}$. Applying (a) of Theorem 2.7 we get (4.7).

Case 1.2: $\{i, j\}$ and $\{2, 3\}$ are disjoint. Then take l to be anyone in $\{2, 3\}$.

Case 1.3: $\{i, j\}$ and $\{2, 3\}$ has one element in common. Take l to be the element of $\{2, 3\}$ which is not in $\{i, j\}$.

In both Cases 1.2 and 1.3 we take $B_1 = A(i, j)$ and $B_2 = A(1, l)$. Clearly, B_1 and B_2 are disjoint. Now taking $x_{1,1} = x_{2,1} = (1, -1)^{\mathrm{T}}$ and applying (a) of Theorem 2.7 on B_1, B_2, we find that (4.7) holds in these cases also. So, the proof for Case 1 is complete.

Case 2: $\lambda_{i,j} = \lambda + 2$. If $\{i, j\} = \{2, 3\}$, then we are done by Lemma 4.4. So, assume $\{i, j\} \neq \{2, 3\}$.

We take Cases 2.2 and 2.3 exactly like 1.2 and 1.3, respectively, and chose l as there. Let m be the other element of $\{2, 3\}$. Taking $B_0 = A(i, j)$, $B_1 = A(1, l)$, $x_{1,1} = (1, -1)^{\mathrm{T}}$ and applying (b) of Theorem 2.7 on B_0, B_1, we find that (4.7) holds. Again, (a) of Theorem 2.7 on $B_2 = A(i, j)$ yields (4.6). Hence, we are done in this case also. □

In view of the above, we assume the following.

$$\text{For } i \neq j, \quad i, j > 1, \quad \lambda_{i,j} \in \{\lambda, \lambda - 1, \lambda + 1\}. \tag{4.8}$$

Using this, we are able to extend Lemma 4.2 as follows.

Lemma 4.6. *Fix $i \geqslant 2$. Let $S = \{j \neq i : \lambda_{i,j} = \lambda - 1\}$ and $T = \{j \neq i : \lambda_{ij} = \lambda + 1\}$. Then the sizes of S and T are equal.*

For $\lambda_{1,i}$'s, we can say more as shown below.

Lemma 4.7. *Let $S = \{i : \lambda_{1,i} = \lambda - 1\}$. If the size of S is $\geqslant 3$ then d is A-worse than d^*.*

Proof. Suppose $|S| \geqslant 3$. W.l.g., let $S = \{2, 3, 4, \ldots\}$. If $\lambda_{i,j} = \lambda + 1$, for some $i, j \in S$ then we are done by Lemma 4.4. So, assume $\lambda_{i,j} \leqslant \lambda$. Further, replace S by its subset $= \{2, 3, 4\}$. Let s be the number of pair (i, j) such that $\lambda_{ij} = \lambda - 1$. Then, $s = 0, 1, 2$, or 3. We take $B_1 = A(1, 2, 3, 4)$ and apply Theorem 2.6. We get

$$\mu_1^{\downarrow} > a + 1 + s/2. \tag{4.9}$$

Now; we consider the different values of s. In each case we apply (a) of Theorem 2.7 with B_1 as above and $u_1 = 1$ or 2 mutually orthogonal vectors among which $x_1 = (3, -1, -1, -1)^T$ is one. Thus, for every s,

$$\mu_1^{\uparrow} \leqslant a - 3 + s/6.$$

So, in view of Corollary 4.1, if $s = 0$ then we are done. If $s = 3$, then the inequality above together with (4.9) yields that $\mu(d)$ is M-worse than $v2$.

Now we consider the cases $s = 1$ and $s = 2$. We assume, w.l.g., that $\lambda_{3,4} = \lambda - 1$ when $s = 1$ and $\lambda_{2,3} = \lambda_{2,4} = \lambda - 1$ when $s = 2$. Let $x_2 = (0, 2, -1, -1)^T$, $x_3 = (0, 0, 1, -1)^T$. Now we apply Theorem 2.7, with the vectors x_1, x_3 if $s = 1$ and x_1, x_2 if $s = 2$. We find that $\mu(d)$ is M-worse than $v4$ if $s = 1$ and $v7$ if $s = 2$. So, our proof is complete. □

In view of the lemma above, we assume the following:

$$\lambda_{1,2} = \lambda_{1,3} = \lambda - 1; \quad \lambda_{1,j} = \lambda, \ j \geqslant 4; \quad \lambda_{2,3} = \lambda \text{ or } \lambda - 1. \tag{4.10}$$

Theorem 4.5. *If $\lambda_{2,3} = \lambda$ then d is A-worse than d^*.*

Proof. By Lemma 4.2, $\exists j_1, j_2$ such that $\lambda_{2,j_1} = \lambda_{3,j_2} = \lambda + 1$.

Case 1: $j_1 = j_2$. W.l.g., let $j_1 = 4$.

By Lemma 4.2, $\exists j_3, j_4$ such that $\lambda_{4,j_3} = \lambda_{4,j_4} = \lambda - 1$. W.l.g., let $j_3 = 5$, $j_4 = 6$. But then $\exists j$ such that $\lambda_{5,j} = \lambda + 1$.

