Sankhyā : The Indian Journal of Statistics
1992, Volume 54, Series A, Pt. 3, pp. 351–376.

DIFFUSIONS AND THE NEUMANN PROBLEM IN THE HALF SPACE

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SUMMARY. The inhomogeneous Neumann problem for certain classes of second order elliptic operators in the half space is investigated using the associated diffusions with normal reflection.

1. Introduction

Consider the Neumann problem

$$\left.\begin{aligned} Lu(x) &= -f(x),\ x \in G \\ \frac{\partial u}{\partial n}(x) &= -\varphi(x),\ x \in \partial G \end{aligned}\right\} \qquad \dots \quad (1.1)$$

where $G \subset \mathbf{R}^d$ is open, L is a second order elliptic operator and n is the direction of the inward normal. If G is a bounded domain, this problem has been investigated using probabilistic methods by several authors. See Ikeda (1961), Watanabe (1964), Brosamler (1976) where L is the generator of a diffusion; see Hsu (1985), Chung and Hsu (1986) for the homogeneous Neumann problem for the Schrodinger operator; Freidlin (1985) gives the stochastic representation for the solutions.

In the case of the bounded domain and when L is the generator of a nondegenerate reflecting diffusion in $\overline{G}$, the concerned diffusion is ergodic; and the transition probability converges to the invariant probability measure μ exponentially fast. Consequently

$$u(x) = \lim_{t\to\infty} E_x\Big[\int_0^t f(X(s))ds + \int_0^t \varphi(X(s))\, d\xi(s)\Big] \qquad \dots \quad (1.2)$$

is well defined, provided f, φ satisfy the compatibility condition

$$\int_{\overline{G}} f(x)\, d\mu(x) + \frac{1}{2}\int_{\partial G} \alpha(x)\varphi(x) d\mu(x) = 0 \qquad \dots \quad (1.3)$$

where ξ denotes the local time at the boundary, and α is a suitable function given in terms of the direction cosines of the normal and the diffusion coefficients. In such a case u is a solution (in a suitable sense) to (1.1); also u is the

Paper received. March 1990; revised October 1990.

AMS (1980) *subject classification*. Primary 60J60; secondary 60H30, 35J25.

Key words and phrases. Diffusions with normal reflection, local time, generator, martingale stochastic solution, invariant measure.

unique solution such that $\int u(x)\, d\mu(x) = 0$; (the latter fact does not seem to be explicitly mentioned in the literature). The compatibility condition (1.3) is also a necessary condition.

The aim of this paper is to investigate using probabilistic methods, the inhomogeneous Neumann problem when $G = \{x \in \boldsymbol{R}^d : x_1 > 0\}$ is the half space and L is the generator of a diffusion process. To our knowledge such an investigation has not been carried out for any unbounded domain. (The homogeneous problem for the Schrodinger-type operator $L+q$ in the half space has been considered by the present author (1992) ; but the results do not apply here as the concerned gauge is infinite). In the cases considered here the $\left(L, \frac{\partial}{\partial x_1}\right)$-diffusion $\{X(t) : t \geqslant 0\}$ can be written as $\{(X_1(t), \hat{X}(t))\}$ where $\{X_1(t)\}$ is a reflecting diffusion in $[0, \infty)$ with generator L_1 and $\{\hat{X}(t)\}$ is $(d-1)-$dimensional diffusion with generator L_2, where the coefficients of L_1 depend only on x_1 and those of L_2 depend on $(x_2, \ldots, x_d)$.

The main difficulty in extending the results to unbounded domains is the lack of information about the rate of convergence of the transition probabilities to the invariant measure.

In Section 2, preliminary results concerning the diffusions in $\overline{G}$ are obtained. In Section 3, we consider stochastic solutions for the Neumann problem when L_1 = Laplacian, L_2 has periodic coefficients and f, φ are periodic in $(x_2, \ldots, x_d)$. So our analysis is essentially over $[0, \infty) \times \boldsymbol{T}^{d-1}$; and the invariant measure is Lebesgue measure on $[0, \infty) \times$ a probability measure on $\boldsymbol{T}^{d-1}$. With the compatibility condition (B 3) which is similar to (1.3), (and two technical conditions) we are able to show that u given by (1.2) is a solution, and is unique in an appropriate class : also the condition (B3) is a necessary condition.

In Section 4, we consider the case when L_1 is self adjoint, L_2 has periodic coefficients and f, φ are periodic in $(x_2, \ldots, x_d)$. Once again the problem is reduced to $[0, \infty) \times \boldsymbol{T}^{d-1}$ with the same invariant measure as in Section 3. But the compatibility condition (C3) is stronger, and perhaps it is not a necessary condition ; (see the remarks at the end of Section 4). However, for the homogeneous problem, (C3) is the same as (B 3) and we get a complete picture.

In Section 5 we consider the case when L is the Laplacian ; here the invariant measure is the Lebesgue measure. The data f, φ are bounded functions having finite second moments and satisfying the compatibility condition (D 3), which again is similar to (1.3). In addition to analogous results as in the preceding sections, we also give, using the spectral representation, a criterion to realise the solution as a continuous function vanishing at infinity.

In the last section L is assumed to be the generator of the Ornstein–Uhlenbeck process, which has a Gaussian invariant measure. Our analysis hinges upon Propositions 2.4 and 2.5 which concern respectively the rate of convergence of $q(t, x, y)$ to $\nu(y)$, and that of $\frac{q(t, x, y)}{\nu(y)}$ to unity, where q is the transition probability density of $O-U$ process and ν is the invariant density.

It may be noted that the existence of a stochastic solution depends on well definedness of u(as given by (1.2)), which in turn depends on the compatibility condition ; and uniqueness depends on $\lim_{t\to\infty} E_x(u(X(t)) = 0$. The latter condition is a natural one from the probabilistic point of view. This is one main reason for investigating the problem using probabilistic methods. though our arguments can be rephrased analytically. (Another reason is that probabilistic method gives an elegant continuous solution for measurable data). It would be interesting if conditions without involving the time parameter t can be put to ensure $\lim_{t\to\infty} E_x(u(X(t))) = 0$; (see e.g. Theorems 4.2, 5.4, 6.1).

Using the estimates given in the following sections, it is easy to establish the continuous dependence of the solution on the given data. Also our results readily extend to those diffusions which are diffeomorphic to any of the cases considered here ; (such diffusions can be easily characterised using Lemma 3.5 of Ramasubramanian (1988)).

Before ending this section we show by an example that, in spite of our (seemingly strong) conditions, the problem can not be reduced to a bounded set or to a lower dimension.

Example. Let $d = 2$, $G = \{(x_1, x_2) : x_1 > 0\}$, $L = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$, $f(.)\equiv 0$, $\varphi(0, x_2) = \cos^3 2\pi x_2$. Clearly φ is a periodic function on ∂G such that $\int_{T^{d-1}} \varphi(x_2)dx_2 = 0$. Suppose there exists a solution $u(x_1, x_2)$ to (1.1) of the form

$$u(x_1, x_2) = u_1(x_1)\, u_2(x_2) \qquad \dots \quad (1.4)$$

Note that (1.4) and the boundary condition imply that $u_2(x_2) = \frac{1}{u_1'(0)}\,\varphi(x_2)$ and hence $u_1'(0) \neq 0$. It now follows from the differential equation that

$u_1(x_1)\varphi''(x_2) = -u_1''(x_1)\varphi(x_2)$. Since $u_1'(0) \neq 0$, there is an x_1 such that $u_1(x_1) \neq 0$. Therefore $\frac{1}{\varphi(x_2)}$ $[\varphi''(x_2)] =$ constant, which is not possible. Thus there can not be a solution of the form (1.4) to the problem (1.1).

2. Diffusions

In this section we put together certain results concerning reflected diffusions in the half space $\overline{G}$, which will be of use in the subsequent sections.

(i) *Self-adjoint×Periodic case.* Let $\overline{G} = \{x \in \mathbf{R}^d : x_1 \geqslant 0\}$ where $d \geqslant 2$. We have the diffusion coefficients a, b satisfying the following conditions.

(A1) : For each $x \in \overline{G}$, $a(x) = ((a_{ij}(x)))_{1\leqslant i,j\leqslant d}$ is a $(d\times d)$ real symmetric positive definite matrix and $b(x) = (b_1(x), \ldots, b_d(x))$ is a d-vector. The functions $a_{ij}(\,.\,), b_i(\,.\,) \in C_b^2(\overline{G})$ for $1 \leqslant i, j \leqslant d$. There exist constants λ_1, λ_2 such that $0 < \lambda_1 \leqslant \lambda_2 < \infty$ and for any $x \in \overline{G}$, any eigenvalue of $a(x) \in [\lambda_1, \lambda_2]$.

(A2) : a_{11}, b_1 are independent of $x_2, \ldots, x_d$; $a_{1j} \equiv a_{j1} \equiv 0$, $j = 2, \ldots, d$; a_{ij}, b_i are independent of x_1 for $2 \leqslant i, j \leqslant d$. Also $b_1(x_1) = \frac{1}{2}\frac{d}{dx_1}a_{11}(x_1)$, and $b_1(0) = 0$.

(A3) : For $i, j = 2, \ldots, d$ the functions $a_{ij}(\,.\,)$, $b_i(\,.\,)$ are periodic in $x_2, \ldots, x_d$ with period 1 in each variable.

Note that the functions a_{11}, b_1 can be extended to the whole of $\mathbf{R}$ by

$$a_{11}(x_1) = a_{11}(-x_1),\ b_1(x_1) = -b_1(-x_1), \text{ if } x_1 < 0. \quad \ldots \quad (2.1)$$

These extensions are again denoted by a_{11}, b_1 respectively. For any $x = (x_1, x_2, \ldots, x_d)$ we shall denote $\hat{x} = (x_2, \ldots, x_d)$ and we shall often identify ∂G with $\mathbf{R}^{d-1}$.