Case 1a: $j = 2$.

Case 1a.1: $\lambda_{3,5} = \lambda - 1$. We take $B_1 = A(1, 2, 3, 4, 5)$, $x_{1,1} = (4, -1, -1, -1, -1)^T$, $x_{1,2} = (0, 0, -1, -1, 2)^T$; $B_2 = A(2, 4, 5)$ and $x_{2,1} = (2, -1, -1)^T$. Now, applying (a) of Theorem 2.7 on B_1 and (b) of the same theorem on B_2 we get the following inequalities:

$$\sum_{i=1}^{l} \mu_i^{\uparrow} \leqslant \sum_{i=1}^{l} z_i,$$

$$\mu_1^{\downarrow} \geqslant a + 5/3.$$

Here $l = 1, 2$, $z_1 = a - 5/2$ and $z_2 = a - 5/3$. So, by Theorem 2.8, $v3 \prec^w \mu(d)$ and this case is settled.

Case 1a.2: $\lambda_{3,5} \geqslant \lambda$. In this case we take B_1 as in Case 1a.1, $x_{1,1} = (8, -3, -3, -1, -1)^T x_{1,2} = (0, 0, 0, -1, 1)^T$, $B_2 = A(2, 3, 4, 5)$, $x_{2,1} = (1, 1, -1, -1)^T$. Now, proceeding as before, we get $v7 \prec^w \mu(d)$ and we are done with Case 1a.

Case 1b: $j > 2$. This means $j > 5$.

We take $B_1 = A(1, 2, 3)$, $B_2 = A(4, 5, 6)$ and $x_{1,1} = x_{2,1} = (2, -1, -1)^T$. In view of Corollary A3, we assume $\lambda_{5,6} = \lambda$, w.l.g. Now, applying (a) of Theorem 2.7 on B_1, B_2 we get the following inequalities:

$$\sum_{i=1}^{l} \mu_i^{\uparrow} \leqslant \sum_{i=1}^{l} z_i, \quad l = 1, 2. \tag{4.11}$$

Here $z_1 = a - 8/3$, $z_2 = a - 4/3$.

Further, we take $B_3 = A(2,3,4)$, $x_{3,1} = (-1,-1,2)^{\mathrm{T}}$, $B_4 = A(5,j)$. Now, applying (b) of Theorem 2.7 on B_3, B_4 we get

$$\sum_{i=1}^{m} \mu_i^{\downarrow} \geqslant \sum_{i=1}^{m} w_i, \quad m = 1, 2, \tag{4.12}$$

where $w_1 = a + 4/3$ and $w_2 = a + 1$. Thus, by Theorem 2.8, $v6 \prec^w \mu(d)$. Hence Case 1 is settled.

Case 2: $j_1 \neq j_2$. W.l.g., we asssume $j_1 = 4$, $j_2 = 5$.

If $\lambda_{3,4} = \lambda + 1$, then we are reduced to Case 1. So, let $\lambda_{3,4} = \lambda$ or $\lambda - 1$, w.l.g. Similarly $\lambda_{2,5} = \lambda$ or $\lambda - 1$.

Case 2a: At least one of $\lambda_{2,5}$ and $\lambda_{3,4}$ is $\lambda - 1$. W.l.g., let $\lambda_{3,4} = \lambda - 1$.

Case 2a.1: $\lambda_{4,5} = \lambda + 1$. We take B_1, B_2 and $x_{1,1}$ same as in Case 1a.2, but $x_{1,2} = (1,-1,1,-1)^{\mathrm{T}}$ and $x_{2,1} = (1,-1,-1,1)^{\mathrm{T}}$. Then, proceeding along the same lines we get the same result as in Case 1a.2.

Case 2a.2: $\lambda_{4,5} \leqslant \lambda$. We take B_1 as in the preceding case, $x_{1,1} = (2,-1,-1,0,0)^{\mathrm{T}}$, $x_{1,1} = (0,1,-1,1,-1)^{\mathrm{T}}$. Applying (a) of Theorem 2.7 on B_1 we get an equations like (4.11) with $l = 1, 2$, the same z_1 but $z_2 = a - 3/2$.

Further, we take $B_3 = A(2,4)$, $B_4 = A(5,j)$. Now, applying (b) of Theorem 2.7 on B_3, B_4 we get two equations like (4.12) with $w_1 = w_2 = a + 1$. Thus, by Theorem 2.8, $v9 \prec^w \mu(d)$. Hence Case 2a is settled.

Case 2b: $\lambda_{2,5} = \lambda_{3,4} = \lambda$.

Case 2b.1: $\lambda_{4,5} = \lambda + 1$.

$\exists j_1, j_2$ such that $\lambda_{5,j_1} = \lambda_{5,j_2} = \lambda - 1$.