Define the elliptic operators L_1, L_2, L respectively on $C^2(\mathbf{R})$, $C^2(\mathbf{R}^{d-1})$, $C^2(\mathbf{R}^d)$ by

$$L_1 h(x_1) = \frac{1}{2}a_{11}(x_1)\frac{d^2h}{dx_1^2}(x_1) + b_1(x_1)\frac{dh}{dx_1}(x_1), \quad \ldots \quad (2.2)$$

$$L_2 g(\hat{x}) = \frac{1}{2}\sum_{i,j=2}^{d} a_{ij}(\hat{x})\frac{\partial^2 g}{\partial x_i\,\partial x_j}(\hat{x}) + \sum_{i=2}^{d} b_i(\hat{x})\frac{\partial g}{\partial x_i}(\hat{x}) \quad \ldots \quad (2.3)$$

$$L\psi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\hat{x})\frac{\partial^2\psi}{\partial x_i\,\partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial\psi}{\partial x_i}(x) \quad \ldots \quad (2.4)$$

Let $\Omega_1 = C([0, \infty) : \boldsymbol{R})$, $\Omega_2 = C([0, \infty) : \boldsymbol{R}^{d-1})$, $\Omega = C([0, \infty) : \boldsymbol{R}^d)$ and $\widetilde{\Omega} = C([0, \infty) : \boldsymbol{T}^{d-1})$ be endowed with the topology of uniform convergence on compacta and the natural Borel structure. (Here $\boldsymbol{T}^{d-1}$ denotes the $(d-1)$-dimensional torus). Let $X(t)$ (sometimes written X_t) denote the t-th coordinate map on Ω; let $\hat{X}(t) = (X_2(t), \ldots, X_d(t))$ and $\widetilde{X}(t) = (X_2(t) \bmod 1, \ldots, X_d(t) \bmod 1)$.

Let $\{P_x : x \in \overline{G}\}$ be the $\left(L, \dfrac{\partial}{\partial x_1}\right)$ diffusion in $\overline{G}$; $\left\{P^{(2)}_{\hat{x}} : \hat{x} \in \boldsymbol{R}^{d-1}\right\}$ be the L_2-diffusion in $\boldsymbol{R}^{d-1}$; $\left\{P^{(1)}_{x_1} : x_1 \geqslant 0\right\}$ be the $\left(L_1, \dfrac{\partial}{\partial x_1}\right)$-difusion in $[0, \infty)$. These are the families of probability measures respectively on Ω, Ω_2, Ω_1 solving the appropriate martingale problems. (It may be mentioned that $\{P_x\}$ and $\{P^{(1)}_{x_1}\}$ are diffusions with normal reflection atthe boundary) Because of our assumptions (A1), (A2) note that L_1 and L_2 are generators of diffusions; also L_1 is self-adjoint.

Under the assumption (A1), there exists a continuous, nondecreasing, nonanticipating process $\xi(t)$ on Ω such that

$$\left.\begin{array}{l} \text{(a)}\quad \xi(t) = \int_0^t I_{\partial G}\,(X(s))\, d\xi(s)\,; \\ \text{(b)}\quad \text{for every } \psi \in C_b^2(\boldsymbol{R}^d), \\ \qquad \psi(X(t)) - \psi(x) - \int_0^t L\psi(X(s))ds - \int_0^t \dfrac{\partial \psi}{\partial x_1}\,(X(s))\, d\xi(s) \\ \qquad \text{is a continuous } P_x\text{-martingale with respect to } \{\mathcal{B}_t\}. \end{array}\right\} \quad \ldots \quad (2.5)$$

where $\mathcal{B}_t = \sigma\{X(s) : 0 \leqslant s \leqslant t\}$. This process, called the local time at the boundary, is uniquely determined. (see Stroock and Varadhan (1971)).

Proposition 2.1. *Let* (A1), (A2) *hold. Then for any* $x = (x_1, x_2, \ldots, x_d)$ *in* $\overline{G}$,

$$P_x = P^{(1)}_{x_1} \times P^{(2)}_{\hat{x}} \quad \ldots \quad (2.6)$$

where $\hat{x} = (x_2, \ldots, x_d)$. *The processes* $\{\xi(s)\}$ *and* $\{\hat{X}(t)\}$ *are independent. Also for* $t > 0$, $x, y \in \overline{G}$,

$$p(t, x, y) = p_1(t, x_1, y_1)\, p_2(t, \hat{x}, \hat{y}) \quad \ldots \quad (2.7)$$

where p, p_1, p_2 *are respectively the transition probability density functions of* $\left(L, \dfrac{\partial}{\partial x_1}\right)$*-diffusion,* $\left(L_1, \dfrac{\partial}{\partial x_1}\right)$*-diffusion and* L_2*-diffusion processes; in particular, the three diffusions are strong Feller.*

Proof. The first two assertions are immediately seen by writing down the stochastic differential equations for the $\left(L, \frac{\partial}{\partial x_1}\right)$–diffusion ; (see Ikeda and Watanabe (1981)). To prove (2.7), extend the coefficients to $\boldsymbol{R}^d$ using (2.1). Consider the diffusion in $\boldsymbol{R}^d$ with generator L ; let $\Gamma(t, \boldsymbol{x}, \boldsymbol{y})$ be the transition probability density function of the L-diffusion $\boldsymbol{R}^d$. Note that p is obtained from Γ by the method of images. In view of our assumptions, (2.7) is now immediate. □

Remark 2.2. Using Green's formula it can be shown that, for any bounded measurable function g on $\partial G(\simeq \boldsymbol{R}^{d-1})$, $\boldsymbol{x} \in \overline{G}$, $t > 0$,

$$E_{\boldsymbol{x}}\left[\int_0^t g(X(s))\, d\xi(s)\right] = \frac{1}{2}\int_0^t \int_{\partial G} a_{11}(\boldsymbol{y}) g(\boldsymbol{y}) p(s, \boldsymbol{x}, \boldsymbol{y})\, d\sigma(\boldsymbol{y})\, ds$$

$$= \frac{1}{2}\int_0^t \int_{\boldsymbol{R}^{d-1}} a_{11}(0) g(\hat{\boldsymbol{y}}) p_1(s, x_1\, 0) p_2(s, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})\, d\hat{\boldsymbol{y}}\, ds \quad \dots \quad (2.8)$$

where $E_{\boldsymbol{x}}$ denotes expectation with respect to $P_{\boldsymbol{x}}$ and $d\sigma(\,.\,)$ denotes the $(d-1)$ –dimensional Lebesgue measure on ∂G ; note that the second equality in the above follows by the preceding lemma. (In what follows, the notation $d\sigma(\boldsymbol{y})$ or $d\hat{\boldsymbol{y}}$ will be used according to convenience).

Proposition 2.3. *Define* $\Psi : \Omega_2 \to \widetilde{\Omega}$ *by* $(\Psi\ \hat{w})(t) = (w_2(t)$ *mod* $1, \dots,$ $w_d(t)$ *mod* $1)$; *put* $\widetilde{\boldsymbol{X}}(t) = (X_2(t)$ *mod* $1, \dots, X_d(t)$ *mod* $1)$. *For* $\hat{\boldsymbol{x}} \in \boldsymbol{T}^{d-1}$ *let* $\widetilde{P}_{\hat{\boldsymbol{x}}} = P^{(2)}_{\boldsymbol{x}}\Psi^{-1}$. *Assume that* (A1)–(A3) *hold. Then* $(\{\widetilde{\boldsymbol{X}}(t)\}$ *is a* $\boldsymbol{T}^{d-1}$ *valued continuous, strong Feller, strong Markov process under* $\{\widetilde{P}_{\hat{\boldsymbol{x}}}\}$; *also*

$$\widetilde{p}_2(t, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \sum_{k \in \boldsymbol{Z}^{d-1}} p_2(t, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}+k) \quad \dots \quad (2.9)$$

is the transition probability density function of $\{\widetilde{\boldsymbol{X}}(t)\}$. *Moreover, there exists a unique twice differentiable periodic function* ρ *on* $\boldsymbol{R}^{d-1}$ *such that*

$$\int_{\boldsymbol{T}^{d-1}} \rho(\hat{\boldsymbol{y}}) d\hat{\boldsymbol{y}} = 1, \quad \dots \quad (2.10)$$

$$L_2^* \rho(\hat{\boldsymbol{y}}) = 0,\ \hat{\boldsymbol{y}} \in \boldsymbol{R}^{d-1}, \quad \dots \quad (2.11)$$

$$\sup_{\hat{\boldsymbol{x}} \in \boldsymbol{T}^{d-1}} \int_{\boldsymbol{T}^{d-1}} |\widetilde{p}_2(t, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) - \rho(\hat{\boldsymbol{y}})|\, d\hat{\boldsymbol{y}} \leqslant c_1 e^{-c_2 t} \quad \dots \quad (2.12)$$

where c_1, c_2 are positive constants independent of t, L_2^* is the formal adjoint of L_2; in other words, under $\widetilde{P}_\rho(\,.\,) = \int_{T^{d-1}} \widetilde{P}_{\hat{x}}(\,.\,)\,\rho(\hat{x})d\hat{x}$ the process $\{\widetilde{X}(t)\}$ is ergodic.

Proof. The first assertion is elementary to prove. Since $\widetilde{X}(t)$ is a Feller continuous diffusion on the torus T^{d-1}, by results in Bensoussan, Lions, Papanicolaou (1978, Chapter 3, Section 3) it follows that there is a unique invariant probability measure $\rho(\hat{y})d\hat{y}$ on T^{d-1} satisfying (2.10)–(2.12); (see also Bhattacharya (1985)). The regularity of ρ follows by the regularity theorems for solutions of second order elliptic equations. □

(ii) *Ornstein–Uhlenbeck process.* We now consider a version of the Ornstein–Uhlenbeck process in $\overline{G}$ with normal reflection at the boundary. In this case the diffusion coefficients are given by $a_{ij}(x) = \delta_{ij}$, $b_i(x) = -x_i$, $1 \leqslant i, j \leqslant d$. The generator is

$$L\psi(x) = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2\psi(x)}{\partial x_i^2} - \sum_{i=1}^{d} x_i \frac{\partial\psi(x)}{\partial x_i} \qquad \ldots \quad (2.13)$$

In this case the transition probability density function is given by

$$q(t, (x_1, \hat{x}), (y_1, \hat{y})) = q_1(t, x_1, y_1)\, q_2(t, \hat{x}, \hat{y}) \qquad \ldots \quad (2.14)$$

where

$$q_1(t, x_1, y_1) = \left[\frac{1}{\pi(1-e^{-2t})}\right]^{\frac{1}{2}}\left[\exp\left\{-\frac{(y_1-e^{-t}x_1)^2}{(1-e^{-2t})}\right\}+\exp\left\{-\frac{(-y_1-e^{-t}x_1)^2}{(1-e^{-2t})}\right\}\right] \qquad \ldots \quad (2.15)$$

$$q_2(t, \hat{x}, \hat{y}) = \left[\frac{1}{\pi(1-e^{-2t})}\right]^{\frac{d-1}{2}} \exp\left\{-\frac{1}{(1-e^{-2t})}\sum_{i=2}^{d}(y_i-e^{-t}x_i)^2\right\} \qquad \ldots \quad (2.16)$$

Note that q_1 is the transition probability density of the O.U. process in $[0, \infty)$ with reflection at 0, and q_2 is the transition probability density of the $(d-1)$-dimensional O.U. process. Let $\{P_x : x \in \overline{G}\}$ denote the corresponding family of probability measures on Ω. By writing down the stochastic differential equations for the O.U. process it can be seen that there is a uniquely determined continuous, nondecreasing, nonanticipating process $\{\xi(t)\}$ on Ω satisfying (2.5). Note that all the assertions of Proposition 2.1 and analogue of (2.8) hold also for the O.U. process $\{P_x : x \in \overline{G}\}$.