We take $B_1 = A(1,2,3)$, $B_2 = A(5,j_1,j_2)$. In view of Corollary A3, we may assume $\lambda_{j_1,j_2} = \lambda$ w.l.g. Applying the same argument as in Case 1b get the same system of two inequalities. Now, we take $B_3 = A(2,3,4,5)$, $x_{3,1} = (1,-1,-1,1)^{\mathrm{T}}$ and apply (b) of Theorem 2.7. We get $\mu_1^{\downarrow} \geqslant 3/2$. Now, using Theorem 2.8, we have $v5 \prec^w \mu(d)$.

Case 2b.2: $\lambda_{4,5} = \lambda - 1$.

We take $B_1 = A(1,2,3,4,5)$, $x_{1,1} = (2,-1,-1,0,0)^{\mathrm{T}}$, $x_{1,2} = (0,1,-1,1,-1)^{\mathrm{T}}$, $B_2 = A(2,3,4,5)$, $x_{2,1} = (1,1,-1,-1)^{\mathrm{T}}$. Then, applying Theorem 2.7: (a) on B_1 and (b) on B_2 we find $v5 \prec^w \mu(d)$.

Case 2b.3: $\lambda_{4,5} = \lambda$. We take $B_1, x_{1,1}$ as in case 1b. Now, by Lemma 4.6, $\exists j, l$ such that $\lambda_{4,j} = \lambda - 1$ and $\lambda_{5,l} = \lambda - 1$.

If $j = l$, we take $B_2 = A(4,5,j)$, $x_{2,1} = (1,1,-2)^{\mathrm{T}}$. If $j \neq l$, we take $B_2 = A(4,j)$, $B_3 = A(5,l)$. We apply (a) of Theorem 2.7 on B_1, B_2 in the former case and B_1, B_2, B_3 in the later case. We obtain (4.11) with the same z_1. In the former case we have the same range for l and $z_2 = a - 4/3$. In the later, $l = 1, 2, 3$; $z_2 = z_3 = a - 1$. Next, we take B_4, B_5 as in Case 2a.2 and get the same equations. Combining these, we see that $v9 \prec^w \mu(d)$ if $j = l$ and $v10 \prec^w \mu(d)$ otherwise.

The proof of this theorem is now complete. □

Now we consider the remaining possibility in the next theorem.

Theorem 4.6. *If $\lambda_{2,3} = \lambda - 1$ then d is A-worse than d^*.*

Proof. By Lemma 4.6, $\exists j_1, j_2, j_3, j_4$ such that $\lambda_{2,j_1} = \lambda_{2,j_2} = \lambda_{3,j_3} = \lambda_{3,j_4} = \lambda + 1$.

Case 1: One element is common between $\{j_1, j_2\}$ and $\{j_3, j_4\}$. W.l.g., let $j_1 = j_3 = 4$ and $j_4 = 5$. By Lemma 4.6, $\exists j$ such that $\lambda_{5,j} = \lambda - 1$.
Case 1a: $j > 5$. Take $B_1 = A(1, 2, 3, 4)$, $x_{1,1} = (3, -1, -1, -1)^{\mathrm{T}}$, $x_{1,2} = (0, 1, -1, 0)^{\mathrm{T}}$, $B_2 = A(5, j)$, $B_3 = A(2, 4)$ and $B_4 = A(3, 5)$. Applying (a) of Theorem 2.7 on B_1, B_2 we see that (4.11) is satisfied with $l = 1, 2, 3$, $z_1 = a - 8/3$, $z_2 = z_3 = a - 1$. Further, applying (b) of the same theorem on B_3, B_4 we find that (4.12) is satisfied with the same values of w_1, w_2. Thus, $v9 \prec^w \mu(d)$. Hence this case is settled.
Case 1b: $j \leqslant 5$. This means $j = 2$. We take $B_1 = A(1, 2, 3, 4, 5)$, $x_{1,1} = (4, -1, -2, -1, 0)^{\mathrm{T}}$, $x_{1,2} = (0, 2, -1, 0, -1)^{\mathrm{T}}$. Applying (a) of Theorem 2.7 on B_1 we see that (4.11) is satisfied with $l = 1, 2$, $z_1 = a - 29/11 < a - 5/2$, $z_2 = a - 5/3$. Again, taking $B_2 = A(2, 3, 4)$, $x_{2,1} = (1, 1, -2)^{\mathrm{T}}$ and applying (b) of the same theorem, we get (4.12) with $m = 1$ and $w_1 = a + 5/3$. Hence, $v3 \prec^w \mu(d)$ and this case is also settled.

Case 2: No element is common between $\{j_1, j_2\}$ and $\{j_3, j_4\}$. W.l.g., we assume $j_1 = 4$, $j_2 = 5$, $j_3 = 6$, $j_4 = 7$.

In this case, we must have

$$\lambda_{2,j} \leqslant \lambda, \quad j \in \{6, 7\}, \tag{4.13}$$

as otherwise it will be same as case 1.