It is easy to check that there is a unique invariant probability measure $\nu(\boldsymbol{y})d\boldsymbol{y}$ on $\overline{G}$ for the O.U. process with normal reflection ; in fact

$$\nu(\boldsymbol{y}) = \nu_1(y_1)\,\nu_2(\hat{\boldsymbol{y}}) \qquad \ldots \quad (2.17)$$

where

$$\nu_1(y_1) = \lim_{t\to\infty} q_1(t, x_1, y_1) = \frac{2}{\sqrt{\pi}}\, e^{-y_1^2} \qquad \ldots \quad (2.18)$$

$$\nu_2(\hat{\boldsymbol{y}}) = \lim_{t\to\infty} q_2(t, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \left(\frac{1}{\pi}\right)^{(d-1)/2} \exp\left(-\sum_{i=2}^{d} y_i^2\right) \ldots \quad (2.19)$$

for $y_1 \geqslant 0$, $\hat{\boldsymbol{y}} \in \boldsymbol{R}^{d-1}$.

Proposition 2.4. *Let $t_0 > 0$. Then*

$$|q(t, \boldsymbol{x}, \boldsymbol{y}) - \nu(\boldsymbol{y})| \leqslant K_1\, e^{-2t} + K_2 |\boldsymbol{x}| e^{-t} \qquad \ldots \quad (2.20)$$

for all $t \geqslant t_0$, $\boldsymbol{x}$, $\boldsymbol{y} \in \overline{G}$, where the positive constants K_1, K_2 are independent of $t \geqslant t_0$, $\boldsymbol{x}$, $\boldsymbol{y}$.

Proof. It is sufficient to prove.

$$\left|\left(\frac{1}{1-e^{-2t}}\right)^{1/2} \exp\left[-\frac{(\alpha-e^{-t}\beta)^2}{(1-e^{-2t})}\right] - e^{-\alpha^2}\right| \leqslant K_1 e^{-2t} + K_2|\beta| e^{-t} \;\ldots\quad (2.21)$$

for $t \geqslant t_0$, $\alpha, \beta \in \boldsymbol{R}$, where K_1, K_2 are positive constants independent of $t \geqslant t_0$, α, β.

Put $\epsilon = e^{-t}$ and set $h(\epsilon) = \dfrac{1}{(1-\epsilon^2)^{\frac{1}{2}}} \exp\left[-\dfrac{(\alpha-\beta\epsilon)^2}{(1-\epsilon^2)}\right]$ where $\alpha, \beta \in \boldsymbol{R}$ are arbitrary but fixed. It is easily verified that

$$|h'(\epsilon)| \leqslant \frac{C_1\,\epsilon}{(1-\epsilon^2)^{3/2}} + \frac{C_2|\beta|}{(1-\epsilon^2)} \qquad \ldots \quad (2.22)$$

for all $0 < \epsilon < 1$, $\alpha, \beta \in \boldsymbol{R}$ where the constants C_1, C_2 are independent of ϵ, α, β. From (2.22) it is simple to obtain the inequality (2.21). This completes the proof. □

Proposition 2.5. *Let $t_0 > 0$ and $H \subsetneq G$ be a compact set. Then*

$$q(t, \boldsymbol{x}, \boldsymbol{y}) \leqslant [1 + e^{-t}(k_0 + k_1|\boldsymbol{y}| + \ldots + k_d|\boldsymbol{y}|^d)]\nu(\boldsymbol{y}) \qquad \ldots \quad (2.23)$$

for all $t \geqslant t_0$, $\boldsymbol{x} \in H$, $\boldsymbol{y} \in \overline{G}$, where the positive constants $k_0, k_1, \ldots, k_d$ depend only on t_0, H.

Proof. It is sufficient to prove that for $t_0 > 0$, $\beta_0 > 0$,

$$\left|\left(\frac{1}{(1-e^{-2t})}\right)^{1/2} \exp\left[-\frac{(\alpha-e^{-t}\beta)^2}{(1-e^{-2t})}\right] - e^{-\alpha^2}\right| \leqslant C(1+|\alpha|)e^{-t}e^{-\alpha^2} \;\ldots\quad (2.24)$$

for all $t \geqslant t_0$, $|\beta| \leqslant \beta_0$, $\alpha \in \boldsymbol{R}$, where the positive constant C depends only on t_0, β_0.

Put $\epsilon = e^{-t}$. It is simple to check that

$$\text{l.h.s. of (2.24)} \leqslant \frac{e^{-\alpha^2}}{(1-\epsilon^2)^{1/2}} \left| 1-\exp\left[\frac{-\beta^2\epsilon^2}{(1-\epsilon^2)} \right]\right| + \frac{\epsilon^2 e^{-\alpha^2}}{(1-\epsilon^2)^{1/2}}$$

$$+ \frac{e^{-\alpha^2}}{(1-\epsilon^2)^{1/2}} \exp\left[-\frac{\beta^2\epsilon^2}{(1-\epsilon^2)} \right] \left| 1-\exp\left[\frac{-(\alpha^2\epsilon^2-2\alpha\beta\epsilon)}{(1-\epsilon^2)} \right]\right| \quad \ldots \quad (2.25)$$

Since the first two terms on the r.h.s. of (2.25) satisfy the required bound, it is enough to prove that the third term also satisfies the required bound.

Set $g(\alpha) = \exp\left[-\dfrac{(\alpha^2\epsilon^2-2\alpha\beta\epsilon)}{(1-\epsilon^2)} \right]$ It is not difficult to verify that for $0 < \epsilon < 1$, $\beta \in \boldsymbol{R}$,

$$\sup\{g(\alpha) : \alpha \in \boldsymbol{R}\} = \exp\left[\frac{\beta^2}{1-\epsilon^2} \right] \quad \ldots \quad (2.26)$$

$$\sup\{|\alpha\, g(\alpha)| : \alpha \in \boldsymbol{R}\}$$

$$= \frac{1}{2\epsilon}\left[\beta+\sqrt{\beta^2+2-2\epsilon^2}\right] \exp\left[\frac{\beta^2+\beta(\beta^2+2-2\epsilon^2)^{1/2}-1+\epsilon^2}{2(1-\epsilon^2)} \right] \ldots \quad (2.27)$$

Using (2.26), (2.27) and the mean value theorem, it is now easy to verify that the third term on the r.h.s. of (2.25) also is dominated by the r.h.s. of (2.24) for all $t \geqslant t_0$, $|\beta| \leqslant \beta_0$, $\alpha \in \boldsymbol{R}$. This completes the proof.

3. Neumann problem : L_1 = Laplacian, L_2 has periodic coefficients

We now consider the inhomogeneous Neumann problem for L in the half space $\bar{G}$. That is, for a measurable function f on $\bar{G}$ and a measurable function φ on ∂G, to find an appropriate function u such that

$$\left.\begin{aligned} Lu(x) &= -f(x),\ x \in G \\ \frac{\partial u}{\partial x_1}(x) &= -\ \varphi(x),\ x \in \partial G \end{aligned}\right\} \quad \ldots \quad (3.1)$$

As in Hsu (1985), Ramasubramanian (1992), a measurable function u on $\bar{G}$ is called a *stochastic solution* to (3.1), if for each $x \in \bar{G}$

$$Z(t) := u(X(t))-u(X(0))+\int_0^t f(X(s))ds+\int_0^t \varphi(X(s))d\xi(s) \quad \ldots \quad (3.2)$$

is a continuous P_x-martingale with respect to $\mathcal{B}_t$, where $\{P_x\}$ is the $\left[L, \dfrac{\partial}{\partial x_1}\right]$-diffusion

Remark 3.1. It can be shown using (2.5) that any classical solution (with appropriate growth condition) is also a stochastic solution to (3.1) Conversely, if f and φ are continuous and $u \in C^2(G) \bigcap C^1(\overline{G})$ is a stochastic solution, then it can be shown that u is a classical solution to (3.1).

In this section we assume that the conditions (A1)–(A3) hold and that $a_{11}(\,.\,) \equiv 1$ and $b_1(\,.\,) \equiv 0$; that is, $\left\{P^{(1)}_{x_1}\right\}$ is the reflected Brownian motion in $[0, \infty)$, L_2 has periodic coefficients, and $\left\{P^{(1)}_{x_1}\right\}$ and $\left\{P^{(2)}_{\hat{x}}\right\}$ are independent diffusions. Note that, in this case

$$p_1(t, x_1, y_1) = \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}}\left[\exp\left\{-\frac{(y_1-x_1)^2}{2t}\right\}+\exp\left\{-\frac{(y_1+x_1)^2}{2t}\right\}\right] \qquad \ldots \quad (3.3)$$

Lemma 3.2. *For* $0 < t_1 < t_2 < \infty$, $\alpha \in \mathbf{R}$,

$$\int_{t_1}^{t_2} s^{-1/2}\left[\exp\left(-\frac{\alpha^2}{2s}\right)-1\right]ds$$

$$= 2t_2^{1/2}\left(\exp\left(-\frac{\alpha^2}{2t_2}\right)-1\right]-2t_1^{1/2}\left[\exp\left(-\frac{\alpha^2}{2t_1}\right)-1\right]$$

$$-\sqrt{2}\,|\alpha|\int_{\alpha^2/2t_2}^{\alpha^2/2t_1} r^{-1/2}\,e^{-r}\,dr \qquad \ldots \quad (3.4)$$

Proof. Put $z = \frac{\alpha^2}{2s}$, use the fact that $e^{-z}-1 = \int_0^z e^{-r}\,dr$; the required result is obtained by a routine computation. □

In this section we shall make the following assumptions on the prescribed data f, φ.

(B1) : f, φ are bounded on compact sets; $\varphi(x_2, \ldots, x_d)$ is a periodic function with period 1 in each variable; $f(x_1, x_2, \ldots, x_d)$ is periodic in $x_2, \ldots, x_d$ with period 1.

(B2) : $H_r \equiv \sup_{\hat{x}\in T^{d-1}} \int_{[0,\infty)} |x_1|^r\,|f(x_1, \hat{x})|\,dx_1 < \infty$, $r = 0, 1, 2$.

(B3) : $\int_{[0,\infty)}\int_{T^{d-1}} f(x_1, \hat{x})\,\rho(\hat{x})d\hat{x}\,dx_1 + \frac{1}{2}\int_{T^{d-1}} \varphi(\hat{x})\,\rho(\hat{x})\,d\hat{x} = 0,$

where ρ is the invariant probability measure for the L_2-diffusion on T^{d-1}.

For $0 \leqslant t_1 < t_2 < \infty$, $x = (x_1, \hat{x}) \in \overline{G}$, put

$$\overline{u}(t_1, t_2\,;\, x_1, \hat{x}) = E_x\left[\int_{t_1}^{t_2} f(X(s))ds+\int_{t_1}^{t_2} \varphi(X(s))d\xi(s)\right] \qquad \ldots \quad (3.5)$$

where ξ is the local time at the boundary as in (2.5). Because of the periodicity assumption we may take $\hat{x} \in T^{d-1}$. Now in view of (2.8), Proposition 2.3 (in particular (2.9)), and condition (B3) we get

$$u(t_1, t_2\,;\, x_1, \hat{x}) = \int_{t_1}^{t_2} \int_{[0,\infty)} \int_{T^{d-1}} f(z_1, \hat{z})\, p_1(s, x_1, z_1) \tilde{p}_2(s, \hat{x}, \hat{z})\, d\hat{z}\, dz_1\, ds$$

$$+ \int_{t_1}^{t_2} \int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z}) p_1(s,\ x_1,\ 0) \tilde{p}_2(s,\ \hat{x},\ \hat{z}) d\hat{z}\ ds$$

$$= \int_{t_1}^{t_2} \int_{[0,\infty)} \int_{T^{d-1}} f(z_1, \hat{z}) p_1\,(s, x_1, z_1)\, [\tilde{p}_2(s, \hat{x}, \hat{z}) - \rho(\hat{z})]\, d\hat{z}\, dz_1\, ds$$

$$+ \int_{t_1}^{t_2} \int_{[0,\infty)} \int_{T^{d-1}} f(z_1, \hat{z})\, \rho(\hat{z}) \left[\ p_1(s, x_1, z_1) - \frac{2}{\sqrt{2\pi s}}\right] d\hat{z}\, dz_1\, ds$$

$$+ \int_{t_1}^{t_2} \int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z})\, p_1\,(s, x_1, 0)\, [\tilde{p}_2(s, \hat{x}, \hat{z}) - \rho(\hat{z})]\, d\hat{z}\, ds$$

$$+ \int_{t_1}^{t_2} \int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z})\, \rho(\hat{z}) \left[\ p_1\,(s, x_1, 0) - \frac{2}{\sqrt{2\pi s}}\ \right] d\hat{z}\, ds$$

$$= I_1\,(t_1, t_2\,;\, x_1, \hat{x}) + I_2(t_1, t_2\,;\, x_1) + I_3(t_1, t_2\,;\, x_1, \hat{x}) + I_4(t_1, t_2\,;\, x_1) \qquad \ldots \quad (3.6)$$

where p_1 is given by (3.3) and $\tilde{p}_2$ is the transition probability density function of the L_2-diffusion on the torus T^{d-1}.

Lemma 3.3. *Let* (A1)–(A3) *hold with* $a_{11}(\,.\,) \equiv 1$, $b_1(\,.\,) \equiv 0$. *Let* f, φ *satisfy conditions* (B1)–(B3). *Then for any* $0 \leqslant t_1 < t_2 < \infty$, $(x_1, \hat{x}) \in \overline{G}$,

$$|I_1\,(t_1, t_2\,;\, x_1, \hat{x})| \leqslant H_0\, c_1 \int_{t_1}^{t_2} s^{-1/2}\, e^{-c_2 s} ds \qquad \ldots \quad (3.7)$$

$$|I_3\,(t_1, t_2\,;\, x_1, \hat{x})| \leqslant \frac{1}{2}\, \|\varphi\|_\infty \frac{c_1}{\sqrt{2\pi}} \int_{t_1}^{t_2} s^{-1/2}\, e^{-c_2 s} ds \qquad \ldots \quad (3.8)$$

where the constants c_1, c_2 *are as in* (2.12) *and the constant* H_0 *is as in* (*B2*). *In particular* I_1 *is bounded.*

Proof. Immediate from condition (B2) and Proposition 2.3. □

Lemma 3.4. *Let the hypotheses be as in the preceding lemma. Then for* $x_1 \in [0, \infty)$, $t > 0$,

$$|I_3(0, t; x_1)| \leq 2H_1 + 2H_0|x_1| + \frac{2\sqrt{2}}{\sqrt{\pi t}}(H_2 + H_0|x_1|^2) \quad \ldots \quad (3.9)$$

$$|I_4(0, t; x_1)| \leq \|\varphi\|_\infty \left\{ |x_1| + \frac{|x_1|^2}{\sqrt{2\pi t}} \right\} \quad \ldots \quad (3.10)$$

where the constants H_0, H_1, H_2 *are as in* (B2). *Moreover for* $\alpha > 0$, $\epsilon > 0$ *one can choose* T *such that*

$$\sup_{t \geq T} |I_3(t, \infty; x_1)| < \epsilon \quad \ldots \quad (3.11)$$

$$\sup_{t \geq T} |I_4(t, \infty; x_1)| < \epsilon \quad \ldots \quad (3.12)$$

for all $|x_1| \leq \alpha$.

Proof. Put $F(z_1, \hat{z}) = \begin{cases} f(z_1, \hat{z}), z_1 > 0 \\ f(-z_1, \hat{z}), z_1 < 0 \end{cases}$

By Lemma 3.2, for $0 < t_1 < t_2 < \infty$, $x_1 \geq 0$, we get

$$I_3(t_1, t_2; x_1) = \int_R \int_{T^{d-1}} \frac{F(z_1, \hat{z})\,\rho(\hat{z})}{\sqrt{2\pi}} J(t_1, t_2, x_1, z_1)\, d\hat{z}\, dz_1 \quad \ldots \quad (3.13)$$

where

$$J(t_1, t_2, x_1, z_1) = 2\sqrt{t_2}\left\{\exp\left(-\frac{(z_1 - x_1)^2}{2t_2}\right) - 1\right\}$$

$$-2\sqrt{t_1}\left\{\exp\left(-\frac{(z_1 - x_1)^2}{2t_1}\right) - 1\right\}$$

$$-\sqrt{2}\,|z_1 - x_1| \int_{\left[\frac{(z_1 - x_1^2)}{2t_2}, \frac{(z_1 - x_1^2)}{2t_1}\right]} \tau^{-1/2} e^{-\tau}\, d\tau \quad \ldots \quad (3.14)$$

Letting $t_1 \to 0$ and taking $t_2 = t$ in the above, we get

$$|I_3(0, t; x_1)| \leq \int_R \int_{T^{d-1}} \frac{|F(z_1, \hat{z})|\,\rho(\hat{z})}{\sqrt{2\pi}} \left\{\frac{|z_1 - x_1|^2}{\sqrt{t}} + \sqrt{2\pi}\,|z_1 - x_1|\right\} d\hat{z}\, dz_1$$

from which (3.9) easily follows.

A similar argument gives

$$I_4(t_1, t_2; x_1) = \frac{1}{\sqrt{2\pi}} J(t_1, t_2, x_1, 0) \int_{T^{d-1}} \varphi(\hat{z})\rho(\hat{z})\, d\hat{z} \quad \ldots \quad (3.15)$$

where J is given by (3.14). The inequality (3.10) is now immediate from (3.15).

Now let $\alpha > 0$, $\varepsilon > 0$ be fixed. Choose $r_0 > 0$ such that

$$\int_{-r_0}^{r_0} \int_{T^{d-1}} \left\{ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\alpha^2 + |z_1|^2) + (\alpha + |z_1|) \right\} |F(z_1, \hat{z})| \rho(\hat{z}) | d\hat{z}\, dz_1 < \frac{1}{2}\varepsilon \quad \ldots \quad (3.16)$$

Such a choice is possible because of (B2). Letting $t_2 \to \infty$, putting $t_1 = t$ in (3.13) and using (3.16) we get for $|x_1| \leqslant \alpha$, and $t > 1$,

$$|I_3(t, \infty; x_1)| \leqslant \int_R \int_{T^{d-1}} \frac{|F(z_1, \hat{z})| \rho(\hat{z})}{\sqrt{2\pi}} |J(t, \infty, x_1, z_1)| d\hat{z}\, dz_1$$

$$\leqslant \frac{1}{2}\varepsilon + \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} H_0(r_0^2 + \alpha^2) + \frac{(\alpha + r_0) H_0}{\sqrt{\pi}} \int_{\left[0, \frac{\alpha^2 + r_0^2}{t}\right]} s^{-\frac{1}{2}} e^{-s}\, ds \quad \ldots \quad (3.17)$$

Clearly one can choose T large enough that the right side of (3.17) $< \varepsilon$ for all $t \geqslant T$. Thus (3.11) is proved.

Using (3.15) in the place of (3.13) and proceeding similarly, (3.12) is proved. This completes the proof of the lemma. □

We now prove the main theorem of this section.

Theorem 3.5. *Let* (A1)–(A3) *hold with* $a_{11}(\cdot) \equiv 1$, $b_1(\cdot) \equiv 0$.