Similarly,

$$\lambda_{3,j} \leqslant \lambda, \quad j \in \{4, 5\}. \tag{4.14}$$

Now, we take $B_1 = A(2, 4, 5)$ and $B_2 = (3, 6, 7)$. In view of Corollary A3 we can assume $\lambda_{4,5} = \lambda_{6,7} = \lambda$. Then, applying Theorem 2.7 on B_1, $x_{1,1} = (2, -1, -1)^{\mathrm{T}}$; B_2, $x_{2,1} = x_{1,1}$ we get the following equation:

$$\sum_{i=1}^{m} \mu_i^{\downarrow} \geqslant m(a + 4/3); \quad m = 1, 2. \tag{4.15}$$

Case 2.1: $\lambda_{3,4} = \lambda - 1$.
Case 2.1a: $\lambda_{4,6} = \lambda + 1$.

Appling Lemma 4.6 on $i = 6$ we find that $\exists l, m$ such that $\lambda_{6,l} = \lambda_{6,m} = \lambda - 1$. W.l.g., we may assume that $l \neq 2$.

We take $B_3 = A(1, 2, 3, 4)$, $x_{3,1} = (3, -1, -1, -1)^{\mathrm{T}}$, $x_{3,2} = (0, 1, -2, 1)^{\mathrm{T}}$; $B_4 = A(6, l)$, $x_{4,1} = (1, -1)^{\mathrm{T}}$. Now, applying (a) of Theorem 2.7 on B_3, B_4 we get equations similar to (4.11) with $l \leqslant 3$, $z_1 = a - 7/3$, $z_2 = a - 5/3$, $z_3 = a - 1$. These, combined with (4.15) gives $v10 \prec^w \mu(d)$. Hence this case is settled.
Case 2.1b: $\lambda_{4,6} \leqslant \lambda$.

We take $B_3 = A(1, 2, 3, 4, 6)$, $x_{3,1} = (2, -1, -1)^{\mathrm{T}}$, $x_{3,2} = (1, -1, 1, -1)^{\mathrm{T}}$. Application of (a) of Theorem 2.7 on B_3 yields inequalities similar to (4.11) with $l = 1, 2$, $z_1 = a - 7/3$, $z_2 = a - 2$. [Recall that $\lambda_{2,6} \leqslant \lambda$.] This with (4.15) says $v11 \prec^w \mu(d)$. Hence this case is also settled.
Case 2.2. $\lambda_{3,4} = \lambda$. [In view of (4.14), this is the remaining case.]
Case 2.2a. $\lambda_{4,6} = \lambda + 1$.

We take $B_3 = A(1, 2, 3, 4, 6)$, $x_{3,1} = (4, -1, -1, -1, -1)^{\mathrm{T}}$, $x_{3,2} = (0, 2, -2, 1, -1)^{\mathrm{T}}$. We get (4.11) with $l \leqslant 2$, $z_1 = a - 13/5$, $z_2 = a - 7/5$. Now, using the fact that $(a - 7/3, a - 5/3) \prec^w (a - 13/5, a - 7/5)$ together with (4.15), we see that $v10 \prec^w \mu(d)$ and the proof is complete for this case.

Case 2.2b. $\lambda_{4,6} \leqslant \lambda$.

Let $t = \lambda - \lambda_{4,6}$. So, $t = 0$ or 1. We take B_3 and $x_{3,1}$ as in Case 2.2a and $x_{3,2} = (0, 1, -1, 1, -1)^T$. Then, proceeding in the same way as that case and using the fact that $(a - 7/3, a - 5/3) \prec^w (a - 5/2, a - 3/2)$, we find that if $t = 0$, $v10 \prec^w \mu(d)$. If $t = 1$, then $v11 \prec^w \mu(d)$. Thus this case is settled and hence the proof of the theorem is complete. □

5. Construction

In this section we present an infinite series of designs d^* [see the statement following (4.1)] with $\lambda = 1$.

Notation 5.1

(a) p is an odd integer, say $p = 2s + 1$. P is the set is the set of integers modulo p. $P^* = P \backslash \{0\}$.
(b) $I = \{0, 1, 2\}$.
(c) $V_0 = I \times P$.
(d) $V = V_0 \cup \{\alpha, \beta\}$ is the set of treatments.

We first construct a Steiner triple $(\tilde{d})$ on the treatment set V_0 as follows.

Let $B = \{B_t : t \in P^*\}$; $B_t = \{(0, t), (0, -t), (1, 0)\}$ and $C = \{(0, 0), (1, 0), (2, 0)\}$. Clearly, $|B| = S \cdot \tilde{d}$ will consist of blocks of two types. The blocks of type one are generated by adding (i, j) to each member of B and those of type two by adding $(0, j)$ to C. Here the addition is mod 3 for the first and mod p to the second co-ordinate. Thus, $\tilde{d}$ has $3sp$ blocks of type one and p blocks of type two and hence altogether $\tilde{b} = 3sp + p = 6s^2 + 5s + 1$ blocks.

Theorem 5.1. *d^* with $\lambda = 1$ exists for all s, whenever P^* defined above can be partitioned into two sets Q, R such that $|Q| = |R| = s$ and for every $u \in Q$, $2u \in R$.*

Proof. Let us assume the condition on P. We construct d^* from $\tilde{d}$ as follows. We take a certain subclass consisting of $p - 1$ blocks of type one and all the p blocks of type two. We replace each one of these $2p - 1$ blocks by two blocks. Finally, we add the block

$$\{(1, 0), (2, 0), \alpha\}.$$

Thus, we get $b^* = \tilde{b} + 2p - 1 + 1 = 6s^2 + 9s + 3$ blocks. We now describe the procedure for replacement.