Let f, φ *satisfy conditions* (B1)–(B3). *For* $x \in \overline{G}$ *define*

$$u(x) = \lim_{T \to \infty} E_x \left[\int_0^T f(X(s))\, ds + \int_0^T \varphi(X(s))\, d\xi(s) \right] \quad \ldots \quad (3.18)$$

Then u is a continuous function on $\overline{G}$ such that

(a) *u is periodic in $x_2, \ldots, x_d$;*

(b) $|u(x_1, \hat{x})| \leqslant K(1 + |x_1|)$, *where K is a constant independent of $x_1, \hat{x}$;*

(c) *u is a stochastic solution to* (3.1);

(d) $\lim_{t \to \infty} \sup_{\hat{x}} |E_{(x_1, \hat{x})}(u(X(t)))| = 0$, *for any $x_1 > 0$.*

Moreover, u is the unique stochastic solution to (3.1) *in the class*

$\mathcal{C}_1 = \{v : \overline{G} \to \mathbf{R}$: (*i*) *v is bounded on compacts* ; (*ii*) *v is periodic in $x_2, \ldots, x_d$* ; (*iii*) $|v(x_1, \hat{\boldsymbol{x}})| \leqslant K(1+|x_1|)$, *for some constant independent of $x_1, \hat{\boldsymbol{x}}$* ; *and* (*iv*) $\lim_{t\to\infty} E_{\boldsymbol{x}}[v(X(t)] = 0\}$. ... (3.19)

Proof. Observe that $u(\boldsymbol{x}) = \lim_{t_2\to\infty} \bar{u}(0, t_2 ; x_1\ \hat{\boldsymbol{x}})$; consequently by the preceding two lemmas it follows that $u(\boldsymbol{x})$ is well defined for each $\boldsymbol{x}$, u is periodic in $x_2, \ldots, x_d$ and that $|u(x_1, \hat{\boldsymbol{x}})| \leqslant K(1+|x_1|)$. To prove continuity, note that for any $\boldsymbol{x} = (x_1, \hat{\boldsymbol{x}}) \in \overline{G}$,

$$u(\boldsymbol{x}) = \bar{u}(0, \delta ; x_1, \hat{\boldsymbol{x}}) + \bar{u}(\delta, T ; x_1\ \hat{\boldsymbol{x}}) + \bar{u}(T, \infty ; x_1, \hat{\boldsymbol{x}})$$

where $\bar{u}$ is defined by (3.5), and $0 < \delta < T$ are to be suitably chosen. For fixed $(x_1, \hat{\boldsymbol{x}}) \in \overline{G}$, $\epsilon > 0$ by the preceding two lemmas, $T > 0$ can be chosen so that $\bar{u}(T, \infty ; y_1, \hat{y}) < \frac{1}{2}\epsilon$ for all $(y, \hat{y})$ in a compact neighbourhood of $(x_1, \hat{\boldsymbol{x}})$. Choose $\delta > 0$ such that

$$\left(\frac{\delta}{2\pi}\right)^{\frac{1}{2}} (2H_0 + \|\varphi\|_\infty) < \frac{1}{2}\ \epsilon.$$

Then it is easily seen that $\sup_y |\bar{u}(0, \delta ; y_1, \hat{y})| < \frac{1}{2}\ \epsilon$. By the strong Feller property, $\bar{u}(\delta, T ; x_1, \hat{\boldsymbol{x}})$ is continuous in $(x_1, \hat{\boldsymbol{x}})$. Continuity of u now follows.

To show that u is a stochastic solution, we have to show that $Z(t)$ is a continuous $P_{\boldsymbol{x}}$-martingale, where $Z(t)$ is given by (3.2). Because of assertion (*b*) of the theorem and condition (B2), it follows that $Z(t)$ is integrable ; continuity in t is clear from continuity of u. For $s, t \geqslant 0$, $w \in \Omega$ put

$$\psi_t(w) \equiv \psi(t, w) = \int_0^t f(X(s, w))\, ds + \int_0^t \varphi(X(s, w))\, d\xi(s, w),$$

$$\theta_t w(s) = w(t+s)$$

As f, φ are bounded on compacts note that ψ is well defined. Since ξ is an additive functional

$$\psi(s, \theta_t w) = \psi(s+t, w) - \psi(t, w) \qquad \ldots \quad (3.20)$$

For $r \leqslant \tau$, put $M_r^\tau = E(\psi(\tau) \mid \mathcal{B}_r)$. By (3.20) and the Markov property, for $t \geqslant 0$, $s > 0$, $\boldsymbol{x} \in \overline{G}$ we have

$$M_t^{s+t} = \psi(t) + E_{X(t)}(\psi(s)),\ a.s.P_{\boldsymbol{x}} \qquad \ldots \quad (3.21)$$

Therefore by Lemmas 3.3 and 3.4 it follows that

$$|M_t^{s+t}| \leqslant |\psi(t)| + |\bar{u}(0, s\,;X_1(t), \hat{X}(t))|$$
$$\leqslant |\psi(t)| + \beta_1 + \beta_2 |X_1(t)| + \beta_3 |X_1(t)|^2, \; a.s. P_x \quad \ldots \quad (3.22)$$

for all $s \geqslant 1$, $t \geqslant 0$, $x \in \overline{G}$ where the constants β_1, β_2, β_3 are independent of $s \in [1, \infty)$, t, x.

Put $N_t = \lim_{s\to\infty} M_t^{s+t}$. By the definitions of u, ψ, M_t^s, N_t, $Z(t)$ and (3.21) it follows that for any $t \geqslant 0$, $x \in \overline{G}$

$$N_t = Z(t) + u(x), \; a.s. P_x.$$

In view of (3.22) it is easily seen that for $t_2 > t_1 \geqslant 0$,

$$E(N_{t_2} | \mathcal{B}_{t_1}) = \lim_{s\to\infty} E(M_{t_2}^{s+t_2} | \mathcal{B}_{t_1})$$
$$= \lim_{s\to\infty} E(E(\psi(s+t_2) | \mathcal{B}_{t_2}) | \mathcal{B}_{t_1})$$
$$= N_{t_1} \; a.s. \; P_x \quad \ldots \quad (3.23)$$

Thus $\{N_t\}$, and hence $\{Z(t)\}$ is a P_x-martingale with respect to $\mathcal{B}_t$. Hence u is a stochastic solution to (3.1).

Note that for $x \in \overline{G}$, $t \geqslant 0$,

$$E_x[u(X(t)] = \int_{[0,\infty)} \int_{T^{d-1}} u(y_1, \hat{y}) p_1(t, x_1, y_1) [\tilde{p}_2(t, \hat{x}, \hat{y}) - \rho(\hat{y})] \, d\hat{y} \, dy_1$$
$$+ \int_{[0,\infty)} \int_{T^{d-1}} u(y_1, \hat{y}) p_1(t, x_1, y_1) \, \rho(\hat{y}) \, d\hat{y} \, dy_1. \quad \ldots \quad (3.24)$$

By assertion (b) of the theorem and Proposition 2.3 it follows that for any $x_1 \geqslant 0$,

$$\lim_{t\to\infty} \sup_{\hat{x}} |\text{first term on the r.h.s. of (3.24)}|$$
$$\leqslant \lim_{t\to\infty} K c_1 e^{-c_2 t} \int_{[0,\infty)} [1 + |y_1|] \, p_1(t, x_1, y_1) \, dy_1 = 0. \quad \ldots \quad (3.25)$$

Next, as the Lebesgue measure on $[0, \infty)$ is the invariant measure for the reflected Brownian motion and ρ is the invariant measure for the L_2-diffusion, by the representation (3.18) for u, we have for $t > 0$, $x_1 \geqslant 0$,

$$\int_{[0,\infty)} \int_{T^{d-1}} u(y_1, \hat{y})\, p_1(t, x_1, y_1)\, \rho(\hat{y})\, d\hat{y}\, dy_1$$

$$= \lim_{T\to\infty} \int_0^T \Big[\int_{[0,\infty)} \int_{T^{d-1}} \int_{[0,\infty)} \int_{T^{d-1}} f(z_1, \hat{z}) p_1(s, y_1, z_1)$$

$$p_1(t, x_1, y_1)\, \tilde{p}_2(s, \hat{y}, \hat{z})\, \rho(\hat{y})\, d\hat{z}\, dz_1\, d\hat{y}\, dy_1$$

$$+ \int_{[0,\infty)} \int_{T^{d-1}} \int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z}) p_1(s, y_1, 0)\, p_1(t, x_1, y_1)\, \tilde{p}_2(s, \hat{y}, \hat{z})\, \rho(\hat{y})\, d\hat{y}\, d\hat{z}\, dy_1 \Big]\, ds$$

$$= \lim_{T\to\infty} \int_0^T \Big[\int_{[0,\infty)} \int_{[0,\infty)} \Big(\int_{T^{d-1}} f(z_1, \hat{z})\, \rho(\hat{z})\, d\hat{z} \Big)\, p_1(s, y_1, z_1)\, p_1(t, x_1, y_1)\, dy_1\, dz_1$$

$$+ \int_{[0,\infty)} \Big(\int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z})\, \rho(\hat{z})\, d\hat{z} \Big)\, p_1(s, y_1, 0)\, p_1(t, x_1, y_1)\, dy_1 \Big]\, ds$$

$$= \lim_{T\to\infty} \int_t^{T+t} \Big[\int_{[0,\infty)} \int_{T^{d-1}} f(z_1, \hat{z})\, p_1(s, x_1, z_1)\, \rho(\hat{z})\, d\hat{z}\, dz_1$$

$$+ \int_{T^{d-1}} \frac{1}{2}\, \varphi(\hat{z})\, p_1(s, x_1, 0)\, \rho(\hat{z})\, d\hat{z} \Big]\, ds \qquad \dots \quad (3.26)$$

Now let F be the extension of f as given in the proof of Lemma 3.4. Since f, φ satisfy the conditions (B2), (B3), from (3.26) we get for $x_1 \geqslant 0$, $t > 0$,

$$\Big| \int_{[0,\infty)} \int_{T^{d-1}} u(y_1, \hat{y}) p_1(t, x_1, y_1)\, \rho(\hat{y})\, d\hat{y}\, dy_1 \Big|$$

$$= \Big| \lim_{T\to\infty} \int_t^{t+T} \Big[\int_R \int_{T^{d-1}} F(z_1, \hat{z})\, \rho(\hat{z}) \frac{1}{\sqrt{2\pi s}} \Big\{ \exp\Big(-\frac{(z_1 - x_1)^2}{2s} \Big) - 1 \Big\}\, d\hat{z}\, dz_1$$

$$+ \int_{T^{d-1}} \frac{1}{2} \varphi(\hat{z})\, \rho(\hat{z})\, \frac{2}{\sqrt{2\pi s}} \Big\{ \exp\Big(\frac{-x_1^2}{2s} \Big) - 1 \Big\}\, d\hat{z} \Big]\, ds \Big|$$

$$\leqslant \Big[\lim_{T\to\infty} \Big(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+T}} \Big) \Big] \frac{1}{\sqrt{2\pi}}\, (2H_2 + 2x_1^2 H_0 + x_1^2\, \|\varphi\|_\infty) \qquad \dots \quad (3.27)$$

From (3.24), (3.25), (3.27) the assertion (d) of the theorem is immediate. In particular $u \in \mathcal{Q}_1$.

Uniqueness in the class $\mathfrak{C}_1$ follows from the following lemma.

Lemma 3.6. *Let* (A1)–(A3) *hold and* f, φ *satisfy* (B1), (B2). *Let* $v \in \mathfrak{C}_1$ *be a stochastic solution to* (3.1). *Then* $v(x) =$ *r.h.s. of* (3.18).

Proof. As v is a stochastic solution,

$$v(x) = E_x(v(X(t)) + E_x\left[\int_0^t f(X(s))ds + \int_0^t \varphi(X(s))d\xi(s)\right];$$

and since $\lim_{t\to\infty} E_x(v(X(t))) = 0$, the conclusion follows. □

We will now prove the necessity of the condition (B3).

Theorem 3.7. *Let* (A1)–(A3) *hold with* $a_{11}(\,.\,) \equiv 1$, $b_1(\,.\,) \equiv 0$; *let* f, φ *satisfy* (B1), (B2). *Suppose there is a stochastic solution in the class* $\mathfrak{C}_1$ *to the problem* (3.1). *Then* f, φ *satisfy the condition* (B3).

Proof. Let $u \in \mathfrak{C}_1$ be a stochastic solution to (3.1). Then by the preceding lemma

$$u(x) = \lim_{T\to\infty} E_x\left[\int_0^T f(X(s))\,ds + \int_0^T \varphi(X(s))\,d\xi(s)\right]$$

Observe that in the derivation of (3.26), the condition (B3) is not used. Therefore by (3.24) and (3.26) we have for any $t > 0$, $x = (x_1, \hat{x})$,

$$E_x(u(X(t))) = \int_{[0,\infty)}\int_{T^{d-1}} u(y_1, \hat{y})p_1(t, x_1, y_1)\,[\tilde{p}_2(t, \hat{x}, \hat{y}) - \rho(\hat{y})]d\hat{y}\,dy_1$$

$$+ \lim_{T\to\infty}\int_t^{t+T}\Bigg[\int_{[0,\infty)}\int_{T^{d-1}} f(z_1, \hat{z})\,\rho(\hat{z})\left\{p_1(s, x_1, z_1) - \frac{2}{\sqrt{2\pi s}}\right\}d\hat{z}\,dz_1$$

$$+ \int_{T^{d-1}}\frac{1}{2}\,\varphi(\hat{z})\,\rho(\hat{z})\left\{p_1(s, x_1, 0) - \frac{2}{\sqrt{2\pi s}}\right\}d\hat{z}\Bigg]ds$$

$$+ \lim_{T\to\infty}\int_t^{t+T}\frac{2}{\sqrt{2\pi s}}\Bigg[\int_{[0,\infty)}\int_{T^{d-1}} f(z_1, \hat{z})\,\rho(\hat{z})\,d\hat{z}\,dz_1$$

$$+ \int_{T^{d-1}}\frac{1}{2}\,\varphi(\hat{z})\,\rho(z)\,d\hat{z}\Bigg]ds \qquad \ldots (3.28)$$

As $u \in \mathfrak{C}_1$, l.h.s. of (3.28) tends to 0 as $t\to\infty$. By (3.25), the first term on the r.h.s. of (3.28) goes to zero as $t\to\infty$. Since f, φ satisfy (B2), note that the second term on the r.h.s. of (3.28) is $0\,(t^{-1/2})$. The desired conclusion now follows. This completes the proof.

4. Neumann problem : self-adjoint × periodic case

In this section we consider the case when L_1 is self-adjoint and L_2 has periodic coefficients. We assume that (A1)—(A3) hold. As before, let p_1 denote the transition probability density function of the $\left(L_1, \frac{\partial}{\partial x_1}\right)$ — diffusion in $[0, \infty)$. Since p_1 is obtained by the method of images from the transition probability density function of the L_1—diffusion in $\boldsymbol{R}$ and since L_1 is self adjoint, by a theorem of Aronson (1967) we have

$$m_1 t^{-1/2} \left[\exp\left\{-\frac{m_2}{t} (y_1-x_1)^2\right\} + \exp\left\{-\frac{m_2}{t} (y_1+x_1)^2\right\}\right]$$

$$\leqslant p_1(t, x_1, y_1)$$

$$\leqslant k_1 t^{-1/2} \left[\exp\left\{-\frac{k_2}{t}(y_1-x_1)^2\right\} + \exp\left\{-\frac{k_2}{t}(y_1+x_1)^2\right\}\right] \qquad \ldots \quad (4.1)$$

where the constants m_1, m_2, k_1, k_2 are independent of t, x_1, y_1.

In this section we make the following assumptions on the prescribed data f, φ.

(C1) : Same as (B_1).

(C2) : $\lim\limits_{x_1 \to \infty} \sup\limits_{\hat{x}} |f(x_1, \hat{x})| = 0$

(C3) : for all $t > 0$, $x_1 \geqslant 0$

$$\int_{[0,\infty)} \int_{T^{d-1}} f(y_1, \hat{y})\, p_1(t, x_1, y_1)\, \rho(\hat{y}) d\hat{y}\, dy_1$$

$$+ \frac{1}{2} \int_{T^{d-1}} a_{11}(0)\, \varphi(\hat{y}) p_1(t, x_1, 0) \rho(\hat{y}) d\hat{y} = 0$$

Lemma 4.1. *Let* (A1)—(A3) *hold ; let* f, φ *satisfy the conditions* (C1)—(C3). *Let* $\varepsilon > 0$. *Then there exists* $r_0 \geqslant 0$ *such that for any* $0 \leqslant t_1 < t_2 < \infty$ *and* $x_1 \geqslant r_0$,

$$\sup_{\hat{x}} \left| \int_{t_1}^{t_2} \left[\int_G f(\boldsymbol{y})\, p(t, (x_1, \hat{x}), \boldsymbol{y}) d\boldsymbol{y} \right.\right.$$

$$\left.\left. + \frac{1}{2} \int_{\partial G} a_{11}(0) \varphi(\hat{y}) p(t, (x_1, \hat{x}), (0, \hat{y})) d\hat{y} \right] dt \right|$$

$$\leqslant 2\, r_0\, k_1\, c_1 \|f\|_\infty \int_{t_1}^{t_2} t^{-1/2} e^{-c_2 t}\, exp\left[-\frac{k_2 |r_0 - x_1|^2}{t}\right] dt$$

$$+ c_1 \varepsilon \int_{t_1}^{t_2} e^{-c_2 t} dt + \frac{1}{2} k_1 c_1 \|\varphi\|_\infty a_{11}(0) \int_{t_1}^{t_2} t^{-1/2} e^{-c_2 t} exp\left(-\frac{k_2 x_1^2}{t}\right) dt \quad \ldots \quad (4.2)$$

where the positive constants c_1, c_2 *are as in* (2.12) *and* k_1, k_2 *are as in* (4.1) ; *also* r_0 *depends only on* ϵ, f.

Proof. By our assumptions note that f is bounded. Let $\epsilon > 0$. By (C2) there exists $r_0 \geqslant 0$ such that

$$\sup_{\hat{y}} |f(y_1, \hat{y})| \leqslant \epsilon \text{ for all } y_1 \geqslant r_0$$

Consequently by (4.1) we get for all $t > 0$, $x_1 \geqslant r_0, \hat{y}$

$$\int_{[0,\infty)} f(y_1, \hat{y}) p_1(t, x_1, y_1) dy_1 | \leqslant \epsilon + \|f\|_\infty \int_{[0,r_0]} p_1(t, x_1, y_1) dy_1$$

$$\leqslant \epsilon + 2r_0 k_1 \|f\|_\infty t^{-1/2} \exp \left[-\frac{k_2}{t} (r_0 - x_1)^2 \right] \quad \ldots \quad (4.3)$$

Note that, because of condition (C3),

$$\text{l.h.s. of (4.2)} = \sup_{\hat{x}} |\overline{u}(t_1, t_2 ; x_1, \hat{x})$$

$$= \sup_{\hat{x}} |I_1(t_1, t_2) ; x_1, x) + I_2(t_1, t_2 ; x_1, x)| \quad \ldots \quad (4.4)$$

where $\overline{u}, I_1, I_2$ are defined analogous to the correspoing objects in Section 3. Applying (4.1) to $p_1(t, x_1, 0)$, using (2.12), (4.3), (4.4) we can now easily prove (4.2). □

Note : For $\epsilon \geqslant \|f\|$, we may take $r_0 = 0$.

Theorem 4.2. *Let* (A1)—(A3) *hold* ; *let* f, φ *satisfy* (C1)—(C3). *Let* $u(x)$ *be defined by* (3.18). *Then*

(*a*) *u is a bounded continuous function on* $\overline{G}$;

(*b*) *u is periodic in* $x_2, \ldots, x_d$;

(*c*) *u is a stochastic solution to* (3.1) ;

(*d*) $\lim_{x_1 \to \infty} \sup_{\hat{x}} |u(x_1, \hat{x})| = 0$.

Moreover u is the unique stochastic solution to (3.1) *in the class*

$$\mathcal{C}_2 = \Big\{ v : \overline{G} \to \mathbf{R} : (i)\ v \text{ is bounded measurable} ;\ (ii)\ v \text{ is periodic in } (x_2, \ldots,$$

$$x_d)), \text{ and } (iii) \lim_{x_1 \to \infty} \sup_{\hat{x}} |v(x_1, \hat{x})| = 0. \Big\} \quad \ldots \quad (4.5)$$

Proof. From the preceding lemma it is clear that u is well defined, bounded, periodic in $(x_2, \ldots, x_d)$. Continuity of u can be proved as in Section 3. The proof of u being a stochastic solution is also similar to the one in Section 3. In view of (4.2), the assertion (d) is easy to establish. In particular $u \in \mathcal{C}_2$.