For $j \in P$, the block $\{(0, j), (1, j), (2, j)\}$ of type two is replaced by the pair of blocks

$$\{(0, j), (1, j), \alpha\} \text{ and } \{(0, j), (2, j), \beta\}, \quad \text{if } j \in Q \cup \{0\},$$

and

$$\{(0, j), (2, j), \alpha\} \text{ and } \{(0, j), (1, j), \beta\}, \quad \text{if } j \in R.$$

Now, let us consider the following set L of $p - 1$ $(= 2s)$ blocks of type 1 of $\tilde{d}$.

$$L = \{L_t : t \in P^*\}; \quad L_t = \{(1, 0), (1, 2t), (2, t)\}.$$

We replace L_t by two blocks, say D_t and E_t, which are as follows:

$$D_t = \{(1, 0), (1, t), (2, t)\}, \quad t \in P^*;$$
$$E_t = \begin{cases} \{(1, 2t), (2, t), \alpha\} & \text{if } t \in Q, \\ \{(1, 2t), (2, t), \beta\} & \text{if } t \in R. \end{cases}$$

It is easy to verify that

(i) every pair of treatments of V_0 appear together exactly once,
(ii) α occurs twice with (1, 0) once with all other treatments of V_0,
(iii) β does not occur with (1,0) and it occurs with all other treatments of V_0 exactly once,
(iv) α, β do not occur together.

These completes the proof of our theorem. □

Remark 4. Regarding the partitioning of P^* in Theorem 5.1, we observe the following. Suppose p is a prime or a prime power.

Case 1: 2 is not a quadratic residue [e.g.: $p = 11, 13, 19$], then we may take Q as the set of quadratic residues.

Case 2: -2 is a primitive element (e.g., $p = 19, 23$). Then, $Q = \{1, \alpha, \ldots, (\alpha)^{(p-1)/2}\}$, $\alpha = -2$ satisfies the condition.

We have also checked that the required partition exist for other odd integers $p \leqslant 23$, except for $p = 7$.

We now present d^* for $s = 1$. This has $v = 11$, $b = 18$, $r = 5$, $k = 3$, $\lambda = 1$. The treatment set is $\{(i, j), i, j = 0, 1, 2\} \cup \{\alpha, \beta\}$ Blocks are as follows:

(0, 1)	(0, 2)	(1, 0)
(0, 2)	(0, 0)	(1, 1)
(0, 0)	(0, 1)	(1, 2)
(1, 1)	(1, 2)	(2, 0)
(1, 0)	(1, 1)	(2, 1)
(1, 2)	(2, 1)	α
(1, 0)	(1, 2)	(2, 2)
(1, 1)	(2, 2)	β
(2, 1)	(2, 2)	(0, 0)
(2, 2)	(2, 0)	(0, 1)
(2, 0)	(2, 1)	(0, 2)
(0, 0)	(1, 0)	α
(0, 0)	(2, 0)	β
(0, 1)	(1, 1)	α
(0, 1)	(2, 1)	β
(0, 2)	(1, 2)	β
(0, 2)	(2, 2)	α
(1, 0)	(2, 0)	α

Remark 5. The design presented above is A-optimal by virtue of Theorem 4.1.

5.1. Concluding remark

So, far we have seen two series of designs with spectrum and $\mathrm{tr}(C_d)^2$ satisfying Jacroux's [9] conditions (see Theorem 3.1). These are $\bar{d}_1$ of MS and d^* of this paper. It is interesting that the spectrum of one is the "opposite" of the other:

$$\text{spectrum}[kC_{\bar{d}_1}] = (a-1)a^{v-3}(a+3), \tag{5.1}$$

$$\text{spectrum}[kC_{d^*}] = (a-3)a^{v-3}(a+1). \tag{5.2}$$

$\bar{d}_1$ has a comparatively bigger value of its minimum eigenvalue and consequently satisfies general optimality (within the binary class) while d^* does not seem to be likely to satisfy general optimality.

Now, does there exist a design with spectrum

$$(a-2)a^{v-3}(a+2)?$$

If it does, then it is very likely to be type 1 optimal as the proofs of this paper indicate.

Acknowledgements

The author is thankful to the referee for all the thoughtful suggestions regarding presentation of the paper.

Appendix A

Consider two non-null $n \times 1$ vectors x, y. To show x is A-worse than y, we need to show $\psi_{x,y} > 0$. [Recall (2.6).] So, we express $\psi_{x,y}$ in the following form:

$$\psi_{x,y} = \sum_{p=1}^{n} n_p c_p (D_p)^{-1}.$$

Here $D_p = (x_p y_p)^{-1}$, $c_p = y_p - x_p$ and n_p is the number of terms in $\psi_{x,y}$ of the form $(x_p)^{-1} - (y_p)^{-1}$. Note that we are using the same notation for expressing different $\psi_{x,y}$'s since it will be clear from the context. Moreover, D_p's are arranged in ascending order except in Lemmas A1 and A2.