Finally, let $v \in \mathcal{C}_2$ be a stochastic solution to (3.1). To prove uniqueness it is enough to show that

$$\lim_{t \to \infty} \sup_{x} |E(v(X(t)))| = 0. \qquad \ldots \quad (4.6)$$

Let $\epsilon > 0$. Since $v \in \mathcal{C}_2$, there is $r_0 \geqslant 0$ such that $\sup_{\hat{y}} |v(y_1, \hat{y})| < \frac{1}{2}\epsilon$ for all $y_1 \geqslant r_0$. Consequently by the upper bound in (4.1) we get for any $(x_1, x_2, \ldots, x_d) \in \overline{G}, t > 0$,

$$|E_x[v(X(t))]| < \frac{1}{2}\epsilon + 2\, k_1 r_0 \|v\|_\infty t^{-1/2}$$

From the above inequality (4.6) is obvious. This completes the proof. □

We now prove the necessity of the condition (C3) for the homogeneous problem.

Theorem 4.3. *Let* (A1)—(A3) *hold ; let* $f \equiv 0$ *and* φ *be a bounded periodic function on* ∂G. *Suppose there is stochastic solution in the class* $\mathcal{C}_2$ *to the problem* (3.1) *Then* $\int_{T^{d-1}} \varphi(\hat{y})\rho(\hat{y})d\hat{y} = 0$

Proof. Note that in the proof of the uniqueness part of Theorem 4.2 we have not used the condition (C3). So, if $u \in \mathcal{C}_2$ is a stochastic solution then by the representation

$$u(x) = u(x_1, \hat{x}) = \lim_{T \to \infty} E_x\left[\int_0^T \varphi(X(s))d\xi(s)\right]$$

$$= \lim_{T \to \infty} \frac{1}{2} \int_0^T \int_{T^{d-1}} a_{11}(0)\varphi(y)p_1(s, x_1, 0)\,[\bar{p}_2(s, x, y) - \rho(y)]dy\, ds$$

$$+ \frac{1}{2} a_{11}(0)\left\{\int_{T^{d-1}} \varphi(\hat{y})\rho(\hat{y})d\hat{y}\right\}\left\{\lim_{T \to \infty} \int_0^T p_1(s, x_1, 0)ds\right\} \qquad \ldots \quad (4.7)$$

By the upper bound in (4.1), and (2.12)

$$\sup_{\hat{x}} |\text{first term on r.h.s. of (4.7)}|$$

$$\leqslant a_{11}(0)\, \|\varphi\|_\infty c_1 k_1 \int_0^\infty t^{-1} e^{-\alpha_2 t} \exp\left[-\frac{k_2}{t} x_1^2\right] dt \to 0 \text{ as } x_1 \to \infty.$$

$$\ldots \quad (4.8)$$

By the lower bound in (4.1)

$$\lim_{x_1 \to \infty} \lim_{T \to \infty} \int_0^T p_1(t, x_1, 0)dt$$

$$\geqslant \lim_{x_1 \to \infty} \lim_{T \to \infty} \int_{x_1^2}^T 2m_1 t^{-1/2} \exp\left[- \frac{m_2 x_1^2}{t} \right] dt$$

$$\geqslant 2 m_1 e^{-m_2} \lim_{x_1 \to \infty} \lim_{T \to \infty} [\sqrt{T} - x_1] \qquad \ldots \quad (4.9)$$

As $u \in \mathcal{C}_2$, l.h.s. of (4.7) tends to 0 as $x_1 \to \infty$. In view of (4.7)—(4.9) this is now possible only if $\int_{T^{d-1}} \varphi(\hat{y})\rho(\hat{y})d\hat{y} = 0$. This completes the proof. □

Remark 4.4. Suppose f, φ satisfy

$$\left.\begin{aligned} &\int_{T^{d-1}} f(y_1, \hat{y})\, \rho(\hat{y})d\hat{y} = 0, \text{ for any } y_1 \geqslant 0. \\ &\int_{T^{d-1}} \varphi(\hat{y})\rho(\hat{y})d\hat{y} = 0 \end{aligned}\right\} \qquad \ldots \quad (4.10)$$

Then clearly f, φ satisfy the condition (C3). Conversely, if f is of the form $f(\hat{y}_1, y) = f_1(y_1)f_2(\hat{y})$, then the condition (C3) implies (4.10) ; for, by (C3)

$$\int_{[0, \infty)} f_1(y_1)p_1(t, x_1, y_1)dy_1 = \text{const.}\ p_1(t, x_1, 0)$$

unless $\int f_2(\hat{y})\rho(\hat{y})d\hat{y} = 0$, and consequently either $f_1 \equiv 0$ or $\int f_2(\hat{y})\rho(\hat{y})d\hat{y} = 0$.

Remark 4.5. In view of the preceding remark, the condition (C3) is not sufficiently general. (In particular, f can not be a function of y_1 alone, unless $f \equiv 0$). A more satisfactory condition would be an analogue of (B3) ; but we have not been able to carry out the analysis under such a condition. However, in the homogeneous case (as (C2) trivially holds), by Theorems 4.2 and 4.3, the condition (C3) is a necessary and sufficient condition for the existence of a unique solution in the class $\mathcal{C}_2$; and the solution is given by

$$u(x_1, \hat{x}) = \lim_{T \to \infty} \tfrac{1}{2} \int_0^T \int_{T^{d-1}} a_{11}(0)\, \varphi(\hat{y})p_1(s, x_1, 0)\tilde{p}_2(s, \hat{x}, \hat{y})\, d\hat{y}\, ds.$$

(Note that, $u \neq 0$ in general. Indeed, using the uniqueness of Doob-Meyer decomposition, sample path continuity, Corollary 2.3 of Stroock and Varadhan (1971), and proceeding as in the proof of Proposition 3.2 of Hsu (1985), it can be shown that, if φ is continuous and $u \equiv 0$ then $\varphi \equiv 0$). It may be noted that, in the homogeneous case, (C3) is the same as (B3). Thus, for the homogeneous problem our analysis gives a complete picture.

5. Neumann problem : L = Laplacian

In this section we assume that $a_{ij}(.) = \delta_{ij}$, $b_i(.) \equiv 0$, $1 \leqslant i, j \leqslant d$; that is, $\{P_x : x \in \overline{G}\}$ is the Brownian motion in $\overline{G}$ with normal reflection it the boundary. For $x = (x_1, \hat{x})$, $y = (y_1, \hat{y})$, $t > 0$ observe that

$$p(t, x, y) = p_1(t, x_1, y_1)\, p_2(t, \hat{x}, \hat{y})$$

where

$$p_1(t, x_1, y_1) = \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} \left[\exp\left\{-\frac{(y_1-x_1)^2}{2t}\right\} + \exp\left\{-\frac{(y_1+x_1)^2}{2t}\right\}\right]$$

$$p_2(t, \hat{x}, \hat{y}) = \left(\frac{1}{2\pi t}\right)^{\frac{d-1}{2}} \exp\left\{-\frac{1}{2t} \sum_{i=2}^{d} (y_i-x_i)^2\right\}$$

The case when f, φ are periodic in the variables $x_2, \ldots, x_d$ has already been dealt with in Section 3. Here we make the following assumptions on the prescribed data f, φ.

(D1) : $f \in L_\infty(\overline{G})$, $\in L_\infty(\partial\overline{G})$;

(D2) : $M_r \equiv \int_{\overline{G}} |y|^r\, |f(y)|\, dy + \int_{\partial G} |\hat{y}|^r\, |\varphi(\hat{y})|\, d\hat{y} < \infty$, $r = 0, 1, 2$;

(D3) : $\int_{\overline{G}} f(y) dy + \frac{1}{2} \int_{\partial G} \varphi(\hat{y}) d\hat{y} = 0$.

For $0 \leqslant t_1 < t_2 < \infty$, $x = (x_1, \hat{x}) \in \overline{G}$, let $\bar{u}(t_1, t_2 ; x_1, \hat{x})$ be defined by (3.5), with $\{P_x\}$ denoting the reflected Brownian motion in $\overline{G}$. Let F be the extension of f as in the proof of Lemma 3.4. By Remark 2.2 and condition (D3) it is seen that

$$\bar{u}(t_1, t_2 ; x_1, \hat{x}) = \int_{t_1}^{t_2} \left(\frac{1}{2\pi s}\right)^{\frac{d}{2}} \left[\int_{R^d} F(z) \left\{ \exp\left(-\frac{|z-x|^2}{2s}\right) - 1 \right\} dz \right.$$

$$\left. + \int_{R^{d-1}} \varphi(\hat{z}) \left\{ e^{-x_1^2/2s} \exp\left(-\frac{|\hat{z}-\hat{x}|^2}{2s}\right) - 1 \right\} d\hat{z} \right] ds \quad \ldots \quad (5.1)$$

In view of the conditions the following lemma can now be proved easily.

Lemma 5.1. *Let* $L = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. *Let* f, φ *satisfy conditions* (D1)—(D3).

Then for any $0 < t_1 < t_2 < \infty$, $(x_1, \hat{x}) \in \bar{G}$,

$$|\bar{u}(t_1, t_2; x_1, \hat{x})| \leqslant 4 \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left[M_2 + |x|^2 M_0\right] \int_{t_1}^{t_2} s^{-\frac{(d+2)}{2}} ds \quad \ldots \quad (5.2)$$

and

$$|\bar{u}(0, t_1; x_1, \hat{x})| \leqslant \|f\|_\infty t_1 + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|\varphi\|_\infty \sqrt{t_1} \quad \ldots \quad (5.3)$$

where the constants M_0, M_2 *are as in* (D2). □

We now have the following theorem.

Theorem 5.2. *Let* $L = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ *and let* f, φ *satisfy conditions* (D1)—(D3). *Let* u *be defined as in* (3.18). *Then*

(*a*) *u is a continuous function on* $\bar{G}$ *such that*

$$|u(x)| \leqslant K(1 + |x|^2), \text{ for all } x \in \bar{G},$$

(*b*) *u is a stochastic solution to* (3.1);

(*c*) $\lim_{t \to \infty} E_x[u(X(t))] = 0$ *uniformly over compact subsets of* G.

Moreover u is the unique stochastic solution to (3.1) *in the class*

$$\mathcal{C}_2 = \{v : \bar{G} \to \mathbf{R} : (i)\, |v(x)| \leqslant K(1 + |x|^2), (ii) \lim_{t \to \infty} E_x[v(X(t))] = 0 \text{ for all } x \in \bar{G}\} \quad \ldots \quad (5.4)$$

Proof. In view of the preceding lemma, all the assertions except (*c*) can be proved as in Section 3.