Lemma A1. *Suppose l is an integer, $\epsilon = \pm 1$ and t_1, t_2 are positive real numbers satisfying*

$$(l+1)t_2 = lt_1 + 2/3.$$

If $a \geqslant 4$ and $t_1, t_2 \leqslant 1/12$ then $x = (a-\epsilon)^2(a-\epsilon t_1)^l$ is A-worse than $y = (a-4\epsilon/3)(a-\epsilon t_2)^{l+1}$.

Proof. We note that $\psi^* = 3\psi_{x,y}D_1D_2D_3/(a-\epsilon)(a-\epsilon t_2) = (\epsilon)^2[(2-3t_2)(1-t_1)(a-4\epsilon/3) - (4/3-t_2)(a-\epsilon t_1)]$ which is > 0 under the given conditions. □

Lemma A2. *Let l, ϵ and t_1, t_2 be as in Lemma* A1 *except for the relation between l and t_1, t_2 which is as follows:*

$$(l+1)t_2 = lt_1 - 2/3.$$

If $a \geqslant 5$ then $x = (a-\epsilon)^2(a+\epsilon t_1)^l$ is A-worse than $y = (a-4\epsilon/3)(a+\epsilon t_2)^{l+1}$.

Proof. We note that $\psi^* = 3\psi_{x,y}D_1D_2D_3/(a-\epsilon)(a+\epsilon t_2) = (2+3t_2)(1+t_1)(a-4\epsilon/3) - (4/3+t_2)(a+\epsilon t_1)$. It is easy to see that this is > 0, under the given conditions. □

Consider a real symmetric matrix A of order n with constant diagonal element a and of the following form:

$$A = \begin{bmatrix} A_1 & B_1 \\ (B_1)^{\mathrm{T}} & A_2 \end{bmatrix}; \quad A_1 = \begin{bmatrix} a & \epsilon & \epsilon \\ \epsilon & a & t \\ \epsilon & t & \epsilon \end{bmatrix},$$

where $\epsilon = \pm 1$ and $t = 0$ or ± 1.

Corollary A3. *Suppose the matrix A described above satisfies the following conditions:*

(a) $\exists$ *an* $m \times 1$ *vector* z, $m \leqslant n-3$ *and an integer* $h \leqslant m$ *such that the eigenvalues of* A_2 *satisfies the following inequalities:*

$$\sum_{i=1}^{j} \mu_i^{\downarrow}(A_2) \leqslant \sum_{i=1}^{j} z_i, \quad 1 \leqslant j \leqslant h$$

and

$$\sum_{i=1}^{j} \mu_i^{\uparrow}(A_2) \geqslant \sum_{i=1}^{j} z_{h+i}, \quad 1 \leqslant j \leqslant m-h.$$

(b) *Let* $s_z = \sum_{i=1}^{m} z_i$, $\delta_1 = s_z + 2(a-\epsilon) - a(m-2)$ *and* $\delta_2 = s_z + (a - 4\epsilon/3) - a(m-1)$ *satisfy the conditions given below:*

(i) δ_1 *and* δ_2 *has the same sign and*
(ii) $\max(|\delta_1|, |\delta_2|) \leqslant l/12$, *where* $l = n - m - 2$.

Then $\mu(A)$ *is A-best if* $t = 0$.

Proof. Let $x_1 = (2, -1, -1)^{\mathrm{T}}$ and $x_2 = (0, 1, -1)^{\mathrm{T}}$. Now we apply Theorem 2.7 on B_1 [(a) or (b) depending on the sign of ϵ], the corresponding vector(s) being x_1 if $t = 0$ or $-\epsilon$ and x_1, x_2 if $t = \epsilon$. It follows from Theorem 2.8 that in the former case $x^* \prec^w \mu(A)$ and in the later case, $y^* \prec^w \mu(A)$, where $x^* = (x|z)$, $y^* = (y|z)$. Here z is as in (a) and the vectors x, y are as in Lemma A1 (respectively, Lemma A2), if the sign of δ_1 is the same as (respectively, different from) ϵ.

Now the result follows from Lemmas A1 and A2. □

Now, we list a few $n \times 1$ vectors $vi = (vi_1, vi_2, \ldots, vi_n) \uparrow$, $1 \leqslant i \leqslant 12$ [see Notation 2.1]. Our aim here is to show that if a is not too small, each vi is A-worse than $v0$ which is the same as $\mu(d^*)$, when $n = v - 1$. [Recall Definition 2.6.] We shall use the notation $\psi_{i,j}$ in place of $\psi_{x,y}$, when $x = vi$ and $y = vj$.

Notation

(0) $v0 = \mu(d^*) = (a-3)^1 a^{n-2}(a+1)$.
(1) $v1 = (a-2)^2(a+t)^{n-3}(a+2n/(n-1))$, where $(n-3)t = \delta = 2/(n-1)$.
(2) $v2 = (a-5/2)(a-1)^2 a^{n-4}(a+5/2)$.
(3) $v3 = (a-5/2)(a-5/3)(a+t)^{n-3}(a+5/3)$, where $(n-3)t = 1/2$.

(4) $v4 = (a-17/6)(a-1)(a+t)^{n-3}(a+3/2)$, where $(n-3)t = 1/3$.
(5) $v5 = (a-8/3)(a-4/3)(a+t)^{n-3}(a+3/2)$, where $(n-3)t = 1/2$.
(6) $v6 = (a-8/3)(a-4/3)(a-t)^{n-4}(a+1)(a+4/3)$, where $(n-4)t = 1/3$.
(7) $v7 = (a-8/3)(a-1)(a-t)^{n-3}(a+2)$, where $(n-3)t = 1/3$.
(8) $v8 = (a-8/3)(a-3/2)(a+t)^{n-4}(a+1)^2$, where $(n-4)t = 1/6$.
(9) $v9 = (a-8/3)(a-1)^2(a+t)^{n-5}(a+1)^2$, where $(n-5)t = 2/3$.
(10) $v10 = (a-7/3)(a-5/3)(a-1)(a+t)^{n-5}(a+4/3)^2$, where $(n-5)t = 1/3$.
(11) $v11 = (a-7/3)(a-2)(a-t)^{n-4}(a+4/3)^2$, where $(n-4)t = 1/3$.
(12) $v12 = (a-2)(a-t)^{n-2}a$, where $(n-2)t = g \geqslant 2$.

Theorem A4. *$v0$ is A-better than $v1$ if either of the following conditions are satisfied:* (i) $a \geqslant 12$, (ii) $a = n+1 \geqslant 10$.

Proof. $\psi_{1,0} = D_2^{-1} - D_4^{-1} - (D_1^{-1} - D_2^{-1})$, where $D_1 = (a-2)(a-3)$, $D_2 = a(a-2)$, $D_3 = a(a-t)$, $D_4 = (a+1)(a+2+\delta)$. From this, after simplification, we get $\psi_{1,0}D_1D_2D_4/(a-2) = 2a^2 - 22a - 12 + \delta(a+1)(a-6)$, which is clearly > 0 under condition (i). Under condition (ii), $\delta(a+1) > 2$ and together with $a \geqslant 10$ makes the expression above > 0. Hence the result. □

Remark 6. In the application, $a = v = n-1$ [see (4.2)], so that condition (ii) is enough. We have presented the lemma in the form above to show that the result holds even when there is no relation between a and n, provided a is big enough.

Theorem A5. *If $a \geqslant 11$, then $v0$ is A-better than vi, $2 \leqslant i \leqslant 12$.*

Proof. We prove the results in four steps:

(a) $v0$ is A-better than vi, $2 \leqslant i \leqslant 5$, $i = 9$ and $i = 12$ whenever $a \geqslant 11$.
(b) $v5$ is A-better than vi, $6 \leqslant i \leqslant 8$, whenever $a \geqslant 7$.
(c) $v9$ is A-better than $v10$, whenever $a \geqslant 10$.
(d) $v6$ is A-better than $v11$, whenever $a \geqslant 7$.

(a) Take $i = 2$. We find that $\psi^* = 2\psi_{2,0}D_1D_2D_3 = 3(D_3 - D_2)D_1 - (D_2 - D_1)D_3$, where $D_1 = (a-3)(a-5/2)$, $D_2 = a(a-1)$, $D_3 = (a+1)(a+5/2)$. Now, since, $D_3 - D_2 > 9a/2 > D_2 - D_1$, $\psi^* > 9a(a^2 - 10a + 10)$, which is > 0, whenever $a \geqslant 9$.

Next, we observe that $\psi_{i,0}$ is ↑ in t, $i = 3, 4, 5, 9$. Thus, $\psi_{i,0} > \psi_{i,0}(t = 0)$ and w.l.g., we can put $t = 0$ in $\psi_{i,0}$, $i = 3, 4, 5, 9$.

Now we compare $v3$ with $v0$. We observe that $6\psi_{3,0}D_1D_2D_3 > \psi^* = -3D_3(D_2 - D_1) + 3(D_3 - D_2)D_1 + 4(D_4 - D_2)D_1$, where $D_1 = (a-3)(a-5/2)$, $D_2 = a(a-5/3)$, $D_3 = a^2$, $D_4 = (a+1)(a+5/3)$.

On simplification, $\psi^* = \frac{65}{6}a^3 - \frac{281}{3}a^2 + \frac{785}{6}a + 50$ which is > 0, if $a \geqslant 7$.

Next we take up $v4$. We define $\psi^* = 6\psi_{4,0}D_1D_2D_3D_4$, where $D_1 = (a-3)(a-17/6)$, $D_2 = a(a-1)$, $D_3 = a^2$, $D_4 = (a+1)(a+3/2)$.

On simplification ψ^* becomes $> \frac{23}{3}a^5 - 67a^4$, which is > 0 whenever $a \geqslant 9$.