In view of (D3), by an argument similar to the derivation of (3.26), (3.27), using Chapman-Kolmogorov equations, we get for $t > 1$.

$$|E_x[u(X(t))]| = |\lim_{T \to \infty} \bar{u}(t, T+t; x_1, \hat{x})|$$

$$\leqslant C(1 + |x|^2)\, t^{-\frac{d}{2}}$$

whence assertion (c) follows. This completes the proof. □

Our next result concerns the necessity of the condition (*D3*).

Proposition 5.3. *Let* $L = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ *and let* f, φ *satisfy* (D1), (D2). *Suppose there is a stochastic solution in the class* $\mathcal{C}_2$ *to the problem* (3.1). *Then* f, φ *satisfy the condition* (D3).

Proof. Note that the condition (D3) is not used in the proof of the uniqueness part of the preceding theorem. So, if $u \in \mathcal{C}_3$ is a stochastic solution then

$$u(x) = u(x_1, \hat{x}) = \lim_{T \to \infty} E_x \left[\int_0^T f(X(s))\, ds + \int_0^T \varphi\, (X(s)) d\xi(s) \right]$$

$$= \lim_{T \to \infty} \int_0^T \left(\frac{1}{2\pi s} \right)^{\frac{d}{2}} \left[\int_{R^d} F(z) \left\{ \exp\left(- \frac{|z-x|^2}{2s} \right) - 1 \right\} dz \right.$$

$$\left. + \int_{R^{d-1}} \varphi(\hat{z}) \left\{ e^{-\frac{x_1^2}{2s}} \exp\left(- \frac{|\hat{z}-\hat{x}|^2}{2s} \right) - 1 \right\} d\hat{z} \right] ds$$

$$+ \lim_{T \to \infty} \int_0^T \left(\frac{1}{2\pi s} \right)^{\frac{d}{2}} \left[\int_G f(y) dy + \frac{1}{2} \int_{\partial G} \varphi(\hat{y}) d\hat{y} \right] ds. \quad \ldots \quad (5.5)$$

By the proof of Lemma 5.1, using only conditions (D1), (D2), it is easily seen that first term on the r.h.s. of (5.5) is well defined. The second term on the r.h.s. of (5.5) is well defined only if (D3) is satisfied. □

In Theorem 5.2 uniqueness is guaranteed in the class $\mathcal{C}_3$ given by (5.4). It would be desirable to replace the condition (*ii*) in the definition of $\mathcal{C}_3$ by a condition not inovlving the parameter t. The following result is in that direction.

Theorem 5.4. *Let L be as before ; let f, φ be integrable functions on $\overline{G}$, ∂G respectively. For $z \in R^d$, put*

$$\hat{F}(z) = \int_{R^d} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} F(y)\, e^{-i < y,\, z >}\, dy.$$

$$\hat{\varphi}\,(z) = \int_{R^{d-1}} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \varphi\,(\hat{y}) \exp\left\{-i < (0, \hat{y}), z >\right\} d\hat{y},$$

$$u(z) = 2\,|z|^{-2}\, \{\hat{F}(z) + \hat{\varphi}(z)\},$$

where
$$F(y) = \begin{cases} f(y_1, y_2, \ldots, y_d),\ y \in \overline{G} \\ f(-y_1, y_2, \ldots, y_d),\ y \notin \overline{G} \end{cases}$$

Suppose u is an integrable function on R^d. Let u be defined as in (3.18). *Then u is the unique bounded continuous function vanishing at infinity, which is a stochastic solution to* (3.1).

Proof. By the spectral representation of the transition probability density of the Brownian motion note that

$$p(s, x, y) = \int_{R^d} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{1}{2}|z|^2 s} e^{i<x, z>} e^{-i<y, z>} dz$$

$$+ \int_{R^d} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{1}{2}|z|^2 s} e^{i<x, z>} e^{-i<y^*, z>} dz$$

for all $s > 0$, $x, y \in \overline{G}$, where $y^* = (-y_1, y_2, \ldots, y_d)$. Consequently under the given assumptions it can easily be verified that

$$u(x) = \int_{R^d} \hat{u}(z)\, e^{i(x, z)}\, dz$$

whence it follows that u is a bounded continuous function vanishing at infinity. It can be established as before that u is a stochastic solution to (3.1).

Suppose v is another such function. Given $\epsilon > 0$ one can find a compact set $K \subseteq \overline{G}$ such that $|v(x)| < \frac{1}{2}\epsilon$ for $x \in K$. Therefore

$$\sup_x |E_x[v(X(t))]| < \frac{1}{2}\epsilon + \|v\|_\infty |K| (2\pi t)^{-d/2}$$

where $|K|$ denotes the Lebesgue measure of K. From the above inequality it follows that $\sup_x |E_x[v(X(t))]| \to 0$ as $t \to \infty$. It is now easily seen that $v \equiv u$, completing the proof. □

Remark 5.5. The hypotheses of the preceding theorem imply that $\hat{F}(0) + \hat{\varphi}(0) = 0$ which is just condition (D3).

6. Neumann problem : $L = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}$

In this section we consider the Neumann problem for L when L is the generator of the Ornstein-Uhlenbeck process ; that is $a_{ij}(x) = \delta_{ij}$, $b_i(x) = -x_i$. The transition probability density is given by (2.14)–(2.16). Unlike the preceding cases, now one has an invariant probability measure $\nu(y)dy$ given by (2.17)–(2.19). We make the following assumptions on the prescribed data f, φ :

(E1) : $M_0 \equiv \int_{\overline{G}} |f(y)|\, dy + \int_{\partial G} |\varphi(\hat{y})|\, d\hat{y} < \infty$;

(E2) : $\int_{\overline{G}} f(y)\, \nu(y) dy + \frac{1}{2}\nu_1(0) \int_{\partial G} \varphi(\hat{y}) \nu_2(\hat{y}) d\hat{y} = 0.$

In what follows $\{P_x\}$ denotes the distribution of the Ornstein-Uhlenbeck process.

Theorem 6.1. *Let* $L = \frac{1}{2}\sum_{i=1}^{d}\left[\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}\right]$, *and* f, φ *satisfy* (E1), (E2). *Let* u *be defined as in* (3.18). *Then* u *is a continuous function on* $\overline{G}$ *such that*

(*a*) $|u(x)| \leqslant K_1 + K_2|x|$, *where the constants* K_1, K_2 *are as in* (2.20) ;

(*b*) u *is a stochastic soluiton to* (3.1) ;

(*c*) $\int_{G} u(x)\,\nu(x)dx = 0$.

Moreover, u is the unique stochastic solution in the class

$$\mathcal{C}_4 = \{h : \overline{G} \to \mathbf{R} \ : \text{(i)}\ |h(x)| \leqslant K(1+|x|),\ \text{(ii)} \int_{G} h(x)\,\nu(x)dx = 0\} \qquad \ldots \quad (6.1)$$

Proof. In view of Proposition 2.4 and conditions (E1), (E2) we have for $0 \leqslant t_1 < t_2 < \infty$, $x \in \overline{G}$,

$$\int_{t_1}^{t_2}\left|\int_{G} f(y)q(t, x, y)dy + \tfrac{1}{2}\int_{\partial G}\varphi(\hat{y})q(t, x, (0, \hat{y}))d\hat{y}\right|dt$$

$$\leqslant \int_{t_1}^{t_2}\int_{G} |f(y)|\,|q(t, x, y) - \nu(y)|\,dy\,dt$$

$$+ \ \tfrac{1}{2}\int_{t_1}^{t_2}\int_{\partial G} |\varphi(\hat{y})|\,|q(t, x, (0, \hat{y})) - \nu_1(0)\nu_2(\hat{y})|\,d\hat{y}\,dt$$

$$\leqslant (K_1 + K_2|x|)\,M_0\int_{t_1}^{t_2} e^{-t}\,dt. \qquad \ldots \quad (6.2)$$

From (6.2) it follows that u is well defined and that (a) holds. Continuity of u, and assertion (b) can be proved as in the earlier sections.

Note that $\nu_1(0) = \int_{[0, \infty)} q_1(t, x_1, 0)\,\nu_1(x_1)dx_1$ for any t. Since ν is the invariant measure, by (E2) we now have

$$\int_{G} u(x)\nu(x)dx = \lim_{T\to\infty}\int_{0}^{T}\int_{G}\int_{G} f(y)q(t, x, y)\nu(x)dy\,dx\,dt$$

$$+ \lim_{T\to\infty}\tfrac{1}{2}\int_{0}^{T}\int_{[0, \infty)}\int_{\partial G}\int_{\partial G}\varphi(\hat{y})q(t, x, (0, \hat{y}))\nu_1(x_1)\nu_2(\hat{x})d\hat{y}\,d\hat{x}\,dx_1\,dt$$

$$= \lim_{T\to\infty}\int_{0}^{T}\left[\int_{G} f(y)\nu(y)dy + \frac{1}{2}\,\nu_1(0)\int_{\partial G}\varphi(\hat{y})\nu_2(\hat{y})dy\right]dt = 0 \qquad \ldots \quad (6.3)$$

establishing (c).

Finally, by Proposition 2.5, for any stochastic solution $h \in \mathcal{C}_4$, note that $|h(y)\, q(t, x, y)| \leqslant$ [polynomial in $|y|$] $\nu(y)$. Consequently by the dominated convergence theorem, $\lim_{t \to \infty} E_x[h(X(t))] = \int h(x)\nu(x)dx = 0$, and the convergence is uniform on compact sets. It can now be proved that $h \equiv u$, completing the proof of the theorem. □

Theorem 6.2. *Let L be as in the preceding theorem ; let f, φ satisfy* (E1). *Suppose there is a stochastic solution in the class* $\mathcal{C}_4$ *to the problem* (3.1). *Then f, φ satisfy the condition* (E2).

Proof. In view of the derivation of (6.3), the theorem can be proved as in the earlier sections. □

Remark 6.3. It may be noted that Propositions 2.4 and 2.5 are the essential ingredients for proving the above theorems. Therefore, for any ergodic diffusion in $\overline{G}$ (with normal reflection at ∂G) such that zero is an isolated point of the spectrum of the generator (on the L_2-space with respect to the invariant probability) and for which Proposition 2.4 and 2.5 hold, our analysis can be extended. However, it is not clear to us for what class of diffusions Propositions 2.4 and 2.5 hold.

Acknowledgement. It is a pleasure to thank V. Pati and A. Sitaram for some useful discussions. Thanks are also due to a referee for a careful reading of the manuscript, comments and suggestions.

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