Next member is $v5$. We find that $\psi^* = 6\psi_{5,0}D_1D_2D_3D_4 > \frac{41}{6}a^5 - 79a^4 + \frac{449}{6}a^3 + 106a^2 + 48a$, which is > 0, whenever $a \geqslant 11$.

Now we look at $v9$. Here $\psi^* = 3\psi_{9,0}D_1D_2D_3D_4 = \frac{10}{3}a^5 - 40a^4 + \frac{182}{3}a^3 + 16a^2$, which is >0, whenever, $a \geqslant 11$.

Finally, we consider $v12$. We see that $\psi^* = \psi_{12,0}D_1D_2 > 2D_1 - D_2$, which is $= a^2 - (10 - t)a + 6$ and so is > 0 if $a \geqslant 10$. Hence, the proof of (a) is complete.

(b) In the comparison between vi with $v5$, we take t_1 to be t of vi, $i = 6, 7, 8$ and $t_2 = t$ of $v5$. Since $\psi_{i,5}$ is increasing in t_2 for each $i = 6, 7, 8$, we can put $t_2 = 0$, w.l.g.

First we compare $v6$ with $v5$. We define $\psi^* = \psi_{6,5}D_1D_2D_3$ where $D_1 = (a - t_1)a$; $D_2 = (a + 1)a$; $D_3 = (a + 4/3)(a + 3/2)$. It is easy to check that $\psi^* > 0$, whenever $a \geqslant 0$.

Next, $v7$ with $v5$. We define $\psi^* = 6\psi_{7,5}D_1D_2D_3$.

It is easy to check that the coefficient of t_1 in ψ^* is always > 0. The part of ψ^* free from t_1 is $\frac{35}{6}a^2(a - 7) + 13a^2 - \frac{35}{2}a + 15$ which is > 0, whenever $a \geqslant 7$.

Finally, we compare $v8$ with $v5$. Since D_i's are in ascending order, $6\psi_{8,5}D_1D_2 > \psi^* = (D_2 - D_1)D_3D_4 + 3(D_3 - D_2)D_1D_4 + (D_4 - D_3)D_1D_2$.

On simplification, $\psi^* > (4/3 - 2t_1)a - 11/2$. Since $t_1 < 1/6$, $\psi^* > 0$, whenever $a \geqslant 6$.

(c) We put $t_1 = t$ of $v10$ and $t_2 = t$ of $v9$. We define $\psi^* = 3\psi_{10,9}D_1D_2D_3D_4$.

Now, since $t_1 + t_2 \leqslant 1$, $D_3 < a(a + 1)$, so that $D_3 - D_1 \leqslant 6(a - 1)$. Again, $D_4 - D_2 = 5a - 1/3$.

Simplifying the expression for ψ^* using the relations above, we see that $\psi^*/(2a(a + 1)) > 2a^3 - 19a^2$, which is > 0 whenever $a \geqslant 10$.

(d) On simplification we find that $\psi^* = 9\psi_{11,6}D_1D_2D_3 > 12a^3 - 90a^2 + 102a$, which is > 0 whenever $a \geqslant 7$. □

References

[1] B. Bagchi, S. Bagchi, Optimality of partial geometric designs, Ann. Statist. 29 (2001) 577–594.

[2] S. Bagchi, A class of non-binary unequally replicated E-optimal designs, Metrika 35 (1987) 1–12.

[3] S. Bagchi, Optimality and construction of some rectangular designs, Metrika 41 (1994) 29–41.

[4] R. Bhatia, Matrix Analysis, Springer, Berlin, 1997

[5] C.S. Cheng, Optimality of certain asymmetrical experimental designs, Ann. Statist. 6 (1978) 1239–1261.

[6] C.S. Cheng, An optimisation problem with application to optimal design theory, Ann. Statist. 15 (1987) 712–723.

[7] C.S. Cheng, R.A. Bailey, Optimality of some two-associate-class partially balanced incomplete block designs, Ann. Statist. 19 (1991) 1667–1671.

[8] G.M. Constantine, Some E-optimal block designs, Ann. Statist. 9 (1981) 886–892.

[9] M. Jacroux, Some sufficient conditions for the type I optimality of block designs, J. Statist. Plann. Inference 11 (1985) 385–396.

[10] J.A. John, T. Mitchell, Optimal incomplete block designs, J. R. Stat. Soc. B 39 (1977) 39–43.

[11] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorisation and its Applications, Academic Press, New York, 1979.

[12] J.P. Morgan, S.K. Srivastav, On the type-1 optimality of nearly balanced incomplete block designs with small concurrence range, Statist. Sinica 10 (2000) 1091–1116.

[13] J.P. Morgan, N. Uddin, Optimal, nonbinary, variance-balanced designs, Statist. Sinica 5 (1995) 535–546.

[14] J.P. Morgan, B. Reck, Optimal designs in irregular settings, J. Statist. Plann. Inference 129 (2005) 59–84.

[15] B.K. Roy, K.R. Shah, On the optimality of a class of minimal covering designs, J. Statist. Plann. Inference 10 (1984) 189–194.